Technical report 16-027

Adaptive asymptotic tracking control of uncertain time-driven switched linear systems*

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Abstract—This paper establishes a novel result for adaptive asymptotic tracking control of uncertain switched linear systems. The result exploits a recently proposed stability condition for switched systems. In particular, a time-varying positive definite Lyapunov function is used to develop a novel piecewise continuous model-reference adaptive law and a dwell-time switching law. In contrast with previous research, where asymptotic tracking was possible only in the presence of a common Lyapunov function for the reference models, in this work asymptotic tracking is shown in a more general setting. Additionally, in the presence of persistence of excitation, the controller parameter estimation errors will converge to zero asymptotically. The main contribution of this work consists in establishing a symmetry between adaptive control of classical non-switched linear systems and adaptive control of switched linear systems. A practical example with an electro-hydraulic system is adopted to illustrate the results.

Index Terms—Adaptive asymptotic tracking control, switched linear systems

I. INTRODUCTION

As a special class of hybrid systems, switched systems have attracted a lot of attention in the last decade. Switched systems can be utilized to model complex systems that are characterized by hybrid dynamics and arise in many fields, such as automotive industry [1], aircraft and air traffic [2], and smart buildings [3].

When controlling such complex systems, a ubiquitous problem is the presence of large parametric uncertainties. Widely used for coping with parametric uncertainties in classical non-switched systems [4], adaptive control of switched systems subject to parametric uncertainties has been also investigated [5], [6], [7], [8] in the later years. The work in [5] and [6] can be cited as representative research for uncertain state-dependent switched systems and uncertain time-driven switched systems, respectively. For state-dependent switched systems, di Bernardo et al. [5] developed an adaptive law based on the so-called minimal control synthesis algorithm, which can guarantee that the plant states asymptotically track the reference trajectory. For time-driven switched systems, Sang and Tao [6] proposed a switching law based on the dwell time and an adaptive law with parameter projection for each subsystem. Two crucial properties of the Lyapunov function are exploited in [6]: an exponential rate of decrease during the active intervals between two consecutive switching instants and a bounded increment at switching instants. Because of this,

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asymptotic stability can be guaranteed only in the presence of a common Lyapunov function for the reference models. For general settings when no common Lyapunov function exists, the control method proposed in [6] can only guarantee (non-asymptotic) stability of the closed-loop switched system and that the tracking error is bounded in a mean square sense. Furthermore, parameter projection is a necessary tool to keep the estimates bounded, even in the absence of any disturbance. These results are not consistent with the well-known results on adaptive tracking control for classical non-switched systems, where parameter projection is not needed in the noiseless case, and asymptotic tracking can be guaranteed [9]. From this point of view, the following questions automatically arise: how to fill the aforementioned theoretical gap between adaptive control of switched linear systems and non-switched linear systems? in other words, can we develop an adaptive law and a switching law for uncertain time-driven switched systems to achieve the same asymptotic stability results of adaptive control of classical non-switched systems? Furthermore, in the presence of a persistently exciting reference input, can we guarantee asymptotic convergence of the controller parameter estimation errors to zero?

Recently, a new asymptotic stability condition for switched linear systems has been proposed based on a dwell-time switching law [10]. There are some distinguishing properties of this new stability condition with respect to those proposed in [11], [12], [13]. In particular, the dwell time guaranteeing the asymptotic stability can be calculated without involving an exponential term. Moreover, instead of a single positive definite matrix, a family of positive definite matrices is associated to each subsystem, which can be used to construct a time-varying positive definite matrix using the linear interpolation method for a quadratic Lyapunov function. The resulting Lyapunov function is decreasing during the intervals between two consecutive switching instants and non-increasing at the switching instants. In light of this, the current work exploits the aforementioned stability result to develop a novel model reference adaptive law for uncertain switched linear systems to guarantee asymptotic stability. The main contributions of the work can be summarized as follows: first, in contrast with previous work involving two properties of the Lyapunov function, the proposed adaptive laws completely remove these requirements; second, there is no need for parameter projection and of the *a priori* knowledge of upper and lower bounds for the parameters when the switched system is not subject to disturbances; finally, asymptotic stability is established for the first time, i.e., the tracking error converges to zero asymptotically, even when no common Lyapunov function for the reference models exists. Furthermore, if the reference is

persistently exciting, we can also guarantee that the parameter estimates of the state-feedback controller converge to the nominal parameters asymptotically, which makes the closed-loop switched system behave like the reference model. In view of these achievements, a symmetry between adaptive control of switched linear systems and adaptive control of non-switched systems is established.

The paper is organized as follows: Section 2 presents the control problem and some preliminaries for later analysis. Section 3 proposes an adaptive law and a switching law to solve the adaptive asymptotic tracking problem. Stability results of closed-loop switched system based on a quadratic Lyapunov function are presented Section 4. Section 5 adopts a practical example to illustrate the proposed results. The paper is concluded in Section 6.

