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Optimistic optimization for model predictive control of max-plus linear systems ⋆

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Abstract

Model predictive control for max-plus linear discrete-event systems usually leads to a nonsmooth nonconvex optimization problem with real valued variables, which may be hard to solve efficiently. An alternative approach is to transform the given problem into a mixed integer linear programming problem. However, the computational complexity of current mixed integer linear programming algorithms increases in the worst case exponentially as a function of the prediction horizon. The focus of this paper is on making optimistic optimization suited to solve the given problem. Optimistic optimization is a class of algorithms that can find an approximation of the global optimum for general nonlinear optimization. A key advantage of optimistic optimization is that one can specify the computational budget in advance and guarantee bounds on the suboptimality with respect to the global optimum. We prove that optimistic optimization can be applied for the given problem by developing a dedicated semi-metric and by proving it satisfies the necessary requirements for optimistic optimization. Moreover, we show that the complexity of optimistic optimization is exponential in the control horizon instead of the prediction horizon. Hence, using optimistic optimization is more efficient when the control horizon is small and the prediction horizon is large.

Key words: Max-plus linear systems; Model predictive control; Optimistic optimization.

1 Introduction

Max-plus algebra is a useful tool to model and analyze discrete-event systems (DES). Maximization and addition are two basic operations in the max-plus algebra. In conventional algebra, DES usually result in nonlinear systems, but there is a class of DES which can lead to linear systems in the max-plus algebra, called max-plus linear (MPL) systems [2,7]. Many results for control of MPL systems have been achieved, e.g. [1,4,8,10–12,14,15].

The model predictive control (MPC) framework has been extended to MPL systems [9]. For some special cases, the MPL-MPC problem can be formulated as a linear programming; however, in general, it results in a nonsmooth nonconvex optimization problem. To solve this problem, one approach is to recast it as a mixed integer linear programming (MILP) problem. Nonetheless, the computational complexity of most MILP algorithms grows in the worst case exponentially if the number of variables increases [19]. For the MILP problem resulting from the MPL-MPC problem, the number of auxiliary binary variables is proportional to the number of max operators (i.e. the prediction horizon and the number of inputs and outputs). Thus, the computation time of the corresponding MILP problem will become unacceptable if the prediction horizon is large. As the period corresponding to the prediction horizon should contain the crucial dynamics of the process, the prediction horizon can be very large for some MPL-MPC problems.

In this paper, we will show that the MPL-MPC problem can be solved efficiently by using optimistic optimization. Optimistic optimization [18] is a class of algorithms that can find an approximation of the global optimal solution for nonlinear optimization problem. This method is called optimistic because the most promising solutions are examined first at each iteration. The main advantage of optimistic optimization is that one can specify the computational budget (e.g. the number of node expansions) in advance and guarantee bounds on the suboptimality with respect to the global optimum. Note that the gap between the result returned by optimistic optimization and the optimal value can be made arbitrarily small as the computational budget increases. In our previous

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conference paper [22], we have applied optimistic optimization to the MPC problem for single-input single-output MPL systems. In the current paper, we study multi-input multi-output MPL systems; we also consider arbitrary piecewise linear output cost functions and two kinds of input cost functions. In addition, we develop an expression for the dedicated semi-metric required by optimistic optimization for each type of objective function. The proposed approach is evaluated with a group of random instances with different parameters settings. Moreover, we show that the computational complexity of optimistic optimization for the MPL-MPC problem depends on the control horizon instead of the prediction horizon.

Situations with a short control horizon and a long prediction horizon are common for DES control and it is useful to have a method to solve the corresponding MPC optimization problem without a significant influence of the prediction horizon. For a given MPL-MPC problem, the method using optimistic optimization in this paper will be more efficient than the MILP method in case of small control horizons and large prediction horizons.

2 Preliminaries

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows: $x \oplus y = \max(x, y)$, $x \otimes y = x + y$, for numbers $x, y \in \mathbb{R}_\varepsilon$; $\left[A \oplus B\right]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$, $\left[A \otimes C\right]_{ij} = \bigoplus_{k=1}^{m} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,m}(a_{ik} + c_{kj})$, for matrices $A, B \in \mathbb{R}_\varepsilon^{n \times m}$ and $C \in \mathbb{R}_\varepsilon^{m \times p}$.

Consider a multi-input multi-output MPL system

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k), \tag{1}$$

$$y(k) = C \otimes x(k) \tag{2}$$

where k is the event counter, $x(k) \in \mathbb{R}_\varepsilon^{n_x}$ is the state, $u(k) \in \mathbb{R}_\varepsilon^{n_u}$ is the input, $y(k) \in \mathbb{R}_\varepsilon^{n_y}$ is the output, and where $A \in \mathbb{R}_\varepsilon^{n_x \times n_x}$, $B \in \mathbb{R}_\varepsilon^{n_x \times n_u}$, and $C \in \mathbb{R}_\varepsilon^{n_y \times n_x}$ are the system matrices. Let $\hat{y}(k+j|k)$, $j = 0, 1, \ldots$ be the estimate of the output at event step $k+j$ based on the information available at event step k. Given a prediction horizon N_p, the estimation of the evolution of the MPL system from event step k up to $k + N_\mathrm{p} - 1$ can be presented as follows

$$\tilde{y}(k) = H \otimes \tilde{u}(k) \oplus g(k) \tag{3}$$

with $\tilde{y}(k) = [\hat{y}^T(k|k) \ \ldots \ \hat{y}^T(k+N_\mathrm{p}-1|k)]^T$, $\tilde{u}(k) = [u^T(k) \ \ldots \ u^T(k+N_\mathrm{p}-1)]^T$ for appropriate $H, g(k)$ (see [9] for details of H, $g(k)$).

