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Chance-Constrained Model Predictive Controller Synthesis for Stochastic Max-Plus Linear Systems

Vahab Rostampour, Dieky Adzkiya, Sadegh Esmaeil Zadeh Soudjani, Bart De Schutter and Tamás Keviczky

Abstract— This paper presents a stochastic model predictive control problem for a class of discrete event systems, namely stochastic max-plus linear systems, which are of wide practical interest as they appear in many application domains for timing and synchronization studies. The objective of the control problem is to minimize a cost function under constraints on states, inputs and outputs of such a system in a receding horizon fashion. In contrast to the pessimistic view of the robust approach on uncertainty, the stochastic approach interprets the constraints probabilistically, allowing for a sufficiently small violation probability level. In order to address the resulting non-convex chance-constrained optimization problem, we present two ideas in this paper. First, we employ a scenario-based approach to approximate the problem solution, which optimizes the control inputs over a receding horizon, subject to the constraint satisfaction under a finite number of scenarios of the uncertain parameters. Second, we show that this approximate optimization problem is convex with respect to the decision variables and we provide a-priori probabilistic guarantees for the desired level of constraint fulfillment. The proposed scheme improves the results in the literature in two distinct directions: we do not require any assumption on the underlying probability distribution of the system parameters; and the scheme is applicable to high dimensional problems, which makes it suitable for real industrial applications. The proposed framework is demonstrated on a two-dimensional production system and it is also applied to a subset of the Dutch railway network in order to show its scalability and study its limitations.

I. INTRODUCTION

Max-Plus Linear (MPL) systems are a class of discrete event systems with a continuous state space characterizing the timing of the underlying sequential discrete events [1]. MPL systems are predisposed to describe the timing synchronization between interleaved processes, without concurrency or choice. MPL systems have been used in the analysis and scheduling of infrastructure networks, such as communication and railway systems [2], production and manufacturing lines [3], [4], or biological systems [5].

Stochastic max-plus linear systems are MPL systems in which the delays between successive events, e.g. the processing or transportation times, are characterized by random

quantities [2], [6], [7]. In practical applications stochastic MPL systems are more realistic than non-probabilistic MPL ones: for instance in a model of a production system, processing times in general depend on machine conditions and in a railway network, transportation times depend on passengers' transfer time conditions. Thus, they are more suitably modeled by random variables, instead of deterministic constants.

Only a few approaches have been developed in the literature to study the steady-state behavior of stochastic MPL systems, for example by employing Lyapunov exponents and asymptotic growth rates [8], [9], [10], [11], [12]. The Lyapunov exponent of a stochastic MPL system is analogous to the max-plus eigenvalue of an autonomous MPL system. The Lyapunov exponent of stochastic MPL systems has been studied in [8] under some assumptions, and later extended to approximate computations in [9, p. 251] under other technical assumptions. Verifying properties of stochastic MPL systems is recently studied in [13], [14], where the authors have proposed approximation algorithms for checking specifications of the MPL system that are encoded through certain logic. This approach uses formal abstraction techniques [15], [16] implemented in the software tool FAUST² [17].

The application of model predictive control (MPC) to MPL systems is initially studied in [18], and consequently, extended to stochastic MPL systems in [19]. In order to decrease the complexity, [20] and [21] leverage variability expansion and raw moments of random variables, respectively. Finally [22] discusses system identification of stochastic MPL systems. Specifically in [19], the authors described an MPC framework to control the stochastic MPL systems and showed their convexity properties with respect to the control input variables. In this paper, we provide a simpler succinct theoretical proof for the convexity of the set of stochastic max-plus-nonnegative-scaling functions. Our proof relies on the convexity definition of a set of functions that simplifies the proof in [19] considerably.