Notation: The notation used in this paper is as follows: $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}^+$ represent the set of real numbers, positive real numbers, and positive natural numbers, respectively. The notation $P = P^T > 0$ indicates a symmetric positive definite matrix. In addition, the superscript T represents the transpose of matrix. The operator $\text{tr}(\cdot)$ represents the trace of a matrix. The notation $\|\cdot\|$ represents the Euclidean norm. The function $\text{sgn}[*]$ takes the sign of $*$. The identity matrix of compatible dimensions is denoted by I. The operator $\lambda_{\max}[P]$ returns the maximum eigenvalue of the square matrix P, and $\text{diag}\{\cdots\}$ represents a block-diagonal matrix.

II. PROBLEM FORMULATION AND PRELIMINARIES

This paper focuses on uncertain time-driven switched linear systems described by the following differential equation:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)}u(t), \quad \sigma(t) \in \mathcal{M} = \{1, \ldots, M\} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, and $u \in \mathbb{R}$ represents some piecewise continuous input. The matrices $A_p \in \mathbb{R}^{n \times n}$ and vectors $b_p \in \mathbb{R}^n$ are assumed to be unknown for all $p \in \mathcal{M}$. The switching law $\sigma(\cdot)$ is a piecewise function taking values in $\mathcal{M}$, and the capital letter M denotes the number of subsystems.

To develop the adaptive tracking scheme, a reference switched system representing the desired behavior of (1) is given as follows:

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}\sigma(t)}x_{\mathrm{m}}(t) + b_{\mathrm{m}\sigma(t)}r(t), \quad \sigma(t) \in \mathcal{M} \quad (2)$$

where $x_{\mathrm{m}} \in \mathbb{R}^n$ is the desired state vector, and $r \in \mathbb{R}$ is a bounded reference input. The matrices $A_{\mathrm{m}p} \in \mathbb{R}^{n \times n}$ and vectors $b_{\mathrm{m}p} \in \mathbb{R}^n$ are known, and $A_{\mathrm{m}p}$ are Hurwitz matrices for $p \in \mathcal{M}$. Suppose that $(A_{\mathrm{m}p}, b_{\mathrm{m}p})$ is controllable for each $p \in \mathcal{M}$ and each subsystem in (1) has its own corresponding reference sub-model. We assume the measurements of $x(t)$ and $x_{\mathrm{m}}(t)$ are available. Hence, the nominal state feedback controller that makes the switched system behave like the reference model is given as follows:

$$u^*(t) = k^{*T}_{\sigma(t)}(t)x(t) + l^*_{\sigma(t)}(t)r(t)$$

where the nominal parameters $k^*_p \in \mathbb{R}^n$ and $l^*_p \in \mathbb{R}$ exist under the assumption that the following matching condition holds [6], [9], [14]:

$$A_p + b_p k^{*T}_p = A_{\mathrm{m}p}, \ b_p l^*_p = b_{\mathrm{m}p}. \quad (3)$$

However, since A_p and b_p are unknown, we cannot obtain k^*_p and l^*_p from (3). In light of this, the state-feedback controller is developed as:

$$u(t) = k^T_{\sigma(t)}(t)x(t) + l_{\sigma(t)}(t)r(t) \quad (4)$$

where k_p and l_p are the estimates of k^*_p and l^*_p, respectively. In addition, we define the tracking error as: $e(t) = x(t) - x_{\mathrm{m}}(t)$.

Substituting (4) into (1), and subtracting (2), the dynamics of the tracking error are as follows:

$$\dot{e}(t) = A_{\mathrm{m}\sigma(t)}e(t) + b_{\sigma(t)}(\tilde{k}^T_{\sigma(t)}(t)x(t) + \tilde{l}_{\sigma(t)}(t)r(t)) \quad (5)$$

where $\tilde{k}_p = k_p - k^*_p$ and $\tilde{l}_p = l_p - l^*_p$ are the parameter estimation errors.

The following definitions will be used in this work:

Definition 1: (**Class $\mathcal{K}$ and $\mathcal{KL}$**) We say that a function $\alpha : [0, \infty) \to [0, \infty)$ is of class $\mathcal{K}$, and write $\alpha \in \mathcal{K}$, when α is continuous, strictly increasing, and $\alpha(0) = 0$. We say that a function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is of *class $\mathcal{KL}$*, and write $\beta \in \mathcal{KL}$ when $\beta(\cdot, t)$ is of *class $\mathcal{K}$* for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$.

Definition 2: (**Dwell-time switching**) Switching laws with the switching sequence $\mathcal{S} := \{t_1, t_2, \ldots\}$ are said to be *dwell-time admissible* if there exists a number $\tau_{\mathrm{d}} > 0$ such that $t_{i+1} - t_i \geq \tau_{\mathrm{d}}$ holds for all $i \in \mathbb{N}^+$. A positive number τ_{d}, for which these constraints hold for all $i \in \mathbb{N}^+$, is called *dwell time*, and the set of dwell-time admissible switching laws is denoted by $\mathcal{D}(\tau_{\mathrm{d}})$.

Definition 3: (**Global asymptotic stability**) [15] A switched system is said to be globally asymptotically stable if there exists a class $\mathcal{KL}$ function such that for all switching signals $\sigma(\cdot)$ and for any initial condition $x(0)$ the following inequality is satisfied: $|x(t)| \leq \beta(|x(0)|, t), \ \forall t \geq 0$.

