Delft University of Technology

Delft Center for Systems and Control

Technical report 15-047

Robust H_∞ switching control techniques for switched nonlinear systems with application to urban traffic control*

M. Hajiahmadi, B. De Schutter, and H. Hellendoorn

If you want to cite this report, please use the following reference instead:
M. Hajiahmadi, B. De Schutter, and H. Hellendoorn, "Robust H_∞ switching control techniques for switched nonlinear systems with application to urban traffic control," *International Journal of Robust and Nonlinear Control*, vol. 26, pp. 1286–1306, Apr. 2016. doi:10.1002/rnc.3504

Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

Robust H_∞ switching control techniques for switched nonlinear systems with application to urban traffic control

Mohammad Hajiahmadi*, Bart De Schutter, Hans Hellendoorn

Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands.

SUMMARY

This paper presents robust switching control strategies for switched nonlinear systems with constraints on the control inputs. A quantization technique is used to relax the constraint on continuous control inputs and the L_2-gain analysis and H_∞ control design problem for switched nonlinear systems are presented. Next, as an alternative method, the switched nonlinear system is approximated by a switched affine system that has autonomous and controlled switching behavior. A robust switching control law is proposed to stabilize the switched affine system. The design procedure involves solving an optimization problem that is nonconvex in a single scalar variable only. Furthermore, we provide the sufficient conditions under which the proposed switching law is able to stabilize the original switched nonlinear system. Finally, the proposed methods are utilized for control of urban traffic networks modeled on a high level. The traffic control objective is translated into a stability and disturbance attenuation problem for the urban network represented by a switched nonlinear system. The switching control approaches are able to reduce congestion in the network and to attenuate the effects of uncertain trip demands. Since the design of the switching laws is performed offline, real-time traffic control is possible with the proposed methods.

KEY WORDS: Switched systems, robust control, linear matrix inequalities, urban traffic control

1. INTRODUCTION

Switched systems are a class of hybrid systems that consist of a set of subsystems and a switching signal selecting the active subsystems. The switching behavior in these systems can be state and/or time dependent and can be autonomous or controlled or a combination of both [1]. In the controlled switching case, the switching laws can be designed in a way that the controlled system achieves a better performance [2, 3]. Stabilization and control synthesis for switched systems and in particular for switched linear systems have been widely studied using common and/or multiple Lyapunov function methods and for time and/or state dependent switching [2, 4, 5, 6, 7]. However, stability analysis of switched nonlinear systems has been investigated for particular classes only [8, 9, 10, 11].

Moreover, the disturbance attenuation problem for switched systems has attracted attention of researchers in recent years. The L_2-gain analysis and H_∞ control have been developed for switched linear systems based on the extension of algebraic Riccati inequalities [12]. For the particular cases of switched nonlinear systems, the H_∞ control problem is solved based on the Hamilton-Jacobi inequalities for nonlinear systems [13, 14, 15]. As an example, in [15] a nonlinear switched system is considered that is affine both in the control input and the disturbance input. The model contains a set of nonlinear subsystems each controlled with an unconstrained continuous control input. Further, a switching signal determines the active subsystem. However, the design procedure for the switching rule and the continuous feedback control is based on the fact that the control input is not constrained. In this paper, we study the stabilization problem for switched nonlinear systems that are affine in

*Correspondence to: m.hajiahmadi@tudelft.nl.

the control and disturbance inputs. The aim is to extend the current results on stabilization and H_∞ control for the constrained control case and moreover, to propose design procedures that are efficiently implementable.

The motivation for this research is based on a practical hybrid model developed for large-scale urban traffic network control [16]. In this model two types of controllers are defined: perimeter control for limiting the flow of vehicles traveling between urban regions and discrete control for switching between the signal timing plans of urban areas. The model is developed based on the existence of macroscopic fundamental diagrams (MFD) for the urban areas [17]. The MFD provides a relationship between the accumulation of vehicles in the network and the network trip completion rate. In fact, the MFD representation makes it possible to efficiently model a large-scale urban network at an aggregate high level and to subsequently develop control strategies that are less computationally complex compared to detailed modeling and control approaches previously proposed by considering individual intersections.

In [16] a mixed integer nonlinear optimization problem is formulated and is solved in the model predictive control (MPC) framework [18] in order to minimize the total delay in the network. Solving nonlinear and nonconvex optimization problems can be time-consuming and finding a global optimum solution is not guaranteed. In the MPC framework, having a prediction of the demand profile is crucial. However, an urban traffic network involves unpredictable human activities and consequently the uncertainties in traffic demands might influence the performance of control schemes. Some approaches are proposed in the literature to consider uncertain traffic demands [19, 20]. In this paper, we represent the network as a switched nonlinear system and we design a robust switching scheme to attenuate the effects of uncertain demands in the network.

More specifically, we aim at designing a new control scheme for urban networks represented by the hybrid model developed in [16] but without having exact knowledge about the traffic demands and at the same time requiring less online computational efforts. Basically, we consider the model in [16] as a switched nonlinear system. We consider the minimization of the total delay as a stabilization and disturbance attenuation problem. Since there are constraints on the feedback control inputs, we propose a model transformation (quantizing the control input) to relax them. The trip demands in the network are considered as exogenous disturbance signals. The main requirement of the proposed approach is that the disturbance is bounded in norm and belongs to the class of square integrable functions. This assumption is valid for finite time intervals (e.g. the peak hours) and since the trip demands inside the urban network are bounded and have a finite average. We propose two robust control design procedures that can be implemented offline and their online computation is limited to simple algebraic operations. This is a major advantage over e.g. the MPC approaches in [16, 21] which require considerable on-line computation.

The two proposed methods are:

1. we directly formulate the robust stabilizing conditions based on the nonlinear dynamics of the switched system. The design conditions are then constructed based on a multiple Lyapunov functions approach [8]. We previously presented a concise version of this method in [22].
2. we first approximate the switched nonlinear system with a switched affine system and next, we formulate the stability conditions based on the affine dynamics. The sufficient conditions for stabilizing the original switched nonlinear system using the designed switching law are presented.

In the first approach, we need to search for Lyapunov functions, which is a difficult task in general. However, for our traffic case study, the choice of quadratic functions of states is sufficient.

As for the second approach, the switched affine system is constructed by approximating each nonlinear subsystem using piecewise affine (PWA) functions. A controllable switching signal orchestrates the switching between piecewise affine subsystems. Note that there also exists an autonomous type of switching between affine functions of each PWA subsystem. This autonomous switching makes the stability analysis and control of such system challenging.

Stabilization of the switched affine system is performed using multiple quadratic Lyapunov functions and a min switching strategy. By fixing one scalar variable, the design conditions will be in the form of linear matrix inequalities. Compared to the existing min switching techniques

[5, 7] which are based on Metzler matrices, the proposed approach is less conservative. The key feature is to replace the elements of a Metzler matrix with matrix variables. This however comes at the price of introducing more variables in the stability conditions. Furthermore, it should be noted that smoothness of the nonlinear functions of the subsystems is a necessary condition for this design method.

Overall, the main contributions of the paper are 1) extending the results of [8, 23] to the robust control design for disturbance attenuation, 2) designing a robust switching law based on an approximate switched affine system with mixed switching and providing sufficient conditions to guarantee the stability of the original switching nonlinear system, 3) formulating the urban congestion control problem under uncertain trip profiles as a robust stabilization problem and proposing an efficient design procedure that is mostly performed offline.

It is worthwhile to mention that as a third approach to tackle the stabilization problem for switched nonlinear systems, one can assume that the nonlinear dynamics are bounded in sector-sets and then uses the sector conditions to obtain efficient control design conditions [11]. Although this method is more computationally efficient than the two proposed methods in this paper, it is in general more conservative due to the use of sector conditions.

The paper is organized as follows. The problem formulation along with a model transformation is presented in Section 2. A general procedure for the design of robust stabilizing switching laws is presented in Section 3. In Section 4, first the switched nonlinear system is approximated by a switched affine system and next, the robust H_∞ control design procedure is presented. The sufficient conditions for stabilizing the switched nonlinear system using the proposed switching law are discussed. Finally, in the case study section, the macroscopic modeling of urban traffic networks is reviewed and the related traffic variables and objectives are defined. Next, two robust stabilizing controllers are designed for a two-region urban network case. The performance of the proposed control schemes are compared with other static state feedback strategies and also with the model predictive control approach.

2. PROBLEM STATEMENT

Consider the following switched nonlinear system:

$$\dot{x}(t) = f_{\sigma(t)}\big(x(t)\big) + g_{\sigma(t)}\big(x(t)\big) \cdot u(t) + H_{\sigma(t)}\omega(t), \qquad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, and $\omega(t) \in \mathbb{R}^{n_\omega}$ is the disturbance input at time t. The superscripts n_x, n_u and n_ω denote the number states, the number of control inputs and the number of disturbance inputs, respectively. The switching signal is denoted by σ and is assumed to be piecewise constant and it is considered as a control input for the system. The variable σ takes values from a pre-defined index set $\{1, \dots, N\}$, and for each value that σ assumes, the state space model (1) is governed by a different set of vector functions $f_i(\cdot)$ and $g_i(\cdot)$ from the following sets:

$$f_{\sigma(t)} \in \{f_1, \dots, f_N\} \tag{2}$$
$$g_{\sigma(t)} \in \{g_1, \dots, g_N\} \tag{3}$$

The vector functions f_i and g_i are continuous functions of states such that $f_i(0) = 0$, $g_i(0) = 0$. Moreover, the control input u is constrained as follows:

$$u(t) \in [0, 1] \tag{4}$$

The aim is to design a state feedback control law together with a switching rule in order to stabilize the system and to reduce the effects of disturbances. However, the constraint (4) on the control input limits the design freedom.

The problem at hand cannot be easily tackled based on the current literature about stabilization of switched nonlinear systems. For instance, the approach presented in [10] provides a systematic way for designing stabilizing control inputs for switched nonlinear systems that are input/output-linearizable. The method is based on the existence of the control Lyapunov functions, which however are not easy to find for the general system (1). Moreover, the design procedures proposed in [24, 25] do not handle any constraint on the control input. Furthermore, in [10] an involved approach is proposed for the stabilization of switched nonlinear systems with the assumption that for each subsystem there exists a control Lyapunov function [26]. The proposed conditions would be simplified in case the system is input-output linearizable and the relative degree of the system is n.

As mentioned before, we assume that the control input $u(t)$ is constrained in $[0, 1]$. In some physical systems, the sensitivity to the variations of the control input is relatively low. In this case, we can limit the control input to take a finite number values from its defined domain. In other words, we can quantize our control input $u \in [0, 1]$ in the following form:

$$u(t) = \sum_{l=0}^{r} U_l \cdot \delta_l(t), \tag{5}$$

with $U_l \in \mathbb{R}$ constant coefficients and $\delta_l(t) \in \{0, 1\}$. The set of possible input values is then finite and its cardinality is 2^{r+1}, while the difference between two consecutive values is determined by the parameters U_l. By quantizing the control input u as in (5) new modes are introduced and therefore we denote the total number of modes by N' with a new set of vector functions $\{f'_1, \ldots, f'_{N'}\}$ that are determined using the functions f_i and g_i and the values that the quantized input u can take.