3 The MPC problem for MPL systems

The MPC framework has been extended to MPL systems [9]. The considered objective function J consists of the weighted sum of an output cost and an input cost: $J = J_\mathrm{out} + \lambda J_\mathrm{in}$ with $\lambda > 0$. Moreover, different objective functions can be designed for different goals. In this paper, we consider four different output cost functions and two input cost functions.

We include both a tardiness and an earliness penalty in the output cost functions with parameters to express the trade-off between the two kinds of penalty. In this paper, parameters α_p, β_p, $p = 1, \ldots, n_y$ with $\alpha_p, \beta_p \geq 0$, $\alpha_p + \beta_p > 0$ are introduced as weighting coefficients for the tardiness and earliness penalties with respect to a reference signal r. Denote $\Phi_{j,p,k} = \max\Big(\alpha_p\big(\hat{y}_p(k+j|k) - r_p(k+j)\big), \beta_p\big(r_p(k+j) - \hat{y}_p(k+j|k)\big)\Big)$. The four output cost functions are: $J_\mathrm{out}^{1,1}(k) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{p=1}^{n_y} \Phi_{j,p,k}$, $J_\mathrm{out}^{1,\infty}(k) = \sum_{j=0}^{N_\mathrm{p}-1} \max_{p=1,\ldots,n_y} \Phi_{j,p,k}$, $J_\mathrm{out}^{\infty,1}(k) = \max_{j=0,\ldots,N_\mathrm{p}-1} \sum_{p=1}^{n_y} \Phi_{j,p,k}$, $J_\mathrm{out}^{\infty,\infty}(k) = \max_{j=0,\ldots,N_\mathrm{p}-1} \max_{p=1,\ldots,n_y} \Phi_{j,p,k}$. It is easy to show that $J_\mathrm{out,1}$ and $J_\mathrm{out,2}$ in [9] are special cases of $J_\mathrm{out}^{1,1}$. In [9] the following input cost functions are introduced:

$$J_\mathrm{in}^1(k) = -\sum_{j=0}^{N_\mathrm{p}-1} \sum_{q=1}^{n_u} u_q(k+j), \tag{4}$$

$$J_\mathrm{in}^2(k) = -\left\|\tilde{u}(k)\right\|_2^2. \tag{5}$$

For an explanation and discussion of the input cost functions, we refer the reader to [9]. Let Δu be the input rate: $\Delta u(k) = u(k) - u(k-1)$. In MPL-MPC, a control horizon N_c with $N_\mathrm{c} < N_\mathrm{p}$ is often introduced and the control input rate is taken to be constant from event step $k + N_\mathrm{c}$ on. Thus, the use of N_c reduces the computational burden. For an in-depth discussion about tuning of N_c, we refer the reader to [21]. Consequently, we assume $\Delta u(k+j) = \Delta u(k+N_\mathrm{c}-1)$, $j = N_\mathrm{c}, \ldots, N_\mathrm{p}-1$. Denote $\bar{u}(k-1) = [u^T(k-1) \cdots u^T(k-1)]^T$ and define $L \in \mathbb{R}^{N_\mathrm{p} n_u \times N_\mathrm{c} n_u}$ as

$$L = \begin{array}{l} \left[\begin{array}{ccccc} I_{n_u} & 0 & \cdots & 0 & 0 \\ I_{n_u} & I_{n_u} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{n_u} & I_{n_u} & \cdots & I_{n_u} & I_{n_u} \\ \hline I_{n_u} & I_{n_u} & \cdots & I_{n_u} & 2I_{n_u} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{n_u} & I_{n_u} & \cdots & I_{n_u} & (N_\mathrm{p} - N_\mathrm{c} + 1) I_{n_u} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} I \\ I \\ \vdots \\ I \end{array}}\right\} N_\mathrm{c} n_u \\ \left.\vphantom{\begin{array}{c} I \\ \vdots \\ I \end{array}}\right\} (N_\mathrm{p} - N_\mathrm{c}) n_u \end{array} \end{array} \tag{6}$$

with I_{n_u} the $n_u \times n_u$ identity matrix. Then

$$\tilde{u}(k) = L \Delta \tilde{u}(k) + \bar{u}(k-1), \tag{7}$$

where $\Delta \tilde{u}(k) = [\Delta u^T(k) \ \ldots \ \Delta u^T(k+N_\mathrm{c}-1)]^T$.

Using (3) to eliminate $\tilde{y}$ from J_out, the eliminated J only depends on $\tilde{u}$ now. Then using (7) to replace $\tilde{u}$ by $\Delta\tilde{u}$, the resulting J only depends on $\Delta\tilde{u}$, denoted as J_Δ.

Simple bound constraints on the input rate are common in practice, meaning that there is a minimum and a maximum separation between input events:

$$a \le \Delta\tilde{u}(k) \le b, \tag{8}$$

with a, b real vectors of size $N_\mathrm{c} n_u \times 1$.