Definition 4: ($\mathcal{L}_2$ **class**) A vector signal $\varphi(\cdot)$ is said to belong to $\mathcal{L}_2$, denoted by $\varphi(\cdot) \in \mathcal{L}_2$, if $\int_0^\infty \varphi(\tau)^T \varphi(t)d\tau < \infty, \ \forall t \geq 0$.

Definition 5: ($\mathcal{L}_\infty$ **class**) A vector signal $\varphi(\cdot)$ is said to belong to $\mathcal{L}_\infty$, denoted by $\varphi(\cdot) \in \mathcal{L}_\infty$, if $\max_{t \geq 0} \|\varphi(t)\| < \infty, \ \forall t \geq 0$.

Definition 6: (**Persistently exciting condition**) [9] Consider a signal vector ν generated, with some abuse of notation, as $\nu(t) = H(s)\xi(t)$ where $\xi \in \mathbb{R}$, and $H(s)$ is a vector whose elements are transfer functions that are strictly proper with stable poles. If the complex vectors $H(j\omega_1), \ldots, H(j\omega_n)$ are linearly independent on the complex space $\forall \omega_1, \ldots, \omega_n$, where $\omega_i \neq \omega_j$ for $i \neq j$, then we say ν is *persistently exciting* if and only if ξ contains at least $n/2$ different nonzero frequencies.

Thus, the problem addressed in this paper is presented as follows:

Problem 1: Develop an adaptive law for k_p and l_p in (4) and a switching law $\sigma(\cdot)$ such that the switched system (1) with the state-feedback controller (4) can asymptotically track the reference switched system (2), i.e., the tracking error $e(t) \to 0$ as $t \to \infty$.

Before presenting the main results, we assume that the sign of l^*_p is known, $\forall p \in \mathcal{M}$, which is widely used in adaptive control problems to ensure the boundedness of signals in closed-loop systems [9].

III. METHODOLOGY

To guarantee that the states x of the uncertain switched system track x_m asymptotically, firstly, we need to develop a dwell-time admissible switching law $\sigma(\cdot)$ to guarantee the global stability of the reference switched system with a bounded reference input r. It has been established that the globally asymptotic stability of the homogeneous system, $\dot{x}_\mathrm{m}(t) = A_{\mathrm{m}p}x_\mathrm{m}(t)$, $p \in \mathcal{M}$, is sufficient to lead to global stability of (2) [15]. Hence, using the stability condition proposed in [10], the following lemma is stated,

Lemma 1: The switched system $\dot{x}_\mathrm{m} = A_{\mathrm{m}p}x_\mathrm{m}$, $p \in \mathcal{M}$, is globally asymptotically stable for any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$ if there exist: a collection of symmetric matrices $P_{p,k} \in \mathbb{R}^{n \times n}$, $p \in \mathcal{M}$, $k = 0, \ldots, K$, and a sequence $\{\delta_k\}_{k=1}^K > 0$ with $\sum_{k=1}^K \delta_k = \tau_\mathrm{d}$ such that the following hold:

$$P_{p,k} > 0 \qquad (6a)$$

$$(P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_{p,\mathscr{K}}A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_{p,\mathscr{K}} < 0 \qquad (6b)$$

$$\text{for } \mathscr{K} = k, k+1; \ k = 0, \ldots, K-1$$

$$P_{p,K}A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_{p,K} < 0 \qquad (6c)$$

$$P_{p,K} - P_{q,0} \geq 0 \qquad (6d)$$

$$\text{for } q = 1, \ldots, p-1, p+1, \ldots M,$$

where K is an integer that may be chosen a priori, according to the allowed computational complexity.

By solving the LMIs in (6), a collection of symmetric matrices $P_{p,k}$ and a dwell time τ_d can be obtained that will be utilized to develop a new adaptive law. Let us define a time sequence $\{t_{i,0}, \ldots, t_{i,K}\}$ with $t_{i,k+1} - t_{i,k} = \delta_{k+1}$, $k = 0, \ldots, K-1$. Note that $t_{i,0} = t_i$ and $t_{i,K} - t_{i,0} = \tau_\mathrm{d}$, as shown in Fig. 1.
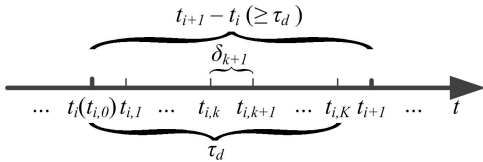

Fig. 1. The time sequence between two consecutive switching instants

Therefore, the adaptive law is proposed as follows:

$$\dot{k}_{\sigma(t)}(t) = -\operatorname{sgn}[l_{\sigma(t)}^*]\Gamma_{\sigma(t)}x(t)e^T(t)P_{\sigma(t)}(t)b_{\mathrm{m}\sigma(t)}$$
$$\dot{l}_{\sigma(t)}(t) = -\operatorname{sgn}[l_{\sigma(t)}^*]\gamma_{\sigma(t)}r(t)e^T(t)P_{\sigma(t)}(t)b_{\mathrm{m}\sigma(t)}, \qquad (7)$$

where $\Gamma_p \in \mathrm{R}^{n \times n}$ and $\gamma_p \in \mathrm{R}$ are given adaptive gains for $p \in \mathcal{M}$ and the time-varying matrix $P_p(t)$ is defined as:

$$P_p(t) = \begin{cases} P_{p,k} + \frac{P_{p,k+1} - P_{p,k}}{\delta_{k+1}}(t - t_{i,k}), & \text{for } t_{i,k} \leq t < t_{i,k+1} \\ P_{p,K}, & \text{for } t_{i,K} \leq t < t_{i+1} \end{cases} \qquad (8)$$