As a result, the system in (1) can be reformulated as:

$$\begin{aligned}\dot{x}(t) &= f_{\sigma(t)}\big(x(t)\big) + g_{\sigma(t)}\big(x(t)\big) \cdot \left[\sum_{l=0}^{r} U_{1,l} \cdot \delta_{1,l}(t), \ldots, \sum_{l=0}^{r} U_{n_u,l} \cdot \delta_{n_u,l}(t)\right]^{\mathrm{T}} + H_{\sigma(t)}\omega(t) \\ &= f'_{\sigma'(t)}\big(x(t)\big) + H_{\sigma'(t)}\omega(t), \end{aligned} \tag{6}$$

where $f'_{\sigma'(t)} \in \{f'_1, \ldots, f'_{N'}\}$.

The current formulation helps to have a concise design procedure as we reflect the effects of the continuous control input u in the switching signal σ' and hence, we have to deal only with one type of control input (switching). In the following sections, we present stabilizing switching control laws for the transformed system (6).

3. ROBUST H_∞ CONTROL OF SWITCHED NONLINEAR SYSTEM

In this section, we design a robust stabilizing switching law for system (6). We assume that the state vector $x(t)$ is available for feedback for all $t \geq 0$. First, we consider the case $\omega \equiv 0$ and we aim to determine a piecewise constant function $r(\cdot) : \mathbb{R}^{n_x} \to \{1, \ldots, N'\}$, such that the switching law:

$$\sigma'(t) = r(x(t)) \tag{7}$$

makes the equilibrium $x = 0$ globally asymptotically stable for (6), with $\omega \equiv 0$. It should be noted that we do not assume that any of the subsystems is locally or globally asymptotically stable. This means that our switching controller would be able to stabilize the switched system even when all subsystems are unstable.

The Lyapunov function $\vartheta(\cdot)$ is constructed as follows:

$$\vartheta(x) := \min_{i=1,\ldots,N'} V_i(x), \tag{8}$$

where $V_1, \ldots, V_{N'}$ are differentiable, positive definite, and radially unbounded functions of x. Furthermore, we use the notion of Metzler matrices [27, 5, 8]. A Metzler matrix is a matrix in which all the off-diagonal components are nonnegative. For our goal, we limit the attention to a subclass of Metzler matrices denoted by $\mathcal{M}$ and containing all matrices $M \in \mathbb{R}^{N' \times N'}$ with elements μ_{ij}, such that:

$$\mu_{ij} \geq 0 \; \forall i \neq j, \quad \sum_{i=1}^{N'} \mu_{ij} = 0, \; \forall j \tag{9}$$

The following theorem, adopted from [8, 23], provides the design procedure for a stabilizing switching rule that makes the switched system asymptotically stable in case $\omega \equiv 0$.

Theorem 1
[8] Assume there exist functions $V_1, \ldots, V_{N'}$, which are all differentiable, positive definite, radially unbounded, and zero at zero. Furthermore, assume there exists matrix $M \in \mathcal{M}$ with elements μ_{ij} that satisfies the Lyapunov-Metzler inequalities:

$$\frac{\partial V_i(x)}{\partial x} f_i'(x) + \sum_{j=1}^{N'} \mu_{ji} V_j(x) < 0, \;\; i \in \{1, \ldots, N'\}, \tag{10}$$

for all $x \neq 0$. Then the switching rule (7) with[†]:

$$r(x(t)) = \arg \min_{i=1,\ldots,N'} V_i(x(t)), \tag{11}$$

makes the equilibrium point $x = 0$ of (6) globally asymptotically stable when $\omega \equiv 0$.

Proof
We present the proof from [8]. Later we will use a similar approach to prove the next theorem. The Lyapunov function (8) is piecewise differentiable, which means that it is not differentiable for all $x \in \mathbb{R}^{n_x}$. Therefore, we need to define the following derivative (see [8, 28]):

$$\mathbf{D}(\vartheta(x(t))) = \lim_{\Delta t \to 0^+} \sup \frac{\vartheta(x(t+\Delta t)) - \vartheta(x(t))}{\Delta t} \tag{12}$$

Assume that at an arbitrary $t \geq 0$, the state switching control is given by $\sigma(t) = r(x(t)) = i$ for some $i \in I(x(t)) = \{i : \; \vartheta(x) = V_i(x)\}$. Hence, from (12) and (6) with $\omega \equiv 0$, we have (using Theorem 1 on page 420 of [29]):

$$\mathbf{D}(\vartheta(x(t))) = \min_{l \in I(x(t))} \frac{\partial V_l}{\partial x} f_i' \leq \frac{\partial V_i}{\partial x} f_i', \tag{13}$$

Since (10) is valid for any $M \in \mathcal{M}$ and $V_j \geq V_i$ for all $j \in \{1, \ldots, N'\} \setminus \{i\}$, using the fact that $i \in I(x(t))$ and by rewriting the Lyapunov-Metzler inequality (10) as follows:

$$\frac{\partial V_i}{\partial x} f_i' < -\sum_{j=1}^{N'} \mu_{ji} V_j, \;\; i \in \{1, \ldots, N'\}, \tag{14}$$

one can obtain:

$$\mathbf{D}(\vartheta(x(t))) \leq \frac{\partial V_i}{\partial x} f_i' < -\sum_{j=1}^{N'} \mu_{ji} V_j \leq -\Big(\sum_{j=1}^{N'} \mu_{ji}\Big) V_i = 0, \;\; \text{for all } x \neq 0 \tag{15}$$

Thus, the switching law (11) makes the equilibrium point $x = 0$ of the switched nonlinear system (6), with $\omega \equiv 0$, globally asymptotically stable. □

[†]Note that in (11), we take the minimum argument, in case of having multiple minima $V_{i,\ell}$ (this can happen as a result of sliding motion).

Remark 1
In order to design the switching law (11), one would need to find appropriate positive definite functions V_i and a Metzler matrix that satisfy the Lyapunov-Metzler inequalities (10) for all $x \neq 0$. Unfortunately, this is a difficult task in general since it includes determination of positive definite *functions*. Fortunately, the choice of quadratic functions works for many cases (e.g. for our case study). Nevertheless, finding the coefficients of the quadratic functions V_i along with the elements of the Metzler matrix constitutes a nonlinear feasibility optimization problem. In some cases, we can recast this problem as a Bilinear Matrix Inequality (BMI) problem [30] and thus, take advantage of the existing solvers for BMIs. But the general case would be a multi-parametric optimization problem. Nonetheless, one can use another approach to tackle the problem of finding the parameters of V_i along with the elements of the Metzler matrix M. By gridding the domain of the state x, one can formulate the Lyapunov-Metzler inequalities for each vertex of the grid. Depending on the characteristics of the system under study and the objectives, one can make grids with different levels of accuracy in a uniform or non-uniform way. Next, the remaining task is to find solutions for the parameters of V_i and the Metzler matrix in order to satisfy all Lyapunov-Metzler inequalities for all grid points. This is a nonlinear optimization problem in which the feasibility of all nonlinear inequality constraints has to be checked. Of course, there might exist multiple solutions for this problem but any feasible solution would work for finding the stabilizing switching law.

Now suppose that the switched nonlinear system is affected by norm bounded disturbances. The model of the system is as follows:

$$\dot{x}(t) = f'_{\sigma'(t)}\big(x(t)\big) + H_{\sigma'(t)}\omega(t), \;\; x(0) = x_0, \tag{16}$$

$$y(t) = C_{\sigma'(t)}x(t), \tag{17}$$

with $y(t) \in \mathbb{R}^{n_y}$ the output vector. Moreover, we assume that the disturbance vector ω belongs to the space of square integrable functions on $[0, T]$, $\forall T \geq 0$, as follows:

$$\|\omega\|_{L_2[0,T]} = \left(\int_0^T \omega^T(t)\omega(t)\mathrm{d}t\right)^{1/2} < \infty \tag{18}$$

System (16) has an L_2-gain $\gamma > 0$ under some switching law σ' if $\|y\|_{L_2[0,T]} \leq \gamma\|\omega\|_{L_2[0,T]}$ for all nonzero $\omega \in L_2[0, T]$ $(0 \leq T < \infty)$ and for initial condition $x(0) = 0$. It follows that:

$$\|y\|_{L_2[0,T]} \leq \gamma\|\omega\|_{L_2[0,T]} \Longleftrightarrow \int_0^T \Big(\|y(t)\|^2 - \gamma^2\|\omega(t)\|^2\Big)\mathrm{d}t \leq 0 \tag{19}$$

for any $T > 0$ when $x(0) = 0$. The aim is to design a switching strategy σ' such that system (16) has L_2-gain γ or equivalently, to have an H_∞ disturbance attenuation level γ.

The approach for H_∞ control of switched nonlinear systems proposed in [15] is not applicable for control of (1), as the input u is constrained in the box $[0, 1]^{n_u}$. Nevertheless, we transformed the model using quantization of the input variable and obtained the model in (6). For this model, the following problem is defined. Assume that a constant $\gamma > 0$ is given, the goal is to design a switching law σ', such that the origin of the closed-loop system is globally asymptotically stable when $\omega(t) = 0, \forall t \geq 0$, and the overall L_2-gain from ω to y on any finite time interval $[0, T]$ is less than or equal to γ. The following theorem provides the design procedure for the switching law (inspired by the linear case in [12]).

Theorem 2
Consider the switched system (6). Assume that there exist positive definite, differentiable, and radially unbounded functions V_i, $i \in \{1, \ldots, N'\}$, a positive scalar γ, and a Metzler matrix M with elements μ_{ij}, such that the following Lyapunov-Metzler inequalities are satisfied:

$$\frac{\partial V_i}{\partial x} f'_i + \frac{1}{2\gamma^2}\frac{\partial V_i}{\partial x} H_i H_i^T \frac{\partial^T V_i}{\partial x} + \frac{1}{2}C_i^T C_i + \sum_{j=1}^{N'} \mu_{ji} V_j < 0 \tag{20}$$

for $i = 1, \ldots, N'$. Then, the system (16) under the switching law

$$\sigma'(t) = r(x(t)) = \arg \min_{i=1,\ldots,N'} V_i(x(t)) \tag{21}$$

has L_2-gain upper bounded by γ. Subsequently, in case $\omega \equiv 0$, the system is asymptotically stable.

Before proceeding with the proof, we emphasize again that the switching signal is assumed to be piecewise constant. In other words, one can define a switching sequence as $\{(t_k, r(x(t_k)))\}_{k=1}^{\infty}$ with $r(x(t_k)) \in \{1, \ldots, N'\}$, while the switching rule remains unchanged in the interval $[t_k, t_{k+1})$.