Therefore, we finally obtain the following MPL-MPC problem, for given $\sigma, \tau \in \{1, \infty\}, \omega \in \{1, 2\}$:

$$\min_{\Delta\tilde{u}(k)} \; J_\Delta^{\sigma,\tau,\omega}(k) = J_\mathrm{out}^{\sigma,\tau}(k) + \lambda J_\mathrm{in}^{\omega}(k) \tag{9}$$

subject to (8). A finite optimal solution of the MPL-MPC problem exists if the feasible set is bounded and closed and the objective function is finite for finite arguments. These conditions hold in general.

4 Optimistic optimization

Optimistic optimization can be used for nonlinear optimization problem with the objective function that is deterministic [17] or stochastic [20]. It was used for the consensus problem of agents with nonlinear dynamics [5]. Optimistic planning [6,13,16] is a class of algorithms related to optimistic optimization.

To introduce optimistic optimization generically, we consider a minimization of a general deterministic function f over a feasible set $\mathcal{X}$. The implementation of optimistic optimization is based on a hierarchical partitioning of $\mathcal{X}$. For any integer $h \in \{0, 1, \ldots\}$, $\mathcal{X}$ is partitioned into K^h sets, called cells $X^{h,d}$ with $d = 0, \ldots, K^h - 1$. This partitioning may be represented by a tree where each cell $X^{h,d}$ corresponds to a node (h, d) of the tree such that each node (h, d) possesses K child nodes $\{(h+1, d_k)\}_{k=1}^K$. In addition, the set of $\{X^{h+1,d_k}\}_{k=1}^K$ forms a partition of the parent cell $X^{h,d}$. The root node of the tree (i.e. the cell $X^{0,0}$) corresponds to the whole domain $\mathcal{X}$. To each cell $X^{h,d}$, we assign a representative point $x^{h,d} \in X^{h,d}$, where f may be evaluated. To use optimistic optimization, some requirements should be satisfied [17].

Definition 1 (Semi-metric). A semi-metric on a set $\mathcal{X}$ is a function $\ell : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ satisfying: 1) $\forall x, y \in \mathcal{X}, \ell(x, y) = \ell(y, x) \ge 0$; and 2) $\forall x, y \in \mathcal{X}, \ell(x, y) = 0$ if and only if $x = y$.

Requirement 1. There exists a semi-metric ℓ on $\mathcal{X}$.

Requirement 2. There exists at least one global optimizer $x^* \in \mathcal{X}$ of f (i.e. $f(x^*) = \min_{x\in\mathcal{X}} f(x)$) and for all $x \in \mathcal{X}$, $f(x) - f(x^*) \le \ell(x, x^*)$.

Requirement 3. There exists a decreasing sequence $\{\delta(h)\}_{h=0}^\infty$ with $\delta(h) > 0$, such that for any depth $h \in \{0, 1, \ldots\}$, for any cell $X^{h,d}$ at depth h, we have $\sup_{x\in X^{h,d}} \ell(x, x^{h,d}) \le \delta(h)$, where $\delta(h)$ is called the maximum diameter of the cells at depth h.

Requirement 4. There exists a scalar $\nu > 0$ such that any cell $X^{h,d}$ at any depth h contains an ℓ-ball of radius $\nu\delta(h)$ centered in $x^{h,d}$.

The requirements guarantee bounds on the suboptimality with respect to the global optimum and on the computational budget. In particular, Requirements 1 and 2 regard the semi-metric ℓ and the local property of the objective function near the optimum with respect to ℓ. Requirements 3 and 4 guarantee that the partitioning of the feasible set generates well-shaped cells that shrink with further partitioning.

5 Optimistic optimization for the MPL-MPC problem

In this section, we show that Requirements 1-4 hold for the MPL-MPC problem (9) with $\sigma = \tau = \omega = 1$, i.e. $J_\Delta^{1,1,1} = J_\mathrm{out}^{1,1} + \lambda J_\mathrm{in}^{1}$. Other cases that can be proved similarly are included in the appendix. Let $\mathcal{X} = \{\Delta\tilde{u}(k) | a \le \Delta\tilde{u}(k) \le b\}$. Before proceeding further, we first present the following result.

Theorem 1 *Let $\Delta\tilde{u}(k)$ is an arbitrary input rate sequence and $\Delta\tilde{u}^*(k)$ be the optimal input rate sequence for problem (9). Then it holds that*

$$\begin{aligned} &J_\Delta^{1,1,1}(\Delta\tilde{u}) - J_\Delta^{1,1,1}(\Delta\tilde{u}^*) \\ &\quad \le \sum_{p=1}^{n_y} \max(\alpha_p, \beta_p) \sum_{j=0}^{N_\mathrm{p}-1} \max_{i=1,\ldots,(j+1)n_u} \left| L_{i,\cdot}(\Delta\tilde{u} - \Delta\tilde{u}^*) \right| \\ &\quad\quad + \lambda \| L(\Delta\tilde{u} - \Delta\tilde{u}^*) \|_1 \end{aligned} \tag{10}$$

where α_p, β_p are as defined in Section 3 and $L_{i,\cdot}$ is the i-th row of L in (6).