The sequence of switch-in instants of subsystem p is represented by $\{t_{p_1}, t_{p_2}, t_{p_3}, \ldots\}$, and the sequence of its switch-out instants is represented by $\{t_{p_1+1}, t_{p_2+1}, t_{p_3+1}, \ldots\}$. Note that the proposed adaptive law (7) is to be implemented as follows: at a switch-in instant of subsystem p the initial conditions of (7) are taken from the estimates available at the previous switch-out instant of the same subsystem, i.e.,

$k_p(t_{p_l}) = k_p(t_{p_{(l-1)}+1})$, and $l_p(t_{p_l}) = l_p(t_{p_{(l-1)}+1})$ for any $l \in \mathbb{N}^+$. Therefore, k_p and l_p evolve continuously.

Remark 1: Compared with adaptive laws proposed in previous research, the following considerations are in order:

- In contrast with [6], [7], [8], projection laws are not necessary in (7) due to non-increasing behavior of the Lyapunov function at the switching instants, as will be demonstrated in the next section. Therefore, the knowledge of *a priori* bounds for $k_p(t)$ and $l_p(t)$ is not needed for (7).
- The adaptive laws introduced in [6], [7], [8] derive from a classical Lyapunov function consisting of quadratic terms of the tracking error and of the parameter estimation errors, where a constant positive definite matrix P_p for each subsystem is adopted. In this paper, we propose a new adaptive law that uses a time-varying positive definite matrix $P_p(t)$ for each subsystem.
- In [6], [7], [8] the adaptive law is derived independently of the switching law (and vice versa). That is, the design of the switching law and of the adaptive law is decoupled. In the approach proposed here adaptive and switching laws are coupled via the solution of (6), which depends on the dwell time.

IV. MAIN RESULTS OF STABILITY

n this section, the stability results of the proposed control scheme will be presented.

Theorem 1: With the adaptive law (7)–(8) and any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$, the tracking error $e(t)$ converges to zero asymptotically as $t \to \infty$.

Proof: Consider the following Lyapunov function:

$$V(t) = e^T(t)P_{\sigma(t)}(t)e(t) + \sum_{p=1}^M \frac{1}{|l_p^*|}\left(\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t)\right) + \sum_{p=1}^M \frac{1}{|l_p^*|}\left(\tilde{l}_p^2(t)\gamma_p^{-1}\right) \qquad (9)$$

which is continuous during any interval between two consecutive switching instants and discontinuous at switching instants considering the fact that $P_{\sigma(\cdot)}(\cdot)$ is continuous during intervals and discontinuous at switching instants. Without loss of generality, let us consider an interval $[t_i, t_{i+1})$ between two consecutive switching instants t_i and t_{i+1} and let $\sigma(t_i) = p$ and $\sigma(t_{i+1}) = q$ with $i \in \mathbb{N}^+$ and $p, q \in \mathcal{M}$. Then, for $t \in [t_i, t_{i+1})$, subsystem p is active and thus k_j and l_j for all $j \in \mathcal{M}/\{p\}$ maintain constant and their values are those at the last switched-out instant of subsystem j before the time instant t_i. Therefore, using (5) and (7), the derivative of $V(t)$ with respect to time is

$$\dot{V}(t) = \dot{e}^T(t)P_p(t)e(t) + e^T(t)P_p(t)\dot{e}(t) + e^T(t)\dot{P}_p(t)e(t)$$
$$+ 2\frac{1}{|l_p^*|}\tilde{k}_p^T(t)\Gamma_p^{-1}\dot{\tilde{k}}_p(t) + 2\frac{1}{|l_p^*|}\tilde{l}_p(t)\dot{\tilde{l}}_p(t)\gamma_p^{-1} \qquad (10)$$
$$= e^T(t)Q_p(t)e(t)$$

with $Q_p(t)$ defined as

$$Q_p(t) = A_{\mathrm{m}p}^T P_p(t) + \dot{P}_p(t) + P_p(t)A_{\mathrm{m}p} \qquad (11)$$

which is continuous for $t \in [t_i, t_{i+1})$ due to the continuity of $P_p(t)$ for $t \in [t_i, t_{i+1})$.

To analyze the properties of $Q_p(t)$ for $t \in [t_i, t_{i+1})$, first we consider $t \in [t_{i,k}, t_{i,k+1})$, $k = 0, \ldots, K-1$. Note that

$$
\begin{aligned}
Q_p(t) &= A_{\mathrm{m}p}^T P_p(t) + (P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_p(t)A_{\mathrm{m}p} \\
&= \eta_1 \left[(P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_{p,k}A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_{p,k} \right] \\
&\quad + \eta_2 \left[(P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_{p,k+1}A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_{p,k+1} \right]
\end{aligned}
$$
(12)

where $\eta_1 = 1 - (t - t_{i,k})/\delta_{k+1}$, $\eta_2 = (t - t_{i,k})/\delta_{k+1}$. According to (6b), it follows from (12) that

$$
Q_p(t) < 0, \quad t \in [t_{i,k}, t_{i,k+1}).
$$
(13)