Proof
Assume that the switching sequence in the interval $[0, T]$ is defined as:

$$\left\{ \Big(t_k, r\big(x(t_k)\big)\Big) \,\Big|\, r\big(x(t_k)\big) \in \{1, \ldots, N'\},\ k = 1, 2, \ldots, l \right\} \tag{22}$$

with $t_1 = 0$ and $t_l \leq T$. Under the switching law (11) in each time interval $[t_k, t_{k+1})$ we have:

$$\frac{\partial V_i}{\partial x} f_i' + \frac{1}{2\gamma^2} \frac{\partial V_i}{\partial x} H_i H_i^T \frac{\partial^T V_i}{\partial x} + \frac{1}{2} C_i^T C_i < -\sum_{j=1}^{N'} \mu_{ji} V_j \leq \Big(-\sum_{j=1}^{N'} \mu_{ji} \Big) V_i = 0 \tag{23}$$

Now following a similar procedure as in [13, 12], we define

$$J = \int_0^T \Big(\frac{1}{2} \|C_{\sigma'(t)} x(t)\|^2 - \frac{\gamma^2}{2} \|\omega(t)\|^2 + \mathbf{D}\big(\vartheta(x(t))\big) \Big) \mathrm{d}t \tag{24}$$

According to the definition of $\mathbf{D}\big(\vartheta(x)\big)$ in (13) and taking into account the switching sequence (22), we obtain:

$$\begin{aligned} J \leq & \sum_{k=1}^{l-1} \int_{t_k}^{t_{k+1}} \big(\frac{1}{2} \|C_{r(x(t_k))} x\|^2 - \frac{\gamma^2}{2} \|\omega\|^2 + \dot{V}_{r(x(t_k))}(x) \big) \mathrm{d}t \\ & + \int_{t_l}^{T} \Big(\frac{1}{2} \|C_{r(x(t_l))} x\|^2 - \frac{\gamma^2}{2} \|\omega\|^2 + \dot{V}_{r(x(t_l))}(x) \Big) \mathrm{d}t \end{aligned} \tag{25}$$

The derivative $\dot{V}_{r(x(t_k))}$ is:

$$\dot{V}_{r(x(t_k))}\big(x(t)\big) = \frac{\partial V_{r(x(t_k))}\big(x(t)\big)}{\partial x} \cdot \Big[f'_{r(x(t_k))}\big(x(t)\big) + H_{r(x(t_k))} \omega(t) \Big] \tag{26}$$

Substitution of (26) in (25) along with adding and subtracting the term:

$$\frac{1}{2\gamma^2} \frac{\partial V_{r(x(t_k))}}{\partial x} H_{r(x(t_k))} H_{r(x(t_k))}^T \frac{\partial^T V_{r(x(t_k))}}{\partial x}, \tag{27}$$

and completing the squares yields (the arguments of the functions are dropped for reducing the complexity):

$$\begin{aligned} & \sum_{k=1}^{l-1} \int_{t_k}^{t_{k+1}} \Big(\frac{\partial V_{r(x(t_k))}}{\partial x} f'_{r(x(t_k))} + \frac{1}{2} \|C_{r(x(t_k))} x(t_k)\|^2 \\ & \qquad + \frac{1}{2\gamma^2} \frac{\partial V_{r(x(t_k))}}{\partial x} H_{r(x(t_k))} H_{r(x(t_k))}^T \frac{\partial^T V_{r(x(t_k))}}{\partial x} - \Big\| \frac{\gamma}{\sqrt{2}} \omega - \frac{1}{\sqrt{2}\gamma} \frac{\partial V_{r(x(t_k))}}{\partial x} H_{r(x(t_k))} \Big\|^2 \Big) \mathrm{d}t \\ & + \int_{t_l}^{T} \Big(\frac{\partial V_{r(x(t_l))}}{\partial x} f'_{r(x(t_l))} + \frac{1}{2} \|C_{r(x(t_l))} x(t_l)\|^2 \\ & \qquad + \frac{1}{2\gamma^2} \frac{\partial V_{r(x(t_l))}}{\partial x} H_{r(x(t_l))} H_{r(x(t_l))}^T \frac{\partial^T V_{r(x(t_l))}}{\partial x} - \Big\| \frac{\gamma}{\sqrt{2}} \omega - \frac{1}{\sqrt{2}\gamma} \frac{\partial V_{r(x(t_l))}}{\partial x} H_{r(x(t_l))} \Big\|^2 \Big) \mathrm{d}t \end{aligned} \tag{28}$$

Referring to (23), we can conclude that (28) is smaller or equal to zero. Hence,

$$J = \int_0^T \Big(\frac{1}{2}\|C_{\sigma'(t)}x\|^2 - \frac{\gamma^2}{2}\|\omega\|^2 + \mathbf{D}(\vartheta)\Big)\mathrm{d}t \leq 0 \tag{29}$$

Note that V_i are positive definite functions with zero value at zero. Thus,

$$\int_0^T \Big(\|C_{\sigma'(t)}x\|^2 - \gamma^2\|\omega\|^2\Big)\mathrm{d}t \leq -2V_i(x(T)) \leq 0, \ \forall i \tag{30}$$

Hence, the system has L_2-gain less than or equal to γ. Moreover, it is easy to show (by utilizing Lemma 3.2.6 of [13]) that the system is asymptotically stable when $\omega \equiv 0$. □

Similar to the procedure explained in Remark 1, a feasibility problem has to be solved in order to find the parameters of the functions V_i along with μ_{ij}. Moreover, the L_2-gain γ can be set either as an unknown parameter to be determined or as a given constant. Basically, one can set a preliminary value for γ and solve the feasibility problem for the given γ. The procedure can be repeated with decreasing values of γ until the problem becomes infeasible and no solution can be obtained for the parameters. By doing this a lower bound for the L_2-gain can be achieved.

4. ROBUST CONTROL DESIGN USING APPROXIMATE SWITCHED AFFINE SYSTEMS

In this section, we propose a robust control design procedure based on the approximation of the switched nonlinear system by a switched affine system. A function $\phi : \Omega \to \mathbb{R}^m$ is PWA if there exists a polyhedral partition $\{\Omega_i\}_{i\in\mathcal{I}}$ $(\cup_{i\in\mathcal{I}}\Omega_i = \Omega,\ \mathrm{int}(\Omega_i) \cap\ \mathrm{int}(\Omega_j) = \varnothing,\ \forall i \neq j)$ of $\Omega \subseteq \mathbb{R}^n$ such that ϕ is affine on each polyhedron Ω_i. By considering a sufficiently large number of regions, one can approximate nonlinear functions f_i by PWA functions with arbitrary accuracy. The piecewise affine (PWA) approximation of f_i will have the following form:

$$f_i(x) \cong \big(A_{i,\ell}\cdot x + b_{i,\ell}\big), \qquad \text{if} \quad x \in \mathcal{X}_{i,\ell}, \tag{31}$$

with $A_{i,\ell}(n\times n)$ and $b_{i,\ell}(n\times 1)$ the PWA matrices, $\mathcal{X}_{i,\ell}$ the corresponding polyhedron, and $\ell \in \mathcal{M}_i = \{1,\ldots,M_i\}$, with M_i the number of polyhedral partitions for function f_i.

Now the switched system (6) can be approximated by the following switched affine system:

$$\dot{x}(t) = A_{\sigma(t),\ell}x(t) + b_{\sigma(t),\ell} + H_{\sigma(t)}\omega(t), \tag{32}$$

$$y(t) = C_{\sigma(t)}x(t), \qquad \text{if} \quad x \in \mathcal{X}_{\sigma(t),\ell}, \tag{33}$$

where the controlled switching signal σ takes values from the set $\mathcal{N} = \{1,\ldots,N\}$, with N the total number of subsystems.

Note that two types of switching are integrated in (32), one associated with switching between affine functions describing the dynamics of each subsystem i; this type of switching is therefore uncontrolled, and the other one is the controlled switching between subsystems driven by σ. In the following sections, the focus is first on the stabilization and robust control of (32) and next, on connecting the obtained results to the stability problem for the original switched nonlinear system (16).

Before presenting the main results, we draw the attention to the fact that functions f_i in (16) may not all be approximated using the same number of affine functions and also, not with the same polyhedral regions. Therefore, even if the number of affine pieces is not the same for all nonlinear functions, we can split the affine functions in such a way that overall, the number of affine functions will be identical for all nonlinear functions f_i and moreover, the polyhedral regions will be common for all piecewise affine subsystems. Taking this into account, each polyhedral region $\mathcal{X}_\ell$ is characterized by [31]:

$$F_\ell x + f_\ell \succeq 0, \qquad \text{iff} \ \ x \in \mathcal{X}_\ell, \tag{34}$$

where the inequality is element-wise. Further, (34) can be reformulated as follows:

$$\bar{F}_\ell \begin{bmatrix} x \\ 1 \end{bmatrix} \succeq 0, \qquad \bar{F}_\ell = \begin{bmatrix} F_\ell & f_\ell \end{bmatrix} \tag{35}$$

Furthermore, the boundary hyperplane for each pair of neighboring regions $\mathcal{X}_\ell$ and $\mathcal{X}_{\ell'}$ is represented by:

$$h_{\ell\ell'}^{\mathrm{T}} x + g_{\ell\ell'} = 0 \Leftrightarrow \underbrace{\begin{bmatrix} h_{\ell\ell'}^{\mathrm{T}} & g_{\ell\ell'} \end{bmatrix}}_{\bar{h}_{\ell\ell'}^{\mathrm{T}}} \begin{bmatrix} x \\ 1 \end{bmatrix} = 0 \tag{36}$$

Moreover, for each polyhedral region $\mathcal{X}_\ell, \ell \in \mathcal{M} = \{1, \ldots, M\}$, with M the total number of polyhedral regions (number of affine functions associated to each subsystem), the following auxiliary functions are defined:

$$V_{i,\ell}(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix}}_{\bar{P}_{i,\ell}} \underbrace{\begin{bmatrix} x \\ 1 \end{bmatrix}}_{\bar{x}}, \quad \forall i \in \mathcal{N}, \forall \ell \in \mathcal{M}. \tag{37}$$

with $P_{i,\ell} \in \mathbb{R}^{n \times n}$ symmetric, $s_{i,\ell} \in \mathbb{R}^n$, and $r_{i,\ell} \in \mathbb{R}$. For each $\mathcal{X}_\ell$, a Lyapunov function is proposed as follows:

$$\mathcal{V}_\ell(x) = \min_{i \in \mathcal{N}} V_{i,\ell}(x), \tag{38}$$

The following theorem presents the design procedure for a stabilizing switching rule that brings the state of the approximate system (16) to the origin, in the absence of disturbances and moreover, ensures the L_2-gain γ for the system exposed to disturbances ω that belong to the L_2 space.