Proof. Assume that $\tilde{y}(k)$ is the estimate of the output sequence corresponding to $\Delta\tilde{u}(k)$ and $\tilde{y}^*(k)$ is defined similarly corresponding to $\Delta\tilde{u}^*(k)$. Let $\tilde{p} = jn_y + p$, thus $\tilde{y}_{\tilde{p}}(k) = \hat{y}_p(k + j|k)$, $\tilde{y}^*_{\tilde{p}}(k) = \hat{y}^*_p(k + j|k)$ and $\tilde{r}_{\tilde{p}}(k) = r_p(k + j)$, for $p = 1, \ldots, n_y$, $j = 0, \ldots, N_\mathrm{p} - 1$. It is easy to verify that, for any $x, y, z \in \mathbb{R}$, $\max\big(\alpha(x - z), \beta(z - x)\big) - \max\big(\alpha(y - z), \beta(z - y)\big) \le \max(\alpha, \beta)|x - y|$, where α, β are non-negative real numbers. Hence,

$$\begin{aligned} &J_\mathrm{out}^{1,1}(\Delta\tilde{u}) - J_\mathrm{out}^{1,1}(\Delta\tilde{u}^*) \\ &\quad \le \sum_{j=0}^{N_\mathrm{p}-1} \sum_{p=1}^{n_y} \max(\alpha_p, \beta_p) \big|\tilde{y}_{\tilde{p}}(k) - \tilde{y}^*_{\tilde{p}}(k)\big|. \end{aligned} \tag{11}$$

From (3), we have $\tilde{y}_{\tilde{p}}(k) = \max\big(H_{\tilde{p},\cdot} \otimes \tilde{u}(k),\; g_{\tilde{p}}(k)\big)$ and $\tilde{y}^*_{\tilde{p}}(k) = \max\big(H_{\tilde{p},\cdot} \otimes \tilde{u}^*(k),\; g_{\tilde{p}}(k)\big)$, where $H_{\tilde{p},\cdot}$ is the $\tilde{p}$-th row of H and $\tilde{u}(k)$ and $\tilde{u}^*(k)$ are the respective input sequences corresponding to $\Delta\tilde{u}(k)$ and $\Delta\tilde{u}^*(k)$. Thus,

$$\big|\tilde{y}_{\tilde{p}}(k) - \tilde{y}^*_{\tilde{p}}(k)\big| \le \big|H_{\tilde{p},\cdot} \otimes \tilde{u}(k) - H_{\tilde{p},\cdot} \otimes \tilde{u}^*(k)\big|. \tag{12}$$

Denote

$$\begin{aligned} H_{\tilde{p},\cdot} \otimes \tilde{u}(k) &= \max_{w=1,\ldots,(j+1)n_u} \big(H_{\tilde{p}w} + \tilde{u}_w(k)\big), \\ &= H_{\tilde{p}w_0} + \tilde{u}_{w_0}(k), \\ H_{\tilde{p},\cdot} \otimes \tilde{u}^*(k) &= \max_{z=1,\ldots,(j+1)n_u} \big(H_{\tilde{p}z} + \tilde{u}^*_z(k)\big) \\ &= H_{\tilde{p}z_0} - \tilde{u}^*_{z_0}(k) \\ &\ge H_{\tilde{p}w_0} - \tilde{u}^*_{w_0}(k). \end{aligned} \tag{13}$$

Then

$$\begin{aligned}
&\left|H_{\tilde{p},\cdot} \otimes \tilde{u}(k) - H_{\tilde{p},\cdot} \otimes \tilde{u}^*(k)\right| \\
&\quad = \left|H_{\tilde{p}w_0} + \tilde{u}_{w_0}(k) - H_{\tilde{p}z_0} - \tilde{u}^*_{z_0}(k)\right| \\
&\quad \overset{(13)}{\leq} \left|H_{\tilde{p}w_0} + \tilde{u}_{w_0}(k) - H_{\tilde{p}w_0} - \tilde{u}^*_{w_0}(k)\right| \\
&\quad \leq \left|\tilde{u}_{w_0}(k) - \tilde{u}^*_{w_0}(k)\right| \\
&\quad \leq \max_{i=1,\ldots,(j+1)n_u} \left|\tilde{u}_i(k) - \tilde{u}^*_i(k)\right| \\
&\quad \overset{(7)}{\leq} \max_{i=1,\ldots,(j+1)n_u} \left|L_{i,\cdot}(\Delta\tilde{u}(k) - \Delta\tilde{u}^*(k))\right|. \qquad (14)
\end{aligned}$$

From (11), (12) and (14), we have

$$\begin{aligned}
&J^{1,1}_{\mathrm{out}}(\Delta\tilde{u}) - J^{1,1}_{\mathrm{out}}(\Delta\tilde{u}^*) \\
&\quad \leq \sum_{p=1}^{n_y} \max(\alpha_p, \beta_p) \sum_{j=0}^{N_\mathrm{p}-1} \max_{i=1,\ldots,(j+1)n_u} \left|L_{i,\cdot}(\Delta\tilde{u} - \Delta\tilde{u}^*)\right|. \qquad (15)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
J^{1}_{\mathrm{in}}(\Delta\tilde{u}) - J^{1}_{\mathrm{in}}(\Delta\tilde{u}^*) &= \sum_{i=1}^{N_\mathrm{p} n_u} \left[\tilde{u}^*_i(k) - \tilde{u}_i(k)\right] \\
&\leq \|\tilde{u}(k) - \tilde{u}^*(k)\|_1 \\
&\leq \|L(\Delta\tilde{u}(k) - \Delta\tilde{u}^*(k))\|_1. \qquad (16)
\end{aligned}$$