Then, let us consider $t \in [t_{i,K}, t_{i+1})$ for the case that $t_{i+1} - t_i > \tau_{\mathrm{d}}$. We have $P_p(t) = P_{p,K}$ according to (7), which indicates by (6c) that

$$
Q_p(t) = A_{\mathrm{m}p}^T P_{p,K} + P_{p,K}A_{\mathrm{m}p} < 0, \quad t \in [t_{i,K}, t_{i+1}).
$$
(14)

Therefore, it follows from (13)–(14) that $Q_p(t) < 0$ due to the continuity of $Q_p(t)$ as $t \in [t_i, t_{i+1})$, which implies that $V(t)$ is strictly decreasing for any $e(t) \neq 0$ for $t \in [t_i, t_{i+1})$, i.e.,

$$
\dot{V}(t) = e^T(t)Q_p(t)e(t) < 0, \quad t \in [t_i, t_{i+1}).
$$
(15)

Since the signals $e(\cdot)$, $\tilde{k}_{\sigma(\cdot)}(\cdot)$, and $\tilde{l}_{\sigma(\cdot)}(\cdot)$ are continuous according to (5) and (7), it follows, at switching instant t_{i+1}, that

$$
\begin{aligned}
&V_{\sigma(t_{i+1})}(t_{i+1}) - V_{\sigma(t_{i+1}^-)}(t_{i+1}^-) \\
&= e^T(t_{i+1})P_{\sigma(t_{i+1})}(t_{i+1})e(t_{i+1}) - e^T(t_{i+1}^-)P_{\sigma(t_{i+1}^-)}(t_{i+1}^-)e(t_{i+1}^-) \\
&= e^T(t_{i+1})(P_{\sigma(t_{i+1})} - P_{\sigma(t_{i+1}^-)})e(t_{i+1}) \\
&= e^T(t_{i+1})\left(P_{q,0} - P_{p,K}\right)e(t_{i+1})
\end{aligned}
$$

which indicates that $V(\cdot)$ is non-increasing at switching instant t_{i+1} considering $P_{q,0} - P_{p,K} \leq 0$ for $p, q \in \mathcal{M}$. Since $V(\cdot)$ is strictly decreasing during any interval between two consecutive switching instants and non-increasing at each switching instant for any $e(t) \neq 0$, now we can conclude that $V(t)$ is strictly decreasing for any $t > 0$ and $e(t) \neq 0$. This implies the boundedness of $V(\cdot)$ and therefore all the signals in the closed-loop switched system according to (9). Integrating (15) from 0 to ∞, we have $\int_0^\infty e^T(t)Q_p(t)e(t)dt < V(0) - V(\infty) < \infty$. Due to the boundedness of $P_p(\cdot)$, $Q_p(\cdot)$ is also bounded, which implies $\int_0^\infty e^T(t)e(t)dt < \infty$, i.e., $e(\cdot) \in \mathcal{L}_2$. According to (9), since $V(\cdot) \in \mathcal{L}_\infty$, we have $e(\cdot) \in \mathcal{L}_\infty$. Additionally, the dynamics of $e(\cdot)$ in (5) gives rise to $\dot{e}(\cdot) \in \mathcal{L}_\infty$. Since $e(\cdot) \in \mathcal{L}_2$ and $\dot{e}(\cdot) \in \mathcal{L}_\infty$, it can be concluded that $e(t) \to 0$ as $t \to \infty$ according to *Barbalat's lemma* [16]. This completes the proof. ∎

Remark 2: Note that $P_p(\cdot)$ is constructed using a family of discrete matrices satisfying (6) for each subsystem. The computational complexity of constructing $P_p(\cdot)$ is dependent on the number K. A larger K leads to a smaller dwell time τ_{d} that can guarantee asymptotic tracking. However, there always exists a constant K^* such that $\forall K > K^*$, the dwell time τ_{d} is equivalent to the result obtained by the stability condition proposed in [12], i.e., for $p \neq q \in \mathcal{M}$,

$$
P_p, P_q > 0, \ P_p A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_p < 0, \ e^{A_{\mathrm{m}p}^T \tau_{\mathrm{d}}} P_q e^{A_{\mathrm{m}p}\tau_{\mathrm{d}}} - P_p < 0 \quad (16)
$$

Remark 3: According to *Remark 2*, a question may arise automatically: why cannot we use the stability condition (16) directly to obtain the result of Theorem 1 instead of condition (6)? The reason is explained in the following.