Note that in the following theorems we use the augmented system matrices and vectors defined as follows [32]:

$$\bar{A}_{i,\ell} = \begin{bmatrix} A_{i,\ell} & b_{i,\ell} \\ 0_{1\times n} & 0 \end{bmatrix}, \ \bar{G}_i = \begin{bmatrix} G_i \\ 0_{1\times n_u} \end{bmatrix},$$
$$\bar{H}_i = \begin{bmatrix} H_i \\ 0_{1\times n_\omega} \end{bmatrix}, \ \bar{C}_i = \begin{bmatrix} C_i & 0_{n_y \times 1} \end{bmatrix} \tag{39}$$

Theorem 3

Assume there exist symmetric matrices $\bar{P}_{i,\ell}$, $\mathcal{T}_{i,j,\ell}$, symmetric matrices U_ℓ, Z_ℓ with nonnegative elements, and vectors $\zeta_{\ell\ell'}$ such that the following optimization problem:

$$\min \gamma \tag{40}$$

s.t.

$$\begin{bmatrix} \bar{P}_{i,\ell}\bar{A}_{i,\ell} + \bar{A}_{i,\ell}^{\mathrm{T}}\bar{P}_{i,\ell} - \sum\limits_{j\in\mathcal{N}, j\neq i} \mathcal{T}_{i,j,\ell} + \bar{F}_\ell^{\mathrm{T}} U_\ell \bar{F}_\ell & \star & \star \\ \bar{H}_i^{\mathrm{T}} \bar{P}_{i,\ell} & -\gamma I & \star \\ \bar{C}_i & 0 & -I \end{bmatrix} < 0,$$
$$\forall i, j \in \mathcal{N},\ i \neq j,\ \forall \ell \in \mathcal{M}, \tag{41}$$

$$\mathcal{T}_{i,j,\ell} < \mu_{\min} \cdot \left(\bar{P}_{i,\ell} - \bar{P}_{j,\ell}\right), \qquad \forall i, j \in \mathcal{N},\ i \neq j,\ \forall \ell \in \mathcal{M}, \tag{42}$$

$$\bar{P}_{i,\ell} - \bar{F}_\ell^{\mathrm{T}} Z_\ell \bar{F}_\ell > 0, \qquad \forall i \in \mathcal{N},\ \ell \in \mathcal{M}, \tag{43}$$

$$\bar{P}_{i,\ell} = \bar{P}_{i,\ell'} + \bar{h}_{\ell\ell'}\zeta_{\ell\ell'}^{\mathrm{T}} + \zeta_{\ell\ell'}\bar{h}_{\ell\ell'}^{\mathrm{T}}, \qquad \forall \ell, \ell' : \mathcal{X}_\ell \cap \mathcal{X}_{\ell'} \neq \emptyset, \qquad \forall i \in \mathcal{N}, \tag{44}$$

has an optimal solution $\gamma^* > 0$ for a given positive scalar $\mu_{\min} > 0$, then the switching rule:

$$\sigma(t) = \arg \min_{i \in \mathcal{N}} V_{i,\ell}(x(t)), \qquad \text{if } x(t) \in \mathcal{X}_\ell \tag{45}$$

with $V_{i,\ell}$ defined as in (37), will asymptotically stabilize system (32)–(33) in case $\omega \equiv 0$, and moreover ensures an upper bound $\sqrt{\gamma^*}$ for the L_2-gain of the system from the disturbance input ω to the output y.

Proof
Suppose that at an arbitrary time instant $t \geq 0$ and based on the polyhedral region ℓ in which the state of the system resides, the switching law is given by $\sigma(t) = r(x(t)) = i$ for some $i \in \mathcal{I}_\ell(x(t)) = \{i : \mathcal{V}_\ell(x) = V_{i,\ell}(x)\}$. Hence, following the definition of the Dini derivative [28, 8], for our system (32), we have:

$$\mathbf{D}\big(\mathcal{V}_\ell(x(t))\big) = \min_{j\in\mathcal{I}_\ell(x(t))} \left[\frac{\partial V_{j,\ell}}{\partial x}\Big(A_{j,\ell}x + b_{j,\ell} + H_j\omega\Big)\right] \leq \frac{\partial V_{i,\ell}}{\partial x}\Big(A_{i,\ell}x + b_{i,\ell} + H_i\omega\Big) \tag{46}$$

where i denotes the index of the active subsystem in region ℓ determined from (45). Pre-multiplying (41) by $[x^{\mathrm{T}}, 1]$ and post-multiplying by its transpose, using (42) and also the fact that for the polyhedral region ℓ, (35) holds, and U_ℓ has nonnegative entries, we obtain

$$\begin{aligned}
\underbrace{\begin{bmatrix}\bar{x}\\ \omega\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\bar{P}_{i,\ell}\bar{A}_{i,\ell}+\bar{A}_{i,\ell}^{\mathrm{T}}\bar{P}_{i,\ell} & \star\\ \bar{H}_i^{\mathrm{T}}\bar{P}_{i,\ell} & 0\end{bmatrix} \begin{bmatrix}\bar{x}\\ \omega\end{bmatrix}}_{\frac{\partial V_{i,\ell}}{\partial x}\left(A_{i,\ell}x+b_{i,\ell}+H_i\omega\right)} &< \sum_{j\in\mathcal{N}, j\neq i} \bar{x}^{\mathrm{T}}\mathcal{T}_{i,j,\ell}\bar{x} - \bar{x}^{\mathrm{T}}\bar{F}_\ell^{\mathrm{T}}U_\ell\bar{F}_\ell\bar{x} - y^{\mathrm{T}}y + \gamma\omega^{\mathrm{T}}\omega\\
&< \bar{x}^{\mathrm{T}} \sum_{j\in\mathcal{N}, j\neq i} \mu_{\min}\big(\bar{P}_{i,\ell} - \bar{P}_{j,\ell}\big)\bar{x} - y^{\mathrm{T}}y + \gamma\omega^{\mathrm{T}}\omega
\end{aligned} \tag{47}$$

Since for the active subsystem i in region ℓ, $V_{i,\ell} \leq V_{j,\ell},\ \forall j \in \mathcal{N}$, we finally have:

$$\mathbf{D}^+\big(\mathcal{V}_\ell(x(t))\big) < -y^{\mathrm{T}}y + \gamma\omega^{\mathrm{T}}\omega \tag{48}$$

On the other hand, we should connect the Lyapunov functions in neighboring polyhedral regions in such a way that the decrease in the overall Lyapunov function is ensured. One way to tackle this problem is to equalize the values of the Lyapunov functions $\mathcal{V}_{i,\ell}$ and $\mathcal{V}_{i,\ell'}$ for the boundary hyperplane of neighboring regions $\mathcal{E}_\ell$ and $\mathcal{E}_\ell'$. Note that at the boundary between polyhedral regions an uncontrolled switching between affine functions of the same subsystem i occurs. Therefore, we only need to connect the Lyapunov functions associated with each subsystem i at the boundary between neighboring regions ℓ and ℓ'. Hence, we have:

$$\bar{x}^{\mathrm{T}}\bar{P}_{i,\ell}\bar{x} = \bar{x}^{\mathrm{T}}\bar{P}_{i,\ell'}\bar{x},\ \forall x :\ \bar{h}_{\ell\ell'}^{\mathrm{T}}\bar{x} = 0 \tag{49}$$

In order to recast (49) as an LMI, we define auxiliary vectors $\zeta_{\ell\ell'}$ and combine the two equalities in (49) in the following way:

$$\bar{x}^{\mathrm{T}}\bar{P}_{i,\ell}\bar{x} = \bar{x}^{\mathrm{T}}\bar{P}_{i,\ell'}\bar{x} + \bar{x}^{\mathrm{T}}\bar{h}_{\ell\ell'}\zeta_{\ell\ell'}^{\mathrm{T}}\bar{x} + \bar{x}^{\mathrm{T}}\zeta_{\ell\ell'}\bar{h}_{\ell\ell'}^{\mathrm{T}}\bar{x}. \tag{50}$$

Since (50) should hold for all x, we can instead check the feasibility of the equality (44). The idea is inspired by the so-called Finsler Lemma [33].

Now since the overall Lyapunov function is continuous over the boundaries between the polyhedral regions, and since asymptotic stability implies $V(x(\infty)) = 0$, using (48), from (48) we obtain (it is assumed that the initial state is zero and $V(x(0)) = 0$. The reason for this assumption is to eliminate the transient response of the system due to nonzero initial conditions) $\|y\|_2 \leq \gamma\|\omega\|_2$.

Finally, the Lyapunov functions (37) are not required to be positive definite in the entire space but only in the active polyhedral region. This is ensured using constraint (43) and it can be easily proved using (35) and the S-procedure [34].

□

Remark 2
In order to solve the optimization problem (40)–(44), one should set a value for $\mu_{\min}$. In fact, $\mu_{\min}$ is

acting like a bound on the elements of a Metzler matrix. Instead of limiting the diagonal elements of a Metzler matrix to be identical (as it is suggested by [5, 8], in order to facilitate with linearizing the stability conditions), we fix a lower bound for all the elements and moreover, we introduce auxiliary matrix variables $\mathcal{T}_{i,j,\ell}$. These two will make the approach less conservative. Hence, the optimization problem (40)–(44) can be recast as a bi-level optimization problem in which on the higher level a bisection search on $\mu_{\min}$ is performed, while on the lower level a convex optimization problem subject to LMI constraints (with fixed $\mu_{\min}$) is solved.

Next, we discuss the stability of the switched nonlinear system (6) using the switching law designed based on the approximated switched affine system (32). For simplicity and without loss of generality we assume that $\omega \equiv 0$. The approximation error can be defined as follows:

$$\epsilon_i(x) = f_i(x) - \big(A_{i,\ell}x + b_{i,\ell}\big) \quad \forall i \in \mathcal{N}, \text{ for } x \in \mathcal{X}_\ell. \tag{51}$$

Suppose that the original switched nonlinear system (6) is controlled by the switching law (45). Therefore when $\sigma(t) = i$, the dynamics of (6) is governed by f_i. Hence, the derivative of the Lyapunov function (38) along the trajectories of (6) is:

$$\dot{V}_\ell = \begin{bmatrix} f_i(x) \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} + \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} f_i(x) \\ 0 \end{bmatrix} \tag{52}$$

for $x \in \mathcal{X}_\ell$ (note that since the continuity of V_ℓ on the boundaries of the polyhedral regions is preserved under conditions of Theorem 3, we therefore only consider the behavior of V_ℓ and $\dot{V}_\ell$ inside the polyhedral regions). Replacing $f_i(x)$ by $\epsilon_i(x) + A_{i,\ell}x + b_{i,\ell}$ yields:

$$\dot{V}_\ell = \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_{i,\ell}^{\mathrm{T}} P_{i,\ell} + P_{i,\ell} A_{i,\ell} & \star \\ b_{i,\ell} P_{i,\ell} + s_{i,\ell}^{\mathrm{T}} A_{i,\ell} & 2b_{i,\ell}^{\mathrm{T}} s_{i,\ell} \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} + 2 \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} \epsilon_i(x) \\ 0 \end{bmatrix}. \tag{53}$$