From (15)-(16), we deduce that (10) holds. □

Based on Theorem 1, we can define $\ell^{1,1,1} : \mathcal{X}\times\mathcal{X} \to \mathbb{R}^+$, such that for any $\Delta\tilde{u}(k), \Delta\tilde{v}(k) \in \mathcal{X}$,

$$\begin{aligned}
&\ell^{1,1,1}(\Delta\tilde{u}(k), \Delta\tilde{v}(k)) \\
&\quad := \ell^{1,1}_{\mathrm{out}}(\Delta\tilde{u}(k), \Delta\tilde{v}(k)) + \lambda\ell^{1}_{\mathrm{in}}(\Delta\tilde{u}(k), \Delta\tilde{v}(k)) \qquad (17)
\end{aligned}$$

with

$$\begin{aligned}
&\ell^{1,1}_{\mathrm{out}}(\Delta\tilde{u}(k), \Delta\tilde{v}(k)) = \\
&\sum_{p=1}^{n_y} \max(\alpha_p, \beta_p) \sum_{j=0}^{N_\mathrm{p}-1} \max_{i=1,\ldots,(j+1)n_u} \left|L_{i,\cdot}(\Delta\tilde{u}(k) - \Delta\tilde{v}(k))\right|, \\
&\ell^{1}_{\mathrm{in}}(\Delta\tilde{u}(k), \Delta\tilde{v}(k)) = \|L(\Delta\tilde{u}(k) - \Delta\tilde{v}(k))\|_1
\end{aligned}$$

where $\lambda > 0$ and α_p, β_p are as defined in Section 3. Because L is not singular, it is easy to verify that the function ℓ defined by (17) is a semi-metric on $\mathcal{X}$. Therefore, Requirements 1-2 are satisfied for $\sigma = \tau = \omega = 1$.

For the partitioning of $\mathcal{X} = \{\Delta\tilde{u}(k) | a \leq \Delta\tilde{u}(k) \leq b\}$, we take the center of $\mathcal{X}$ as the starting point (i.e. the root node of the tree). At each iteration, we bisect each dimension of $\mathcal{X}$; so the number of branches K equals $2^{N_\mathrm{c} n_u}$. From (8), we have, for any $\Delta\tilde{u}(k) \in X^{h,d}$ with its center denoted as $\Delta\tilde{u}^{h,d}(k)$,

$$\|\Delta\tilde{u}(k) - \Delta\tilde{u}^{h,d}(k)\|_\infty \leq \frac{1}{2^{h+1}}\|b - a\|_\infty, \qquad (18)$$

$$\|\Delta\tilde{u}(k) - \Delta\tilde{u}^{h,d}(k)\|_1 \leq \frac{1}{2^{h+1}}\|b - a\|_1. \qquad (19)$$

Optimistic optimization expands a leave with the best b-value $b^{h,d} \overset{\mathrm{def}}{=} f(x^{h,d}) - \delta(h)$ by adding its K children to the current tree (i.e. splitting the corresponding cell into K sub-cells). We now derive the expression $\delta^{1,1,1}(h)$ for $\delta(h)$ corresponding to $\ell^{1,1,1}$.

Theorem 2 *Define*

$$\delta^{1,1,1}(h) = \frac{1}{2^{h+1}}\left[\delta^{1,1}_{\mathrm{out}} + \lambda\delta^{1}_{\mathrm{in}}\right] \qquad (20)$$

for $h \in \{0, 1, \ldots\}$ with

$$\delta^{1,1}_{\mathrm{out}} = \frac{N_\mathrm{p}(N_\mathrm{p}+1)\|b-a\|_\infty}{2} \sum_{p=1}^{n_y} \max(\alpha_p, \beta_p), \qquad (21)$$

$$\delta^{1}_{\mathrm{in}} = \|L(b-a)\|_1. \qquad (22)$$

Then for any $h \in \{0, 1, \ldots\}$, $d \in \{0, \ldots, K^h - 1\}$, it holds that $\sup_{\Delta\tilde{u}(k)\in X^{h,d}} \ell^{1,1,1}(\Delta\tilde{u}(k), \Delta\tilde{u}^{h,d}(k)) \leq \delta^{1,1,1}(h)$, where $\Delta\tilde{u}^{h,d}(k)$ is the center of cell $X^{h,d}$.