The differences between the stability conditions in (6) and (16) have a significant impact on the derivative of the Lyapunov function (9). Note that it is not necessary to develop the derivative in (10) during the intervals between two consecutive switches into an exponential decay formulation, i.e., $\dot{V}(t) \leq -\alpha V(t)$ with a compatible number $\alpha > 0$, which is needed in the approach followed in [6], [7], [8]. On the other hand, using (16), the following classical Lyapunov function as in [6] is considered:

$$
\begin{aligned}
V(t) &= e^T(t)P_{\sigma(t)}e(t) + \sum_{p=1}^M \frac{1}{|l_p^*|} \left(\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t) \right) \\
&\quad + \sum_{p=1}^M \frac{1}{|l_p^*|} \left(\tilde{l}_p^2(t)\gamma_p^{-1} \right)
\end{aligned}
$$
(17)

whose derivative is, for $t \in [t_i, t_{i+1})$

$$
\begin{aligned}
\dot{V}(t) &= e^T(t)(P_p A_{\mathrm{m}p} + A_{\mathrm{m}p}P_p)e(t) \\
&\quad + \frac{1}{|l_p^*|}(\tilde{k}_p^T(t)\Gamma_p^{-1}f_{xp}(t) + \gamma_i^{-1}\tilde{l}_p(t)f_{rp}(t)) \\
&\leq -V(t)/\rho - (V(t) - \mathcal{B})/(s\rho)
\end{aligned}
$$
(18)

where f_{xp} and f_{rp} are projection laws, and the positive numbers ρ, s, and $\mathcal{B}$ can be calculated as shown in [6]. The derivative in (18) can be shown to be decreasing at an exponential rate only when $V(t) \geq \mathcal{B}$. According to condition in (16), asymptotic stability can be guaranteed only if the Lyapunov function in (17) is decreasing at an exponential rate for $t \in \mathbb{R}^+/\mathcal{S}$ which cannot be satisfied according to (18). In light of this, we cannot utilize (16) obtain the result of Theorem 1 due to the presence of the exponential term $e^{A_{\mathrm{m}p}\tau_{\mathrm{d}}}$ in (16), while it does not appear in (6).

Remark 4: In the literature on adaptive control of switched systems, the switching laws based on dwell time [6], average dwell time [7], or mode-dependent dwell time [8] are designed based on the following two properties of the Lyapunov function: an exponential decreasing rate during active intervals between two consecutive switching instants, and a bounded increment at switching instants. Note that switching laws based on these two properties and adaptive laws with constant positive definite matrices P_p prevent asymptotic tracking from being achieved, following the same reasoning as in *Remark 3*.

Remark 5: Since subsystem matrices are necessary to calculate the dwell time using Theorem 1 in [10], the method proposed in [10] can only guarantee asymptotic stability of switched systems with uncertainties residing in a known polytope. However, the Lyapunov function (9) exploits the matrices of the reference modes. As a consequence, the proposed adaptive laws (7) with time-varying matrices $P_p(\cdot)$ can achieve asymptotic stability of switched systems with more general (possibly non-polytopic) uncertainties.

Theorem 2: If the reference input signal $r(\cdot)$ is persistently exciting with respect to system (2) (i.e., $r(\cdot)$ has at least $(n+1)/2$ different frequencies), then $\tilde{k}_p(t)$, $\tilde{l}_p(t)$, $p \in \overline{\mathcal{M}}$, and $e(t)$

converge to zero asymptotically as $t \to \infty$, with the adaptive law (7)–(8) and any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_d)$, where $\overline{\mathcal{M}}$ represents the set of subsystems that are active intermittently over infinite intervals.

Proof: Define $\tilde{\theta}_p(t) = [\tilde{k}_p^T(t) \ \tilde{l}_p(t)]^T$ for $t \in T_p$, where $T_p = \cup[t_{p_l}, t_{p_l+1})$ with $l \in \mathbb{N}^+$ denoting the total time period when subsystem p is active. Then, we can express (7) as:

$$\dot{\theta}_p(t) = -\text{sgn}[l_p^*]\overline{\Gamma}_p\omega(t)b_{mp}^T P_p(t)e(t)$$

where $\overline{\Gamma}_p = \text{diag}\{\Gamma_p, \ \gamma_p\}$ and $\phi(t) = [x(t) \ r(t)]^T$. Define $\chi(t) = [e^T(t) \ \tilde{\theta}^T(t)]^T$. Then we have

$$\dot{\chi}(t) = \overline{A}_p(t)\chi(t), \quad e(t) = C_p^T\chi(t)$$

where

$$\overline{A}_p(t) = \begin{bmatrix} A_{mp} & b_p\phi^T(t) \\ -\text{sgn}[l_p^*]\overline{\Gamma}_p\phi(t)b_{mp}^T P_p(t) & 0 \end{bmatrix}, \ C_p = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Consider the time-varying positive definite matrix $\overline{P}_p(t) = \text{diag}\left\{P_p(t), \frac{1}{|l_p^*|}\overline{\Gamma}_p^{-1}\right\}$, where $P_p(t)$ is defined in (8). We consider the following Lyapunov function:

$$V_p(t) = \chi^T(t)\overline{P}_p(t)\chi(t)$$
$$= e^T(t)P_p(t)e(t) + \frac{1}{|l_p^*|}\left(\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t) + \tilde{l}_p^2(t)\gamma_p^{-1}\right).$$

For an interval $[t_{p_l}, t_{p_l+1})$ with $l \in \mathbb{N}^+$ when subsystem p is active, the derivative of the Lyapunov function is given by

$$\dot{V}_p(t) = \chi^T(t)\left(\overline{A}_p^T(t)\overline{P}_p(t) + \overline{P}_p(t)\overline{A}_p(t) + \dot{\overline{P}}_p(t)\right)\chi(t)$$
$$= e^T(t)\left(A_{mp}^T P_p(t) + \dot{P}_p(t) + P_p(t)A_{mp}\right)e(t)$$
$$= e^T(t)Q_p(t)e(t).$$