Stochastic MPC is an alternative approach to achieve a less conservative solution compared to a robust MPC formulation [23], [24]. Robust MPC provides a control law that satisfies the problem constraints for all admissible uncertain variables by assuming that the uncertainty is bounded. Moreover, it treats all uncertainty realizations equally. In stochastic MPC the constraints are treated in a probabilistic sense (chance constraints), meaning that the constraints need to be satisfied only probabilistically up to a pre-assigned level. The resulting optimization problem, however, is in general hard to solve and nonconvex. One way to approximate such

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problems is to employ randomized algorithms that require substituting the chance constraints with a finite number of hard constraints corresponding to samples of the uncertainty set. Interestingly, there is no need for any assumptions on the uncertainty set and its distribution using a randomization approach.

In this paper, we first present the convexity property of max-plus-nonnegative-scaling functions using a different approach to the existing results, which yields a considerably simpler proof. This leads to a finite-horizon optimal control problem for stochastic MPL systems with a convex feasible set. However, the convexity of the feasible set in the problem is typically lost when chance constraints are introduced. To address this issue a randomized technique is employed to tackle the finite-horizon chance-constrained optimal control problem. This leads to a computationally tractable approximation to the aforesaid optimization problem in which only finitely many uncertainty realizations are considered due to the fact that the underlying problem is a convex program with respect to the decision variables. We then investigate the performance and the improvements of the proposed framework for an uncertain production system as a case study by a comparison with a benchmark approach by means of Monte Carlo simulation. The main contributions of this paper are as follows:

- An introduction on Max-Plus algebra together with convexity proof of a set of max-plus-nonnegative-scaling functions. We then describe the MPL models with the extension into the uncertain systems, which leads to the stochastic MPL models.
- An overview on the stochastic control problem formulation for the general stochastic systems. Then, a detailed steps toward stochastic MPC problem formulation is provided by having defined a probabilistically feasible system trajectories for such a dynamical system using a chance constraint representation. However, this formulation is intractable due to the involvement with probabilistic constraints.
- A tractable framework is developed using a randomization technique in order to approximate the underlying probabilistic nature of the problem. Although MPL systems are naturally having a nonlinear behavior, they are inherently having the convexity properties with respect to their control input signal. Therefore, we are able to provide a-priori probabilistic performance guarantees for the feasibility of obtained solution with high confidence level.
- Simulation study for two different cases, namely production system case and Dutch railways system case studies are provided. The former is to demonstrate steps of the proposed framework on a small-scale problem and the latter is to illustrate applicability of such a framework for the large-scale problems as real industrial applications.

The layout of this paper is as follows. After a preliminaries section that consists of the notations used throughout the

paper and an introduction to the basics of max-plus algebra, in Section II, we describe a general stochastic MPC framework for stochastic (perturbed) MPL systems. Section III provides a tractable framework using a randomization technique to approximate the proposed chance-constrained optimization problem together with theoretical probabilistic performance guarantees on the feasibility of the solution. In Section IV, the proposed framework is applied to an uncertain production system as the first case study to show detailed steps of the proposed framework. To demonstrate the efficiency and practical improvements of the proposed methodology, we then implement it to the Dutch railway case study as a large-scale problem and compare it against a deterministic approach. We conclude the paper in Section V and provide some future research directions.

PRELIMINARIES

In this section, we introduce the notations used throughout the paper and describe the fundamental concepts of max-plus linear algebra. We also present simple steps to show that max-plus-nonnegative-scaling functions are convex functions which is used in Section III to provide a tractable scheme.

Notations

The symbols \mathbb{N} , \mathbb{N}_n , \mathbb{R} and \mathbb{R}^+ represent the set of positive integers $\{1, 2, \dots\}$, the first n positive integers $\{1, 2, \dots, n\}$, the set of real numbers and the nonnegative real numbers, respectively. Furthermore \mathbb{R}_ε and ε are defined respectively as $\mathbb{R} \cup \{\varepsilon\}$ and $-\infty$. For $\alpha, \beta \in \mathbb{R}_\varepsilon$, introduce the two operations $\alpha \oplus \beta = \max\{\alpha, \beta\}$ and $\alpha \otimes \beta = \alpha + \beta$, where the element ε is considered to be absorbing w.r.t. \otimes [1, Def. 3.4]. In this paper, the conventional multiplication symbol \times is usually omitted, whereas the max-algebraic multiplication symbol \otimes is always written explicitly. The rules for the order of evaluation of the max-algebraic operators correspond to those in the conventional algebra: max-algebraic multiplication has a higher precedence than max-algebraic addition [1, Sec. 3.1].