Proof. For any $\Delta\tilde{u}(k) \in X^{h,d}$, we have [1]

$$\begin{aligned}
&\ell^{1,1}_{\mathrm{out}}(\Delta\tilde{u}(k), \Delta\tilde{u}^{h,d}(k)) \\
&\quad \leq \sum_{p=1}^{n_y} \max(\alpha_p, \beta_p) \sum_{j=0}^{N_\mathrm{p}-1} \Big[\max_{i=1,\ldots,(j+1)n_u} \|L_{i,\cdot}\|_1 \\
&\qquad\qquad \|\Delta\tilde{u}(k) - \Delta\tilde{u}^{h,d}(k)\|_\infty \Big] \\
&\quad \overset{(18)}{\leq} \sum_{p=1}^{n_y} \max(\alpha_p, \beta_p) \frac{N_\mathrm{p}(N_\mathrm{p}+1)}{2} \frac{\|b-a\|_\infty}{2^{h+1}} \\
&\quad \overset{(21)}{\leq} \frac{1}{2^{h+1}} \delta^{1,1}_{\mathrm{out}},
\end{aligned}$$

and

$$\begin{aligned}
\ell^{1}_{\mathrm{in}}(\Delta\tilde{u}(k), \Delta\tilde{u}^{h,d}(k)) &\overset{(19)}{\leq} \frac{1}{2^{h+1}}\|L(b-a)\|_1 \\
&\overset{(22)}{\leq} \frac{1}{2^{h+1}}\delta^{1}_{\mathrm{in}}.
\end{aligned}$$

Thus if we define $\delta^{1,1,1}(h)$ as in (20), then

$$\sup_{\Delta\tilde{u}(k)\in X^{h,d}} \ell^{1,1,1}(\Delta\tilde{u}(k), \Delta\tilde{u}^{h,d}(k)) \leq \delta^{1,1,1}(h). \quad □$$

Theorem 3 *Choose $\nu^{1,1,1}$ such that*

$$0 < \nu^{1,1,1} \leq \frac{\rho \min\limits_{i=1,\ldots,N_\mathrm{c} n_u} (b_i - a_i)}{\delta^{1,1}_{\mathrm{out}} + \lambda\delta^{1}_{\mathrm{in}}}.$$

Then any cell $X^{h,d}$ at any depth h contains an ℓ-ball $\mathcal{B}^{h,d}$ of radius $\nu^{1,1,1}\delta^{1,1,1}(h)$ centered in $\Delta\tilde{u}^{h,d}$ where $0 < \rho < 1$ and $\delta^{1,1,1}(h), \delta^{1,1}_{\mathrm{out}}, \delta^{1}_{\mathrm{in}}$ are as defined in (20)-(22).

Proof. According to Theorem 2, we can define a decreasing sequence $\{\delta^{1,1,1}(h)\}_{h=0}^\infty$ as in (20). Select a real number ρ with $0 < \rho < 1$. From (8), the ℓ-ball $\mathcal{B}^{h,d}$ of radius $\nu^{1,1,1}\delta^{1,1,1}(h)$ centered in $\Delta\tilde{u}^{h,d}$ is inside the cell $X^{h,d}$, if we choose $\nu^{1,1,1}$ such that $\nu^{1,1,1}\delta^{1,1,1}(h) \leq \rho\frac{\min_{i=1,\ldots,N_\mathrm{c} n_u} (b_i - a_i)}{2^{h+1}}$. Then ν can be chosen as

$$\nu^{1,1,1} \leq \frac{\rho \min\limits_{i=1,\ldots,N_\mathrm{c} n_u} (b_i - a_i)}{\delta^{1,1}_{\mathrm{out}} + \lambda\delta^{1}_{\mathrm{in}}}. \quad □$$

[1] For every $x, y \in \mathbb{R}^n$, we have $|x^T y| \leq \|x\|_1\|y\|_\infty$.

Up to now, we have proved that the Requirements 1-4 are satisfied for $\sigma = \tau = \omega = 1$. In a similar way, we can obtain corresponding results for other cases. The dedicated semi-metrics ℓ and the $\delta(h)$ expressions for different σ, τ, ω are presented in the appendix, where $\ell^{\sigma,\tau,\omega} = \ell_{\mathrm{out}}^{\sigma,\tau} + \lambda \ell_{\mathrm{in}}^{\omega}$ and $\delta^{\sigma,\tau,\omega}(h) = \frac{1}{2^{h+1}}\left[\delta_{\mathrm{out}}^{\sigma,\tau} + \lambda \delta_{\mathrm{in}}^{\omega}\right]$.

Remark 1. The computational complexity of optimistic optimization in our implementation is exponential in the control horizon N_{c}. On the other hand, the MPL-MPC problem can also be formulated as an MILP problem [9,3]. The number of auxiliary binary variables that are used to convert the max operator into linear equations is proportional to the prediction horizon N_{p}. As a result, the complexity of state-of-the-art MILP algorithms is in the worst case exponential in N_{p} [19]. Therefore, optimistic optimization will be more efficient if $N_{\mathrm{c}} \ll N_{\mathrm{p}}$.

6 Examples

In this section, we illustrate the approach and the statement in Remark 1 with random experiments and an application to an industrial manufacturing system.