Based on the proof of Theorem 1, it is clear that $V_p(t)$ is strictly decreasing for any $e(t) \neq 0$, i.e., $\dot{V}_p(t) < 0$ for $t \in [t_{p_l}, t_{p_l+1})$. Therefore, there exist positive constants $\upsilon_p = -\sup\{\lambda_{\max}[Q_p(t)]\}$ such that

$$\dot{V}_p(t) \leq -\upsilon_p e^T(t)e(t) \tag{19}$$
$$\leq -\upsilon_p\chi(t)^T C_p C_p^T\chi(t), \quad t \in [t_{p_l}, t_{p_l+1}).$$

Furthermore, since the reference input $r(\cdot)$ is persistently exciting and (A_{mp}, b_{mp}) is controllable, it follows that $\phi_m(\cdot) := [x_m(\cdot) \ r(\cdot)]^T$ is also persistently exciting [17], which, together with (19), implies that $\phi(\cdot)$ is weakly persistently exciting (according to Definition 3 in [18]). This only leads to asymptotic convergence to zero of the system $\dot{\chi}(t) = \overline{A}_p(t)\chi(t)$ (see Theorem 4 of [18]) for $t \in [t_{p_l}, t_{p_l+1})$. Next, we compare the value of $V_p(t)$ at the switch-out instant t_{p_l+1} and the switch-in instant $t_{p_{(l+1)}}$ of subsystem p. Due to $k_p(t_{p_{(l+1)}}) = k_p(t_{p_l+1})$, and $l_p(t_{p_{(l+1)}}) = l_p(t_{p_l+1})$, and the result that $V(t)$ in (9) is strictly decreasing for any $e(t) \neq 0$, we have

$$V_p(t_{p_{(l+1)}}) - V_p(t_{p_l+1}) = V(t_{p_{(l+1)}}) - V(t_{p_l+1})$$
$$= e(t_{p_{(l+1)}})^T P_p(t_{p_{(l+1)}})e(t_{p_{(l+1)}}) \tag{20}$$
$$\quad - e(t_{p_l+1})^T P_p(t_{p_l+1})e(t_{p_l+1})$$
$$< 0$$

which shows that $V_p(t)$ is strictly decreasing for all $t \in T_p$ together with (19). Now, we construct a continuous time line $\bar{t}$

by joining the intervals when subsystem p is active, i.e., taking $t_{p_{(l+1)}} = t_{p_l+1}$ for all $l \in \mathbb{N}^+$ and $\bar{t}_0 = t_{p_0}$. Therefore, according to (19)–(20), it holds that the system $\dot{\chi}(\bar{t}) = \overline{A}_p(t)\chi(\bar{t})$ is asymptotically stable for the time line $\bar{t}$, that is, $\chi(\bar{t}) \to 0$ as $\bar{t} \to \infty$, which indicates that $e(t)$, $\tilde{k}_p(t)$, and $\tilde{l}_p(t)$ converge to zero asymptotically, as $t \in T_p \to \infty$. ∎

V. EXAMPLE

In this section, an electro-hydraulic system [19], as shown in Fig. 2, is used to demonstrate the effectiveness of the proposed adaptive asymptotic tracking control scheme. Two operating conditions with respect to different supply pressures, 11.0 MPa and 1.4 MPa, are selected, and the corresponding transfer functions are:

$$G_1(s) = \frac{62.4}{s(s+4.58)}, \quad G_2(s) = \frac{47.2}{s(s+9.19)}$$

which can be written in canonical form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -4.58 \end{bmatrix}x + \begin{bmatrix} 0 \\ 62.4 \end{bmatrix}u(t)$$
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -9.19 \end{bmatrix}x + \begin{bmatrix} 0 \\ 47.2 \end{bmatrix}u(t)$$

where $x = [x_1 \ x_2]^T$ with x_1, x_2 representing the displacement of arm and the velocity of the arm, respectively; the input is the control voltage.
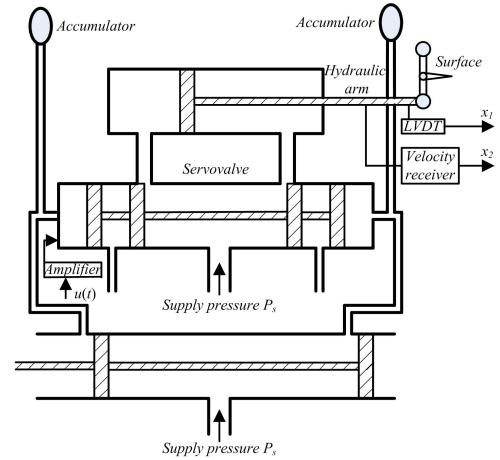

Fig. 2. The schematic diagram of the electro-hydraulic system

The desired behavior is represented by:

$$\dot{x}_m = \begin{bmatrix} 0 & 1 \\ -15 & -8 \end{bmatrix}x_m + \begin{bmatrix} 0 \\ 31.2 \end{bmatrix}r(t), \quad 11.0 \text{ MPa}$$
$$\dot{x}_m = \begin{bmatrix} 0 & 1 \\ -27 & -12 \end{bmatrix}x_m + \begin{bmatrix} 0 \\ 23.6 \end{bmatrix}r(t), \quad 1.4 \text{ MPa}.$$