For the analysis in Section III, given a metric space Δ , its Borel σ -algebra is denoted by $\mathfrak{B}(\Delta)$. Throughout the paper, measurability always refers to Borel measurability. In a probability space $(\Delta, \mathfrak{B}(\Delta), \mathbb{P})$, we denote the N -Cartesian product set of Δ by Δ^N with the respective product measure by \mathbb{P}^N .

Max-Plus Algebra

The max-plus algebraic operations over scalars are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$, $C \in \mathbb{R}_\varepsilon^{m \times p}$, $D \in \mathbb{R}_\varepsilon^{p \times n}$ and $\alpha \in \mathbb{R}_\varepsilon$, then

$$\begin{aligned} [\alpha \otimes A]_{ij} &= \alpha \otimes A_{ij} = \alpha + A_{ij} , \\ [A \oplus B]_{ij} &= A_{ij} \oplus B_{ij} = \max\{A_{ij}, B_{ij}\} , \\ [C \otimes D]_{ij} &= \bigoplus_{k=1}^p C_{ik} \otimes D_{kj} = \max_{k \in \mathbb{N}_p} \{C_{ik} + D_{kj}\} , \end{aligned}$$

for each $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$. Note the analogy between \oplus , \otimes and respectively $+$, \times for matrix and vector operations in

the conventional algebra. In this paper, the following notation is adopted for reasons of convenience. A vector with each component being equal to 0 and $-\infty$ is also denoted by 0 and ε , respectively. Let $\mathcal{S}_{\text{mpns}}$ be the set of max-plus-nonnegative-scaling functions [19], i.e. functions f of the following form

$$f(z) = \max_{i \in \mathbb{N}_m} \{ \alpha_{i,1} z_1 + \cdots + \alpha_{i,n} z_n + \beta_i \}, \quad (1)$$

where $z \in \mathbb{R}_\varepsilon^n$, $\alpha_{i,\cdot} \in \mathbb{R}^+$ and $\beta_i \in \mathbb{R}$. In the sequel, we write $f(z) \in \mathcal{S}_{\text{mpns}}$ to emphasize that f is a max-plus-nonnegative-scaling function of z . The set $\mathcal{S}_{\text{mpns}}$ is closed under the operations \oplus , \otimes and scalar multiplication by a nonnegative scalar [19, Lemma 1]. In the following proposition, we show that $\mathcal{S}_{\text{mpns}}$ is a convex set and its elements are convex functions.

Proposition 1. *Given $\mathcal{S}_{\text{mpns}}$ a max-plus-nonnegative-scaling function of z with each element $f(z) \in \mathcal{S}_{\text{mpns}}$ is in the form of (1). Then for any $\theta \in [0, 1]$:*

1) *the set $\mathcal{S}_{\text{mpns}}$ is a convex set, if*

$$\forall g(z), h(z) \in \mathcal{S}_{\text{mpns}} \Rightarrow \theta g(z) + (1 - \theta)h(z) \in \mathcal{S}_{\text{mpns}}.$$

2) *$f(\cdot)$ is a convex function in the convex set $\mathcal{S}_{\text{mpns}}$, if for all $v, w \in \mathbb{R}_\varepsilon^n$*

$$f(\theta v + (1 - \theta)w) \leq \theta f(v) + (1 - \theta)f(w).$$

Proof. As a consequence of the fact that the set $\mathcal{S}_{\text{mpns}}$ is closed under the operations of $\max\{\cdot, \cdot\}$, plus and multiplication with a nonnegative scalar, then $\theta g(z) + (1 - \theta)h(z) \in \mathcal{S}_{\text{mpns}}$ and therefore, $\mathcal{S}_{\text{mpns}}$ is a convex set.