6.1 Random systems

Consider the MPL system (1)-(2) with $n_u = n_y = 1$. We will consider $n_x = 5, 10, 20$. To verify the statement in Remark 1, we will perform experiments for $N_{\mathrm{c}} = 3, 4, 5$ and $N_{\mathrm{p}} = N_{\mathrm{c}} + 1, \ldots, 60$. Assume that $\lambda = 0.01$, $u(0) = 0$, $\sigma = \tau = \omega = 1$, and $-15 \leq \Delta u(k) \leq 15$ for all k. The elements of $A, B, C, x(0)$ are selected as random integers uniformly distributed in the interval $[0, 10]$, but some elements of $A, B, C, x(0)$ may be equal to ε with a probability 0.2. The increments of the reference sequence r are random integers uniformly distributed in the interval $[0, 10]$. For each $n_x \in \{5, 10, 20\}$, we generate 20 random $(A, B, C, x(0))$ combinations. For each choice of $(A, B, C, x(0))$, we generate 10 random reference sequences $\{r(k)\}_{k=1}^{N_{\mathrm{p}}}$. We compare the efficiency of our method with the MILP solvers `cplex` and `glpk` for solving the problem (9). This comparison is fair because our method and the MILP solvers are all implemented in object code. The computational budget of optimistic optimization is set to 200 node expansions. The `cplex` solver is from the Tomlab toolbox and the `glpk` solver is from the Multi-Parametric Toolbox. All experiments are performed in Matlab. The CPU time for each method are plotted using logarithmic scale in Figure 1. We can see that, in Figure 1(a), for $N_{\mathrm{c}} = 3$, the mean CPU time curves of optimistic optimization (`oo`) and the MILP solvers intersect at $N_{\mathrm{p}} = 6$. For $N_{\mathrm{c}} = 4$ and $N_{\mathrm{c}} = 5$, the intersections of the mean CPU time curves for `oo` and for the MILP solvers occur at respectively $N_{\mathrm{p}} = 9$ and $N_{\mathrm{p}} = 14$ as shown in Figure 1(b-c). Thus optimistic optimization is faster than MILP when N_{p} is about two or three times as large as N_{c}. We can also see that the computation time of the MILP solvers is exponential in N_{p}, while N_{p} has no significant influence on the computation time of optimistic optimization.

We also compute the relative error between the objective function value obtained by optimistic optimization and the best value among the MILP solvers (see Figure 1(d)). The difference between the objective function values provided by the MILP solvers are negligible, so it is not plotted. For each n_x and each combination of $A, B, C, x(0)$ and $r(k)$, the relative error of optimistic optimization is computed. The plotted relative errors are the average values over all instances. We can see that for each value of N_{c} considered, the average relative errors are less than 3.5×10^{-3}.

6.2 Industrial manufacturing system

Now we consider the manufacturing unit for producing rubber tubes for automobile equipment presented in [23]. The dynamic behavior of this system is described by an MPL system with 19 states, 2 inputs, and 1 output (see [23] for details). Let $N_{\mathrm{c}} = 2$. We run experiments for $N_{\mathrm{p}} = N_{\mathrm{c}} + 1, \ldots, 40$ with $\lambda = 0.0001$, $\sigma = \tau = \omega = 1$, $u(0) = [0\ 0]^T$ and $2 \leq \Delta u_i(k) \leq 8$, $i = 1, 2$, for all k. The increments of the reference sequence r are random integers uniformly distributed in the interval $[2, 10]$. We use optimistic optimization and the `cplex` solver to solve the corresponding MPL-MPC problem (The `glpk` solver is not used for comparison because `cplex` is much faster than `glpk` when solving the resulting MILP problem for this example). The computational budget of optimistic optimization is set to 700 node expansions. Figure 2 shows the CPU time for optimistic optimization and `cplex` and the relative error between the values of $J_{\Delta}^{1,1,1}$ provided by both methods. We can see that optimistic optimization is faster than `cplex` after $N_{\mathrm{p}} = 14$ and the relative error between the objective function values is less than 9%.

7 Conclusions

We have considered model predictive control for max-plus linear systems which usually results in a nonsmooth nonconvex optimization problem. We have extended optimistic optimization to solve the given problem and derived expressions for the required parameters. Based on the theoretical analysis, we found that the complexity of the proposed approach increases exponentially in the control horizon instead of the prediction horizon. Moreover, the worst-case complexity of the mixed integer linear programming (MILP) method is exponential in the prediction horizon. As illustrated by the numerical results, optimistic optimization is more efficient than MILP when the prediction horizon is large and the control horizon is small.