Now since the inequalities in (41) of Theorem 3 are strict, it implies that if (41) holds, there should exist a positive scalar variable denoted by α such that:

$$\begin{bmatrix} P_{i,\ell} A_{i,\ell} + A_{i,\ell}^{\mathrm{T}} P_{i,\ell} & \star \\ b_{i,\ell}^{\mathrm{T}} P_{i,\ell} + s_{i,\ell}^{\mathrm{T}} A_{i,\ell} & 2b_{i,\ell}^{\mathrm{T}} s_{i,\ell} \end{bmatrix} - \sum_{j \in \mathcal{N}, j \neq i} \mathcal{T}_{i,j,\ell} + \bar{F}_\ell^{\mathrm{T}} U_\ell \bar{F}_\ell < -\alpha I,$$
$$\forall i, j \in \mathcal{N},\ i \neq j,\ \forall \ell \in \mathcal{M} \tag{54}$$

Now if (54) holds, we obtain:

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} A_{i,\ell} + A_{i,\ell}^{\mathrm{T}} P_{i,\ell} & \star \\ b_{i,\ell}^{\mathrm{T}} P_{i,\ell} + s_{i,\ell}^{\mathrm{T}} A_{i,\ell} & 2b_{i,\ell}^{\mathrm{T}} s_{i,\ell} \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < -\alpha \|\bar{x}\|_2^2, \tag{55}$$

for the active subsystem i in (32). Therefore, for (53) we have:

$$\dot{V}_\ell < -\alpha \|\bar{x}\|_2^2 + 2 \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} \epsilon_i(x) \\ 0 \end{bmatrix}, \tag{56}$$

for $x \in \mathcal{X}_\ell$. Therefore, in order to have $\dot{V}_\ell < 0$ for the switched nonlinear system, we need to have:

$$2 \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_{i,\ell} & \star \\ s_{i,\ell}^{\mathrm{T}} & r_{i,\ell} \end{bmatrix} \begin{bmatrix} \epsilon_i(x) \\ 0 \end{bmatrix} < \alpha \|\bar{x}\|_2^2 \tag{57}$$

The following proposition provides the sufficient condition for stabilization of the switched nonlinear system (6) using switching law (45).

Proposition 1
Assume there exist matrices $P_{i,\ell}$ and $\mathcal{T}_{i,j,\ell}$, vectors $s_{i,\ell}$, $\zeta_{\ell\ell'}$, scalars $r_{i,\ell}$, $\alpha > 0$ and symmetric matrices U_ℓ, Z_ℓ with nonnegative elements that satisfy (41)–(44) and (54) for a given positive scalar

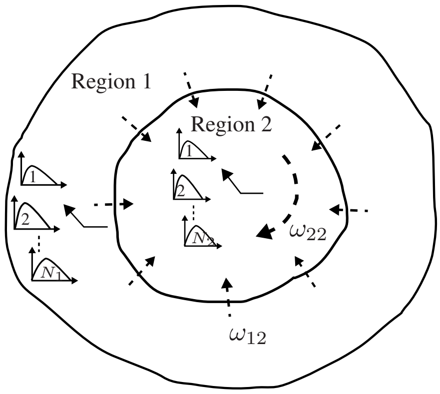


Figure 1. Schematic two-region urban network.

$\mu_{\min} > 0$. Then the switching rule (45) asymptotically stabilizes (6) provided that the norm of the PWA approximation error is bounded by:

$$\|\epsilon_i(x)\|_2 < \frac{\alpha\|\bar{x}\|_2}{2a_{\max}(\bar{P}_{i,\ell})}, \qquad \forall i \in \mathcal{N}, \text{ for } x \in \mathcal{X}_\ell, \tag{58}$$

where $a_{\max}(\bar{P}_{i,\ell})$ denotes the largest singular value of $\bar{P}_{i,\ell}$.

Proof
First, it can be easily proved that:

$$\bar{x}^{\mathrm{T}}\bar{P}_{i,\ell}\begin{bmatrix}\epsilon_i(x)\\0\end{bmatrix} \le \|\bar{x}\|_2 a_{\max}(\bar{P}_{i,\ell})\|\epsilon_i(x)\|_2 \tag{59}$$

Therefore, using (58) we obtain:

$$2\bar{x}^{\mathrm{T}}\bar{P}_{i,\ell}\begin{bmatrix}\epsilon_i(x)\\0\end{bmatrix} \le 2\|\bar{x}\|_2 a_{\max}(\bar{P}_{i,\ell})\|\epsilon_i(x)\|_2 \le \alpha\|\bar{x}\|_2^2 \tag{60}$$

which yields $\dot{V}_\ell < 0$ as in (56) and hence, asymptotic stability of the switched nonlinear system (6) is ensured. □

Remark 3
As can be inferred from (58), the upper bound on the approximation error $\epsilon_i(x)$ depends on the maximum singular values of the $\bar{P}_{i,\ell}$ matrices. Therefore, the upper bound on the approximation error can be further relaxed if a search for $\bar{P}_{i,\ell}$ matrices that satisfy (42)–(44) and (54), and with minimized maximum singular values is performed.

In the next section, the obtained control design rules are implemented and evaluated for an urban network case study.

5. CASE STUDY

The dynamics of a heterogeneous large-scale urban network can be modeled as multiple homogeneous regions with the macroscopic fundamental diagram (MFD) representation [35], as illustrated in Figure 1. For network regions with homogeneously distributed congestion, the MFD (as depicted in Figure 2) provides a unimodal, low-scatter relationship between network vehicle accumulation and network space-mean flow.

Using the proposed robust switching control strategies presented in Sections 3 and 4, we aim at stabilizing this system. In the context of urban network control, resolving the congestion and

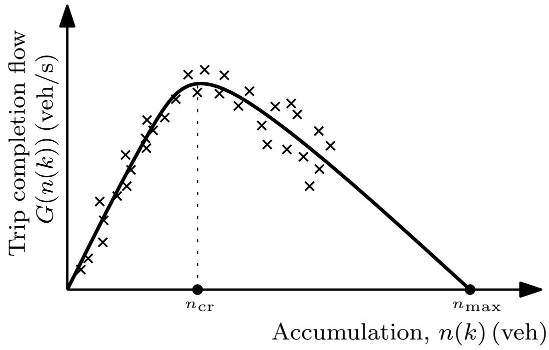


Figure 2. A well-defined macroscopic fundamental diagram.

reducing the effects of uncertain trip demands in the network can be recast as an asymptotic stabilization and robust disturbance rejection problem [22]. In the following, two benchmark case studies are presented. In the first one, the performance of the approach presented in Section 3 is evaluated for the hybrid macroscopic control of an urban network, while in the second case, the method proposed in Section 4 is implemented. Different traffic scenarios are discussed in the two examples, and the performance of the proposed approaches are compared with other strategies such as greedy feedback control and model predictive control.

5.1. *Example 1*

For an urban network divided into two regions: the city center (region 2) and the periphery (region 1), the following macroscopic model is proposed (based on the two-region model in [36]):

$$\dot{n}_1(t) = -G_{1,j}(n_1(t)) \cdot u(t) + \omega_{12}(t), \tag{61}$$

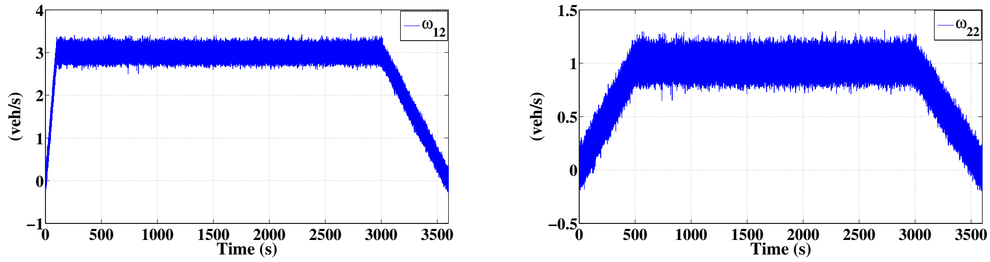
$$\dot{n}_2(t) = -G_{2,j}(n_2(t)) + G_{1,j}(n_1(t)) \cdot u(t) + \omega_{22}(t), \tag{62}$$

where $n_i(t)$, $i = 1, 2$, is the total number vehicles in region i at time t. The trip completion flow $G_{i,j}(n_i(t))$ (veh/s) is defined as the rate of vehicles reaching their destinations and it is a function of total number of vehicles in the region. In fact, $G_{i,j}(n_i(t))$ constitutes the MFD representation of the region corresponding to a signal timing plan for intersections. The index j denotes a particular MFD for region i. The signal timing plans for intersections inside each region can be altered. Consequently, instead of one MFD, a set of MFDs (each corresponds to a different timing plan) can be defined. The total number of MFDs defined for each region i is denoted by N_i and $j \in \{1, \dots, N_i\}$.

Further, the perimeter control $u \in [0, 1]$ may restrict vehicles to transfer between regions (in our case, the flow of vehicles is restricted from the periphery to the city center). The perimeter control can be realized by e.g. coordinating green and red durations of signalized intersections placed on the border between two regions.

In this example, we assume that the city center has two pre-defined timing plans and therefore two MFDs ($N_1 = 2$). Each MFD is modeled by a 3rd-order polynomial $G_{2,j}(n_2) = 1/3600 \cdot (a_{2,j}n_2^3 + b_{2,j}n_2^2 + c_{2,j}n_2)$ with coefficients $a_{2,1} = 1.4877 \cdot 10^{-7}$ $(1/(\text{veh}^2 \cdot \text{h}))$, $b_{2,1} = -2.98 \cdot 10^{-3}$ $(1/(\text{veh} \cdot \text{h}))$, $c_{2,1} = 15.091$ $(1/\text{h})$, $a_{2,2} = 2.57 \cdot 10^{-7}$ $(1/(\text{veh}^2 \cdot \text{h}))$, $b_{2,2} = -4.47 \cdot 10^{-3}$ $(1/(\text{veh} \cdot \text{h}))$, $c_{2,2} = 18.98$ $(1/\text{h})$. For the periphery, we assume that there exists only one timing plan and thus one MFD ($N_2 = 1$). The MFD of periphery is denoted by $G_1 = G_{1,1}$ and has $a_{1,1} = a_{2,1}$, $b_{1,1} = b_{2,1}$, $c_{1,1} = c_{2,1}$ as its parameters. The values for the parameters are inspired by the observed MFDs in [21, 36, 16].

Furthermore, the perimeter control input u can be limited to take values from a finite set. This is not a conservative assumption as in reality perimeter control is realized by manipulating the green to red duration of traffic signals and investigations have shown that the evolution of flows is not very sensitive to small changes in the perimeter signal [16]. Therefore, we use the quantization technique



(a) (b)

Figure 3. Demand profiles in Example 1.

presented in Section 2 in order to achieve a complete switching system as follows:

$$\dot{n}_1(t) = -G'_{1,j'}(n_1(t)) + \omega_{12}(t), \tag{63}$$

$$\dot{n}_2(t) = -G'_{2,j'}(n_2(t)) + G'_{1,j'}(n_1(t)) + \omega_{22}(t), \tag{64}$$

where the perimeter control input can take values from the set $\{0.1, 0.35, 0.65, 0.9\}$. The number of modes introduced by performing the quantization is $2 \cdot 4 = 8$ and therefore $j' \in \{1, \dots, 8\}$.