With $K = 1$ we have a dwell time $\tau_d = 1$ and the matrices obtained by solving the LMIs in (6) are:

$$P_{1,1} = \begin{bmatrix} 1.2605 & 0.0546 \\ 0.0546 & 0.0540 \end{bmatrix}, \ P_{1,2} = \begin{bmatrix} 2.0107 & 0.0491 \\ 0.0491 & 0.1216 \end{bmatrix}$$
$$P_{2,1} = \begin{bmatrix} 1.3832 & 0.0305 \\ 0.0305 & 0.0443 \end{bmatrix}, \ P_{2,2} = \begin{bmatrix} 2.1008 & 0.0279 \\ 0.0279 & 0.0818 \end{bmatrix}.$$

Without loss of generality, we select the switching interval $t_{i+1} - t_i = \tau_\mathrm{d}$ for all i of the switching law $\sigma(\cdot)$. Therefore, the time-varying positive matrix $P_p(t)$ for $p \in \{1, 2\}$ can be calculated by $P_p(t) = (t - \tau_\mathrm{d} \cdot \mathrm{floor}(t/\tau_\mathrm{d})) \cdot (P_{p,2} - P_{p,1})/\tau_\mathrm{d} + P_{p,1}$, where $\mathrm{floor}(t/\tau_\mathrm{d})$ rounds t/τ_d to the nearest integer less than or equal to t/τ_d. The initial conditions are chosen as: $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $x_\mathrm{m}(0) = \begin{bmatrix} 3 & 0 \end{bmatrix}^T$, $l_p(0) = 0.5 l_p^*$ and $k_p(0) = 0.5 k_p^*$, $p \in \{1, 2\}$. The adaptive gains $\Gamma_p = 10I$, $\gamma_p = 5$, $p \in \{1, 2\}$, are selected. In addition, a persistently exciting reference input $r(t) = 3\sin(\pi t) + 2\cos(2t)$ is chosen. The simulation results are shown in Fig. (3)–(5), which indicate that the tracking error converges to 0 and the parameter estimates of the controller $k_1(t)$, $l_1(t)$, $k_2(t)$ and $l_2(t)$ converge to $k_1^* = \begin{bmatrix} -0.2404 & -0.0548 \end{bmatrix}^T$, $l_1^* = 0.5$, $k_2^* = \begin{bmatrix} -0.5720 & -0.0595 \end{bmatrix}^T$, and $l_2^* = 0.5$ asymptotically, respectively.
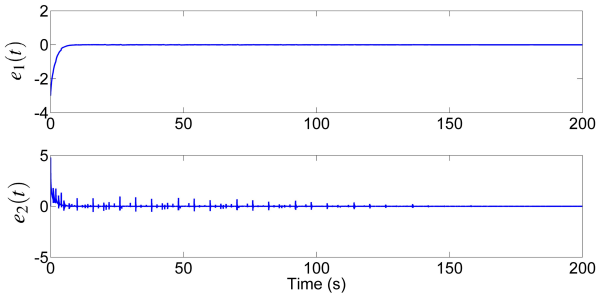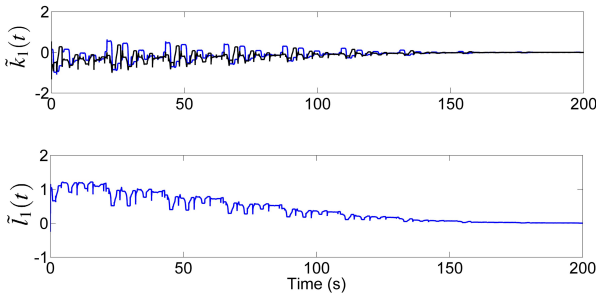

Fig. 3. The tracking error

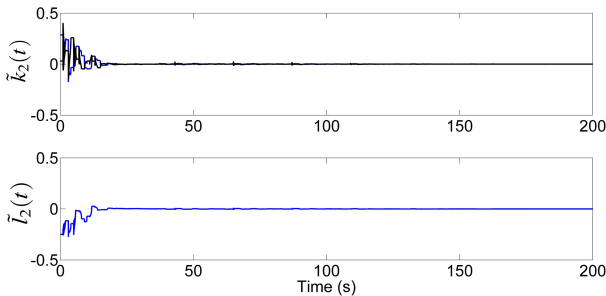Fig. 4. The parameter estimation errors of the controller for subsystem 1

Fig. 5. The parameter estimation errors of the controller for subsystem 2

VI. Conclusion

The adaptive asymptotic tracking control problem of uncertain switched linear systems has been investigated. An adaptive law based on a time-varying positive definite matrix and a dwell-time switching law have been developed. The proposed control scheme can guarantee the asymptotic stability of the tracking error. Furthermore, with the presence of a persistently exciting reference input, the parameter estimates of the controller converge to the real values asymptotically. In light of this, the proposed method has filled the theoretical gap between adaptive control of switched linear systems and non-switched linear systems. A practical example of an electro-hydraulic system has demonstrated the effectiveness of the proposed adaptive control scheme.

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