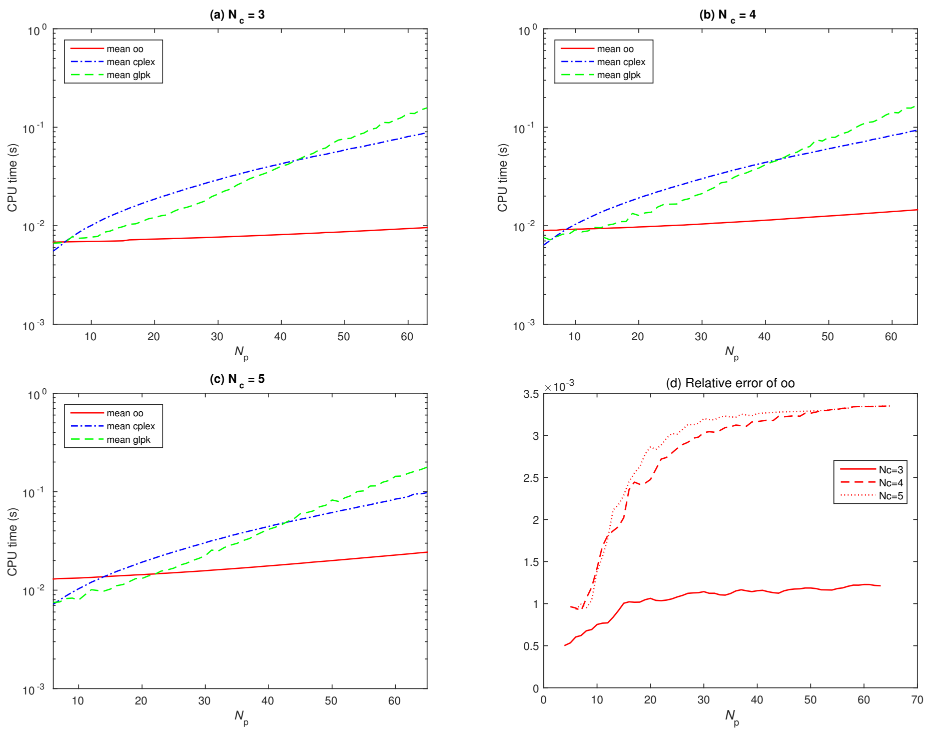

Fig. 1. (a-c) The CPU time for optimistic optimization (`oo`), `cplex`, and `glpk` for $N_c = 3, 4, 5$; (d) Relative error between `oo` and the MILP solvers.

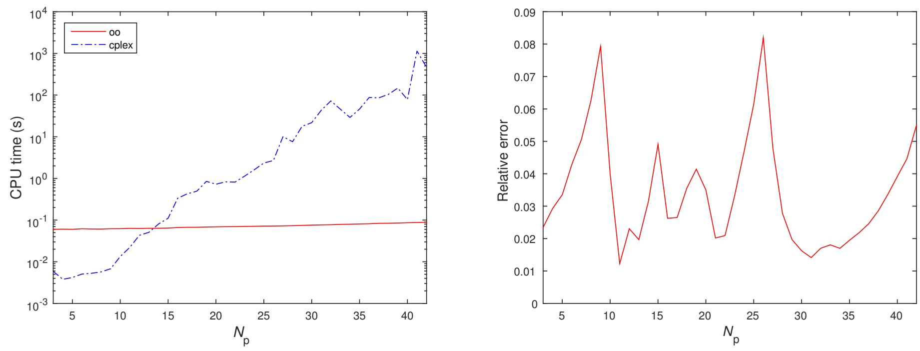

Fig. 2. (a) The CPU time for optimistic optimization (`oo`) and `cplex`; (b) Relative error between `oo` and `cplex`.

We only considered the simple bounds on the input rate. The case with general linear constraints on inputs and outputs will be considered in the future. We will also consider leveraging the nonexpansivity property of max-plus linear systems to further reduce the complexity and to get tighten expressions for the parameters of Requirements 1-4.

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Appendix

The main results in Section 5 are for the case $\sigma = \tau = \omega = 1$. Now we present the dedicated results for other cases. The proofs are similar.

1) $\sigma = 1$, $\tau = \infty$

$$\ell_{\text{out}}^{1,\infty}(\Delta\tilde{u}(k), \Delta\tilde{v}(k)) = \Big[\max_{p=1,\ldots,n_y} \max(\alpha_p, \beta_p) \sum_{j=0}^{N_\text{p}-1} \max_{i=1,\ldots,(j+1)n_u} \Big| L_{i,\cdot}(\Delta\tilde{u}(k) - \Delta\tilde{v}(k)) \Big| \Big],$$

$$\delta_{\text{out}}^{1,\infty} = \frac{N_\text{p}(N_\text{p}+1)\|b-a\|_\infty}{2} \max_{p=1,\ldots,n_y} \max(\alpha_p, \beta_p).$$

2) $\sigma = \infty$, $\tau = 1$

$$\ell_{\text{out}}^{\infty,1}(\Delta\tilde{u}(k), \Delta\tilde{v}(k)) = \Big[\sum_{p=1}^{n_y} \max(\alpha_p, \beta_p) \max_{j=0,\ldots,N_\text{p}-1} \max_{i=1,\ldots,(j+1)n_u} \Big| L_{i,\cdot}(\Delta\tilde{u}(k) - \Delta\tilde{v}(k)) \Big| \Big],$$

$$\delta_{\text{out}}^{\infty,1} = N_\text{p}\|b-a\|_\infty \sum_{p=1}^{n_y} \max(\alpha_p, \beta_p).$$

3) $\sigma = \infty$, $\tau = \infty$

$$\ell_{\text{out}}^{\infty,\infty}(\Delta\tilde{u}(k), \Delta\tilde{v}(k)) = \Big[\max_{p=1,\ldots,n_y} \max(\alpha_p, \beta_p) \max_{j=0,\ldots,N_\text{p}-1} \max_{i=1,\ldots,(j+1)n_u} \Big| L_{i,\cdot}(\Delta\tilde{u}(k) - \Delta\tilde{v}(k)) \Big| \Big],$$

$$\delta_{\text{out}}^{\infty,\infty} = N_\text{p}\|b-a\|_\infty \max_{p=1,\ldots,n_y} \max(\alpha_p, \beta_p).$$

4) $\omega = 2$

$$\ell_{\text{in}}^{2}(\Delta\tilde{u}(k), \Delta\tilde{v}(k)) = 2\big\|Lb + \bar{u}(k-1)\big\|_2 \big\|L(\Delta\tilde{u}(k) - \Delta\tilde{v}(k))\big\|_2,$$

$$\delta_{\text{in}}^{2} = 2\big\|Lb + \bar{u}(k-1)\big\|_2 \big\|L(b-a)\big\|_2.$$