Here, we assume that the scenario simulates a morning peak in which a high trip demand ω_{12} from the periphery (region 1) to the city center (region 2) exists while there is also a demand ω_{22} for trips inside the center. To take into account the uncertainty around the demands, we add a zero mean white Gaussian noise with variance 0.1 (veh/s) to the base profiles as shown in Figure 3.

In order to determine the switching law σ, we use quadratic functions $V_i(n_i) = 1/2(\alpha_i n_1^2 + \beta_i n_2^2)$. Thus the switching rule is defined as

$$\sigma(t) = r(n_i(t)) = \arg \min_{i \in \{1,\dots,8\}} 1/2(\alpha_i n_1^2 + \beta_i n_2^2) \tag{65}$$

The parameters α_i and β_i along with a feasible attenuation level γ are determined using (20) and the gridding technique described in Remark 1 (the nonlinear feasibility problem is solved using the *fmincon* function inside the *Tomlab* toolbox of MATLAB). The obtained parameters are as follows:

$$\begin{aligned}(\alpha_i, \beta_i) \in \big\{ &(3.8014, 2.9193), (6.5982, 4.3430), \\ &(9.9993, 5.7571), (5.4335, 6.2613), (7.2388, 3.2234), \\ &(4.5741, 0.2113), (8.4626, 0.2899), (4.8048, 1.0877) \big\}\end{aligned}$$

with $\gamma = 0.8 \cdot 3600$. The initial accumulations are $n_1(0) = 6200$ (veh), $n_2(0) = 5200$ (veh). The states are measured and plugged into the switching law (65) in order to find the active subsystem (corresponding to a specific MFD and perimeter value). The closed-looped system is simulated for one hour. In order to show the effectiveness of the proposed control strategy, results of the some simple control strategies along with a model predictive controller are presented in Figure 4. It can be observed that the switching H_∞ control is able to stabilize the system and also significantly reduce the effects of the trip demands (disturbances), while in almost all the simple control strategies either one or both regions end up in the gridlock situation (as the states grow unboundedly in the figures). Only in one state feedback case, when timing plan 2 is chosen for the center, the accumulations eventually decrease by the end of simulation time (Figure 4-(g)). Moreover, the results are compared also with a hybrid MPC scheme constructed based on the approaches in [16]. In the MPC framework, the optimal perimeter and timing plans are determined by solving a mixed integer nonlinear optimization problem in the receding horizon manner. The MPC controller has the knowledge about the average demand profile (noiseless) throughout the simulation period. As can be observed from Figure 4(h), the performance of the MPC controller is better than the all other cases, including the robust H_∞ controller. This is due to the fact that the MPC controller is

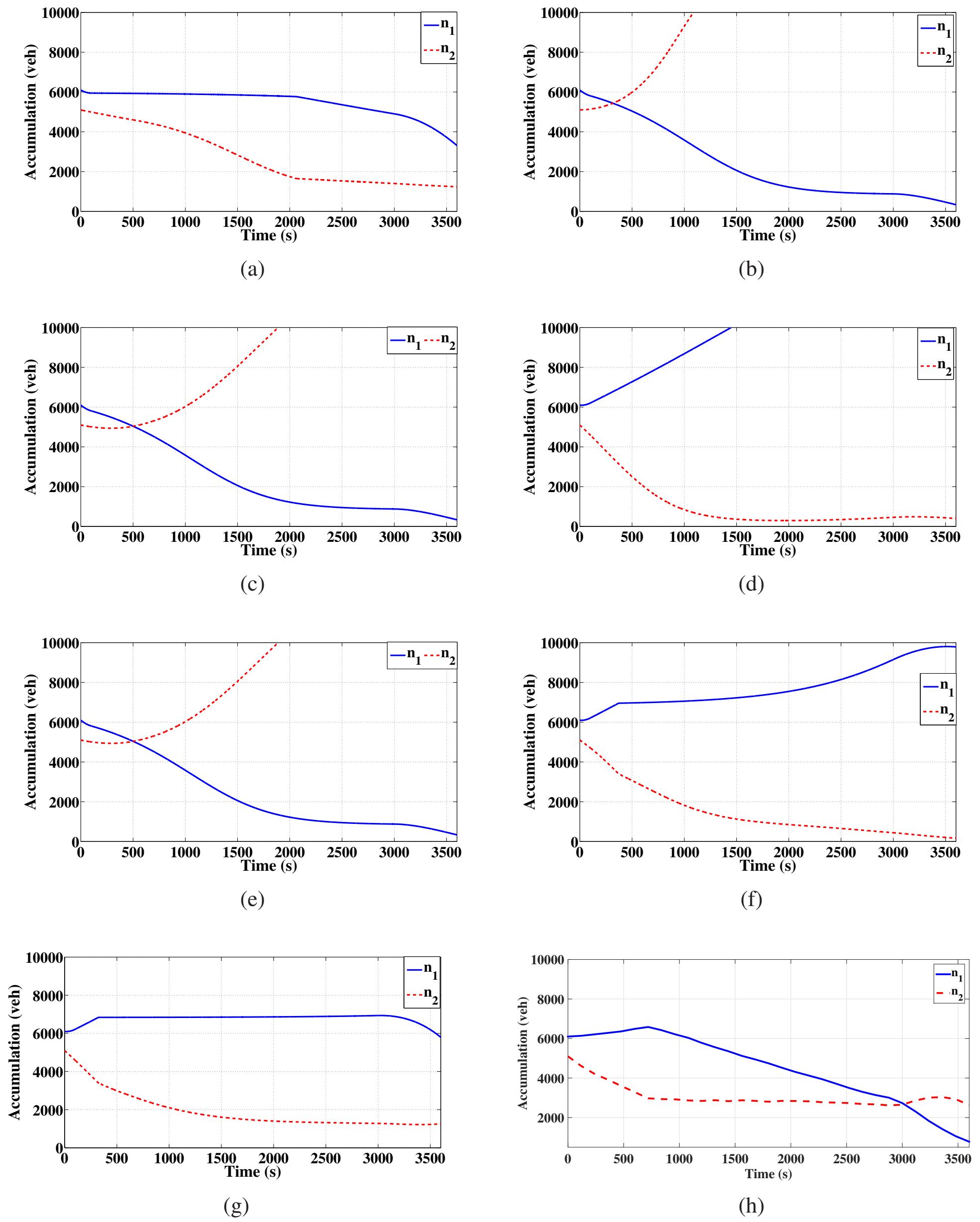


Figure 4. Example 1: comparing the accumulations, (a) robust H_∞ switching scheme, (b) $u = 1$ and timing plan 1 ($G_{2,1}$), (c) $u = 1$ and timing plan 2 ($G_{2,2}$), (d) $u = 0.1$ and timing plan 1 ($G_{2,1}$), (e) $u = 0.1$ and timing plan 2 ($G_{2,2}$), (f) $u = 0.1$ when $n_2 > n_{\mathrm{cr}}$, otherwise $u = 1$, along with $G_{2,1}$, and (g) $u = 0.1$ when $n_2 > n_{\mathrm{cr}}$, otherwise $u = 1$, along with $G_{2,2}$, (h) Hybrid model predictive control [16].

supplied with the information about the trip demands, although the average profiles. However, the computation time required by the nonlinear mixed integer optimization algorithm is considerable (see [16] for detailed CPU times for different scenarios and optimization parameters). This may become problematic for more complex traffic cases studies which would need more computational effort. On the other hand, the proposed robust control strategy is computationally efficient and can be implemented in real-time since the switching law (65) is computed in a very short time (with 16 multiplications, 8 additions, and a minimum operation). This is a great advantage over the MPC method [21, 16]. Furthermore, in the robust switching control approach having a knowledge of the demand profile is not necessary.

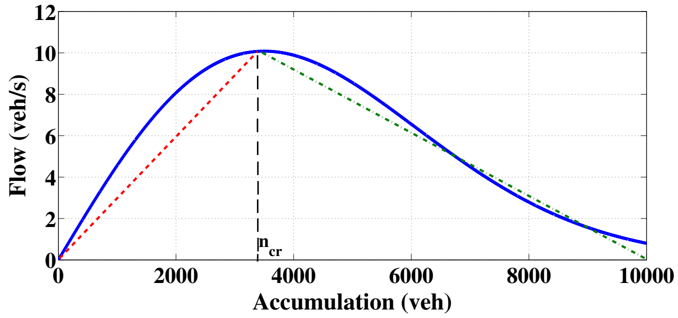


Figure 5. Piecewise affine approximation of the trip completion flow function $G_{i,j}(n_i)$).

Moreover, the L_2-gain of the controlled system can be determined by setting the initial conditions to zero and by using (18) (the output of the system is defined as $y = (n_1\ n_2)^{\mathrm{T}}$. The achieved gain $\|y\|_{L_2}/\|\omega\|_{L_2}$ for the assumed demand profile is $0.1691 \cdot 3600$.

Note that in order to calculate (65), the number of vehicles in both regions must be estimated online. For this purpose, there are different approaches, e.g. using data from several loop detectors in the network and averaging techniques [37], and using the data from GPS and in-car navigation systems [37, 38].

5.2. Example 2

In this example, we follow the robust control design method presented in Section 4. The model is the same as in (61)–(62). The trip completion flow functions $G_{i,j}$ can be approximated by piecewise affine functions, as illustrated in Figure 5. Moreover, in this example, we assume that both regions have only one MFD associated to them, but the size of the MFDs are different (with different maximum flow and critical accumulation). Moreover, the demand profiles are selected different from the ones in Example 1, to show that the proposed methodologies do not depend on the demand scenario.

Moreover, we assume that u can take values from the set $\{0.1, 0.35, 0.65, 0.9\}$. Doing this along with approximating the trip flow functions will result in a switched affine system with mixed controlled and uncontrolled switching behavior and with the following system matrices:

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ F_2 = \begin{bmatrix} 1 & 0 & -n_{1,\mathrm{cr}} \\ 0 & -1 & n_{2,\mathrm{cr}} \\ 0 & 1 & 0 \\ -1 & 0 & n_{1,\mathrm{jam}} \end{bmatrix},$$

$$F_3 = \begin{bmatrix} -1 & 0 & n_{1,\mathrm{cr}} \\ 0 & 1 & -n_{2,\mathrm{cr}} \\ 1 & 0 & 0 \\ 0 & -1 & n_{2,\mathrm{jam}} \end{bmatrix},\ F_4 = \begin{bmatrix} 1 & 0 & -n_{1,\mathrm{cr}} \\ 0 & 1 & -n_{2,\mathrm{cr}} \\ -1 & 0 & n_{1,\mathrm{jam}} \\ 0 & -1 & n_{2,\mathrm{jam}} \end{bmatrix},$$

$$\bar{h}_{12} = \bar{h}_{34} = \begin{bmatrix} 1 & 0 & -n_{1,\mathrm{cr}} \end{bmatrix}^{\mathrm{T}},$$
$$\bar{h}_{13} = \bar{h}_{24} = \begin{bmatrix} 0 & 1 & -n_{2,\mathrm{cr}} \end{bmatrix}^{\mathrm{T}},$$
$$A_{i,1} = \frac{1}{3600} \cdot \begin{bmatrix} -u_i \cdot 10.28 & 0 \\ u_i \cdot 10.28 & -8.4 \end{bmatrix}, \; b_{i,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_{i,2} = \frac{1}{3600} \cdot \begin{bmatrix} u_i \cdot 6.4 & 0 \\ -u_i \cdot 6.4 & -8.4 \end{bmatrix}, \; b_{i,2} = \begin{bmatrix} -u_i \cdot 16.22 \\ u_i \cdot 16.22 \end{bmatrix},$$
$$A_{i,3} = \frac{1}{3600} \cdot \begin{bmatrix} -u_i \cdot 10.28 & 0 \\ u_i \cdot 10.28 & 4.5 \end{bmatrix}, \; b_{i,3} = \begin{bmatrix} 0 \\ -10.75 \end{bmatrix},$$
$$A_{i,4} = \frac{1}{3600} \cdot \begin{bmatrix} u_i \cdot 6.4 & 0 \\ -u_i \cdot 6.4 & 4.5 \end{bmatrix}, \; b_{i,4} = \begin{bmatrix} -u_i \cdot 16.22 \\ u_i \cdot 16.22 - 10.75 \end{bmatrix}$$

with $n_{1,\mathrm{cr}} = 3500$ (veh), $n_{2,\mathrm{cr}} = 3000$ (veh), $n_{1,\mathrm{jam}} = 10000$ (veh), $n_{2,\mathrm{jam}} = 9000$ (veh) and $u_i \in \{0.1, 0.35, 0.65, 0.9\}$.

The demand scenario is depicted in Figure 6. The matrices of the Lyapunov functions along with the minimum L_2-gain are determined using Theorem 3. Note that we use the LMI solver SeDuMi and the Yalmip toolbox along with bisection search on $\mu_{\min}$ and gridding $\beta_{\min}$. The Lyapunov matrices are presented in Box I.

The measured accumulations are supplied to (45) to determine the active subsystem (to obtain the proper perimeter input). The results are depicted in Figure 7. As inferred from Figure 7(a), the switching control resolves the initial congestion in the network and also significantly reduces the effects of the high-level trip demands, while in the no control case and also by using a greedy feedback strategy ($u = u_{\min}$ when $n_2 > n_{2,\mathrm{cr}}$, otherwise $u = u_{\max}$), one or both accumulations grow unboundedly, as shown in Figure 7(b)-(c). Moreover, the results are compared with the perimeter control using MPC scheme implemented based on [21]. In this example, one can see that the performance of the MPC scheme is worse than the robust H_∞ control. We can increase the prediction and control horizons for the MPC controller to achieve slightly better results. However, the computation time would increase significantly and real time implementation would not be possible. Moreover, the online computation of the switching H_∞ control law is performed in real time, as it basically needs simple adding, multiplication and minimum operations. Nevertheless, it should be noted that in the H_∞ control case, the perimeter signal has fast switching behavior which may not be desirable in real traffic application. This can be avoided by modifying the stabilization conditions in Section 4, and by imposing a minimum dwell time between switching instants as is done in [39, 40] for other switched systems cases.

Furthermore, setting the initial accumulations to zero, the actual L_2-gain is $0.0881 \cdot 3600$ which is lower than the theoretical value $0.1332 \cdot 3600$ obtained by solving the optimization problem.

As a final remark, note that the proposed robust switching control scheme is suitable for high-level congestion control in urban networks. To be more precise, the proposed schemes can be used in a hierarchical traffic control scheme and at the top level, while in the lower levels local controllers are used to realize the reference signals obtained from the robust high-level control scheme. Moreover, at the lower level, each urban region can be further partitioned into subregions and the flows in subregions can be controlled such that the traffic congestion is homogeneously distributed over the entire region. Overall, the number of regions and the number of timing plans at the high-level is maintained at a small number. Therefore, the (mainly offline) computational complexity of the proposed robust switching strategies for a real traffic control scenario will not be significantly higher than in the scenarios presented here. Nevertheless, decomposition and coordination techniques can be used to efficiently solve a larger-scale version of the proposed optimization problems.

6. CONCLUSION AND FUTURE WORK

The L_2-gain analysis and the H_∞ control design procedure for switched nonlinear systems has been presented. We have used a model transformation in order to overcome the bound constraint on the

$$P_{1,1} = \begin{bmatrix} 15.8293 & 2.1235 \\ 2.1235 & 4.5335 \end{bmatrix}, P_{1,3} = \begin{bmatrix} 9.8171 \cdot 10^{-7} & 5.7418 \cdot 10^{-7} & -5.4657 \cdot 10^{-4} \\ 5.7418 \cdot 10^{-7} & 2.7229 \cdot 10^{-7} & 1.7718 \cdot 10^{-3} \\ -5.4657 \cdot 10^{-4} & 1.7718 \cdot 10^{-3} & 3.8305 \end{bmatrix},$$

$$P_{2,1} = \begin{bmatrix} 15.8000 & 2.1260 \\ 2.1260 & 4.5335 \end{bmatrix}, P_{2,3} = \begin{bmatrix} 9.8085 \cdot 10^{-7} & 5.7385 \cdot 10^{-7} & -5.4408 \cdot 10^{-4} \\ 5.7385 \cdot 10^{-7} & 2.7229 \cdot 10^{-7} & 1.7718 \cdot 10^{-3} \\ -5.4408 \cdot 10^{-4} & 1.7718 \cdot 10^{-3} & 3.8306 \end{bmatrix},$$

$$P_{3,1} = \begin{bmatrix} 15.7649 & 2.1291 \\ 2.1291 & 4.5335 \end{bmatrix}, P_{3,3} = \begin{bmatrix} 9.7980 \cdot 10^{-7} & 5.7347 \cdot 10^{-7} & -5.4113 \cdot 10^{-4} \\ 5.7347 \cdot 10^{-7} & 2.7229 \cdot 10^{-7} & 1.7718 \cdot 10^{-3} \\ -5.4113 \cdot 10^{-4} & 1.7718 \cdot 10^{-3} & 3.8306 \end{bmatrix},$$

$$P_{4,1} = \begin{bmatrix} 15.7358 & 2.1317 \\ 2.1317 & 4.5334 \end{bmatrix}, P_{4,3} = \begin{bmatrix} 9.7886 \cdot 10^{-7} & 5.7316 \cdot 10^{-7} & -5.3873 \cdot 10^{-4} \\ 5.7316 \cdot 10^{-7} & 2.7228 \cdot 10^{-7} & 1.7718 \cdot 10^{-3} \\ -5.3873 \cdot 10^{-4} & 1.7718 \cdot 10^{-3} & 3.8304 \end{bmatrix},$$

$$P_{1,2} = \begin{bmatrix} 3.9867 \cdot 10^{-5} & 8.4224 \cdot 10^{-5} & 0.7128 \\ 8.4224 \cdot 10^{-5} & 5.3187 \cdot 10^{-5} & -0.6935 \\ 0.7128 & -0.6935 & 32.7528 \end{bmatrix},$$

$$P_{2,2} = \begin{bmatrix} 3.9820 \cdot 10^{-5} & 8.4245 \cdot 10^{-5} & 0.7139 \\ 8.4245 \cdot 10^{-5} & 5.3190 \cdot 10^{-5} & -0.6937 \\ 0.7139 & -0.6937 & 16.3897 \end{bmatrix},$$

$$P_{3,2} = \begin{bmatrix} 3.9779 \cdot 10^{-5} & 8.4272 \cdot 10^{-5} & 0.7151 \\ 8.4272 \cdot 10^{-5} & 5.3190 \cdot 10^{-5} & -0.6939 \\ 0.7151 & -0.6939 & -2.2494 \end{bmatrix},$$

$$P_{4,2} = \begin{bmatrix} 3.9828 \cdot 10^{-5} & 8.4306 \cdot 10^{-5} & 0.7154 \\ 8.4306 \cdot 10^{-5} & 5.3190 \cdot 10^{-5} & -0.6942 \\ 0.7154 & -0.6942 & -12.4139 \end{bmatrix},$$

$$P_{1,4} = \begin{bmatrix} -8.6688 \cdot 10^{-6} & 9.2617 \cdot 10^{-6} & 1.3909 \cdot 10^{-2} \\ 9.2617 \cdot 10^{-6} & -1.1040 \cdot 10^{-5} & 1.2609 \cdot 10^{-2} \\ 1.3909 \cdot 10^{-2} & 1.2609 \cdot 10^{-2} & 2.1053 \end{bmatrix},$$

$$P_{2,4} = \begin{bmatrix} -8.6925 \cdot 10^{-6} & 9.2750 \cdot 10^{-6} & 1.4019 \cdot 10^{-2} \\ 9.2750 \cdot 10^{-6} & -1.1039 \cdot 10^{-5} & 1.2486 \cdot 10^{-2} \\ 1.4019 \cdot 10^{-2} & 1.2486 \cdot 10^{-2} & 2.0895 \end{bmatrix}$$

$$P_{3,4} = \begin{bmatrix} -8.7211 \cdot 10^{-6} & 9.2911 \cdot 10^{-6} & 1.4151 \cdot 10^{-2} \\ 9.2911 \cdot 10^{-6} & -1.1039 \cdot 10^{-5} & 1.2338 \cdot 10^{-2} \\ 1.4151 \cdot 10^{-2} & 1.2338 \cdot 10^{-2} & 2.0632 \end{bmatrix},$$

$$P_{4,4} = \begin{bmatrix} -8.7452 \cdot 10^{-6} & 9.3046 \cdot 10^{-6} & 1.4263 \cdot 10^{-2} \\ 9.3046 \cdot 10^{-6} & -1.1039 \cdot 10^{-5} & 1.2214 \cdot 10^{-2} \\ 1.4263 \cdot 10^{-2} & 1.2214 \cdot 10^{-2} & 2.0362 \end{bmatrix}.$$

Box I

control inputs. As an alternative approach, we have proposed a robust control design approach based on the approximation of the system with piecewise affine subsystems and composing a switched affine system with mixed autonomous and controlled switching behavior. The design conditions for stabilization and disturbance attenuation have been formulated as a bi-level optimization problem that can be efficiently solved using bisection search along with a convex optimization algorithm.

The proposed robust control schemes have been implemented for high-level control of a two-region urban network case and the obtained results have shown good performance of our approaches in case of uncertain demand profiles. Moreover, as the Lyapunov functions required for the feedback switching law are determined off-line, the proposed methods have major advantages over the existing MPC schemes for real-time implementation and for treating uncertain demand profiles.

Possible extensions of the current work are: 1) directly incorporating the constraints on the control inputs into the design conditions (rather than quantizing the control inputs), 2) reducing

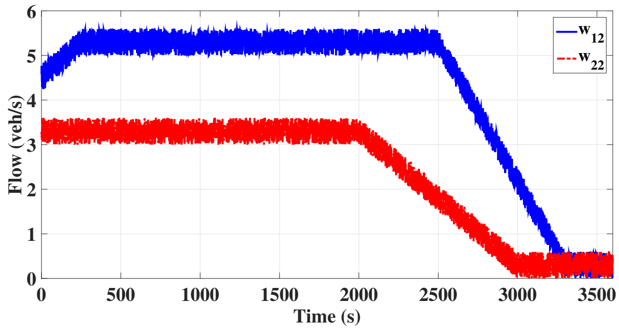


Figure 6. Trip demands in Example 2, region 1 to 2 (ω_{12}), and inside region 2 (ω_{22}).

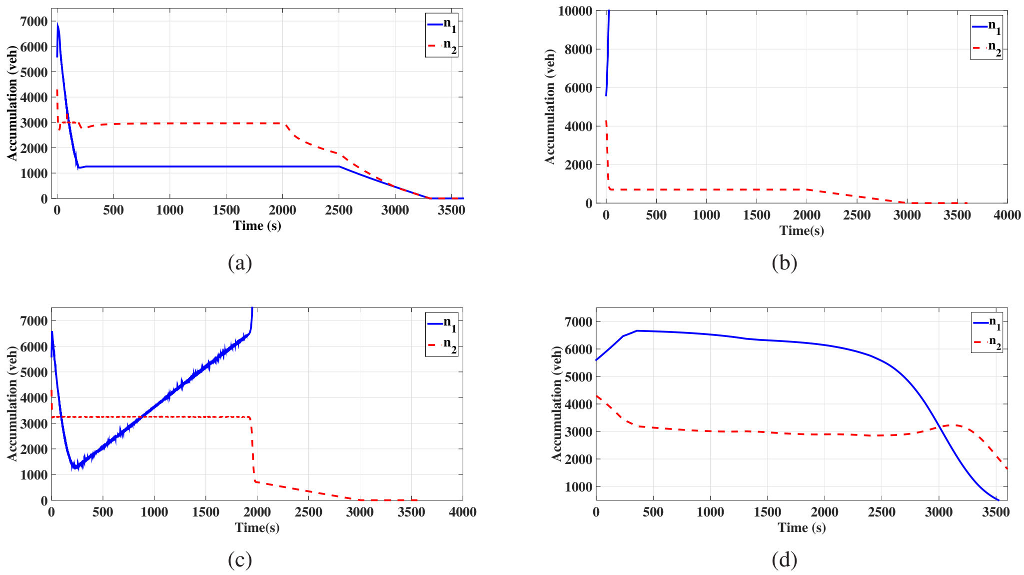


Figure 7. Accumulations in Example 2: (a) robust H_∞ switching control, (b) uncontrolled case (always $u = 1$), (c) greedy feedback control, (d) Model predictive perimeter control.

the conservatism of the second approach (namely approximation by a switched affine system) by relaxing the continuity of the Lyapunov functions over the boundaries of polyhedral regions and/or by using a joint time- and state-based switching strategy and the concept of average dwell-time [4, 39, 40], 3) design of robust stabilizing switching controllers based on the dual stability approaches [41], 4) investigate the controllability ([42, 43]) of the switched affine system with mixed switching types.

ACKNOWLEDGEMENTS

This work has been supported by the European 7th Framework Network of Excellence “Highly-complex and networked control systems (HYCON2)” under grant agreement no. 257462, and by the European COST Action TU1102.

REFERENCES

1. Liberzon D. *Switching in Systems and Control*. Birkhäuser: Boston, 2003.
2. Daafouz J, Riedinger P, Iung C. Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach. *IEEE Transactions on Automatic Control* 2002; **47**(11):1883–1887.

3. Liberzon D, Morse A. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine* 1999; **19**(5):59–70.
4. Hespanha JP. Uniform stability of switched linear systems: Extensions of LaSalle's invariance principle. *IEEE Transactions on Automatic Control* 2004; **49**(4):470–482.
5. Geromel J, Colaneri P. Stability and stabilization of continuous-time switched linear systems. *SIAM Journal on Control and Optimization* 2006; **45**(5):1915–1930.
6. Lin H, Antsaklis P. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control* 2009; **54**(2):308–322.
7. Geromel J, Deaecto G, Daafouz J. Suboptimal switching control consistancy analysis for switched linear systems. *IEEE Transactions on Automatic Control* 2013; **58**(7):1857–1861.
8. Colaneri P, Geromel JC, Astolfi A. Stabilization of continuous-time switched nonlinear systems. *Systems & Control Letters* 2008; **57**(1):95–103.
9. Zhao J, Dimirovski G. Quadratic stability of a class of switched nonlinear systems. *IEEE Transactions on Automatic Control* 2004; **49**(4):574–578.
10. El-Farra NH, Mhaskar P, Christofides P. Output feedback control of switched nonlinear systems using multiple Lyapunov functions. *Systems & Control Letters* 2005; **54**(12):1163–1182.
11. Hajiahmadi M, De Schutter B, Hellendoorn H. Stabilization and robust H_∞ control for sector-bounded switched nonlinear systems. *Automatica* 2014; **50**(10):2726–2731, doi:10.1016/j.automatica.2014.08.015.
12. Long F, Fei S, Fu Z, Zheng S, Wei W. H_∞ control and stablization of switched linear systems with linear fractional uncertainties via output feedback. *Nonlinear Analysis: Hybrid Systems* 2008; **2**(1):18–27.
13. van der Schaft A. *L_2-Gain and Passivity Techniques in Nonlinear Control*. Springer: New York, 2000.
14. Hespanha JP. $\mathcal{L}_2$-induced gains of switched linear systems. *Unsolved Problems in Mathematical Systems & Control Theory*, Blondel VD, Megretski A (eds.). Princeton University Press: Princeton, NJ, 2003; 131–133.
15. Zhao J, Hill DJ. On stability, L_2-gain and H_∞ control for switched systems. *Automatica* 2008; **44**(5):1220–1232.
16. Hajiahmadi M, Haddad J, De Schutter B, Geroliminis N. Optimal hybrid perimeter and switching plans control for urban traffic networks. *IEEE Transactions on Control Systems Technology* 2015; **23**(2):464–478, doi:10.1109/TCST.2014.2330997.
17. Daganzo C. Urban gridlock: Macroscopic modeling and mitigation approaches. *Transportation Research Part B* 2007; **41**(1):49–62.
18. Rawlings J, Mayne D. *Model Predictive Control: Theory and Design*. Nob Hill Publishing: Madison, WI, United States, 2009.
19. Ukkusuri SV, Ramadurai G, Patil G. A robust transportation signal control problem accounting for traffic dynamics. *Computers and Operations Research* 2010; **37**(5):869–879.
20. Zhang L, Yin Y, Lou Y. Robust signal timing for arterials under day-to-day demand variations. *Transportation Research Record* 2010; **2192**:156–166.
21. Geroliminis N, Haddad J, Ramezani M. Optimal perimeter control for two urban regions with macroscopic fundamental diagrams: A model predictive approach. *IEEE Transactions on Intelligent Transportation Systems* 2013; **14**(1):348–359.
22. Hajiahmadi M, De Schutter B, Hellendoorn H. Robust H_∞ control for switched nonlinear systems with application to high-level urban traffic control. *Proceedings of the 52nd IEEE Conference on Decision and Control*, Florence, Italy, 2013; 899–904.
23. Prajna S, Papachristodoulou A. Analysis of switched and hybrid systems–Beyond piecewise quadratic methods. *Proceedings of the American Control Conference*, Denver, CO, United States, 2003; 2779–2784.
24. Niu B, Zhao J. Robust H_∞ control for a class of switched nonlinear cascade systems via multiple Lyapunov functions approach. *Applied Mathematics and Computation* 2012; **218**(11):6330–6339.
25. Li LL, Zhao J, Dimirovski G. Robust H_∞ control for a class of uncertain switched nonlinear systems using constructive approach. *Proceedings of the American Control Conference*, Seattle, United States, 2008; 5068–5073.
26. Khalil H. *Nonlinear Systems*. Prentice Hall: Upper Saddle River, NJ, USA, 2002.
27. Berman A, Plemmons RJ. *Nonnegative Matrices in the Mathematical Sciences*. SIAM: New York,USA, 1994.
28. Garg K. *Theory of Differentiation: A Unified Theory of Differentiation via New Derivate Theorems and New Derivatives*. Wiley-Interscience: New York, United States, 1998.
29. Lasdon L. *Optimization Theory for Large Systems*. Macmillan: New York, 1970.
30. Van Antwerp J, Braatz R. A tutorial on linear and bilinear matrix inequalities. *Journal of Process Control* 2000; **10**(1):363–385.
31. Johansson M. *Piecewise Linear Control Systems*. Springer-Verlag: Berlin Heidelberg, 2003.
32. Rantzer A, Johansson M. Piecewise linear quadratic optimal control. *IEEE Transactions on Automatic Control* 2000; **45**(4):629–637.
33. Kailath T. *Linear Systems*. Prentice Hall International: New Jersey, United States, 1998.
34. Boyd S, Ghaoui LE, Feron E, Balakrishnan V. *Linear Matrix Inequalities in Systems and Control Theory*. SIAM: Philadelphia, United States, 1994.
35. Geroliminis N, Daganzo CF. Existence of urban-scale macroscopic fundamental diagrams: some experimental findings. *Transportation Research Part B* 2008; **42**(9):759–770.
36. Haddad J, Geroliminis N. On the stability of traffic perimeter control in two-region urban cities. *Transportation Research Part B* 2012; **46**(1):1159–1176.
37. Ortigosa J, Menendez M, Tapia H. Study on the number and location of measurement points for an MFD perimeter control scheme: a case study of Zurich. *Euro Journal on Transportation and Logistics* 2013; **1**(3):245–266.
38. Keyvan-Ekbatani M, Kouvelas A, Papamichail I, Papageorgiou M. Urban congestion gating control based on reduced operational network fundamental diagrams. *Transportation Research Part C* 2013; **33**:74–87.
39. Allerhand L, Shaked U. Robust stability and stabilization of linear switched systems with dwell time. *IEEE Transactions on Automatic Control* 2011; **56**(2).

40. Duan C, Wu F. Analysis and control of switched linear systems via dwell-time min-switching. *Systems & Control Letters* 2014; **70**:8–16.
41. Rantzer A. A dual to lyapunov's stability theorem. *Systems & Control Letters* 2001; **42**(3):161–168.
42. Xie G, Zheng D, Wang L. Controllability of switched linear systems. *Automatic Control, IEEE Transactions on* Aug 2002; **47**(8):1401–1405.
43. Camlibel M, Heemels W, Schumacher J. *On the controllability of bimodal piecewise linear systems*, *Springer*, vol. 2993. 2004; 250–264.