As for the second part of the proof, consider $f(z) = \max\{f_1(z), \dots, f_m(z)\} \in \mathcal{S}_{\text{mpns}}$ where $f_i(z) = \alpha_{i,1} z_1 + \cdots + \alpha_{i,n} z_n + \beta_i$. Note that $f_i(z)$ is an affine function of each element of z and thus, it is a convex function. Now define $z = \theta v + (1 - \theta)w$, where variables $v, w \in \mathbb{R}_\varepsilon^n$ and $\theta \in [0, 1]$. The following relation leads to the required convexity condition of $f(z)$:

$$\begin{aligned} f(z) &= f(\theta v + (1 - \theta)w) \\ &= \max \left\{ f_1(\theta v + (1 - \theta)w), \dots \right. \\ &\quad \left. \dots, f_m(\theta v + (1 - \theta)w) \right\} \\ &\leq \max \left\{ \theta f_1(v) + (1 - \theta)f_1(w), \dots \right. \\ &\quad \left. \dots, \theta f_m(v) + (1 - \theta)f_m(w) \right\} \\ &\leq \max \left\{ \theta f_1(v), \dots, \theta f_m(v) \right\} \\ &\quad + \max \left\{ (1 - \theta)f_1(w), \dots, (1 - \theta)f_m(w) \right\} \\ &= \theta \max \left\{ f_1(v), \dots, f_m(v) \right\} \\ &\quad + (1 - \theta) \max \left\{ f_1(w), \dots, f_m(w) \right\} \\ &= \theta f(v) + (1 - \theta)f(w). \end{aligned}$$

The proof is completed by noting that the first inequality is using the convexity property of each element f_i , and the

second inequality is due to the simple fact that if $a, b, c, d \in \mathbb{R}^+$, then $\max\{a + c, b + d\} \leq \max\{a, b\} + \max\{c, d\}$. \square

II. PROBLEM STATEMENT

In this section, we first introduce the mathematical model description of MPL systems and further, the extension to stochastic MPL systems. Then a stochastic control problem formulation for stochastic MPL systems is provided. In particular, we formulate a stochastic MPC problem.

A. Mathematical Model Description

Consider an MPL system in the following form:

$$\begin{aligned} x_{k+1} &= A \otimes x_k \oplus B \otimes u_k, \\ y_k &= C \otimes x_k, \end{aligned} \quad (2)$$

where $A \in \mathbb{R}_\varepsilon^{n \times n}$, $B \in \mathbb{R}_\varepsilon^{n \times m}$, $C \in \mathbb{R}_\varepsilon^{\ell \times n}$. The independent variable k denotes an increasing occurrence index. The variables x_k , u_k and y_k define the (continuous) time of the k -th occurrence of the internal, input and output events, respectively. Since the real system is subject to a measurement noise (uncertainty), we extend the MPL system (2) to the following stochastic MPL system (3) where the uncertainties caused by disturbances and errors in the estimation of physical variables are gathered in a vector δ_k as random variables.

A stochastic MPL system is an extension of event-invariant MPL system (2) where the system matrices are uncertain. This system is described as

$$\begin{aligned} x_{k+1} &= A(\delta_k) \otimes x_k \oplus B(\delta_k) \otimes u_k, \\ y_k &= C(\delta_k) \otimes x_k, \end{aligned} \quad (3)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input vector, $y_k \in \mathbb{R}^\ell$ is the output vector, $\delta_k \in \Delta \subseteq \mathbb{R}^d$ is the random vector defined on a probability space (Δ, \mathbb{P}) for $k \in \mathbb{N} \cup \{0\}$. Next we describe the assumptions. First it is assumed that Δ is endowed with a σ -algebra \mathcal{D} , and that \mathbb{P} is a probability measure defined over \mathcal{D} . Furthermore the augmented matrix $A(\delta_k) \in \mathcal{S}_{\text{mpns}}^{n \times n}(\delta_k)$, $B(\delta_k) \in \mathcal{S}_{\text{mpns}}^{n \times m}(\delta_k)$ and the output matrix $C(\delta_k) \in \mathcal{S}_{\text{mpns}}^{\ell \times n}(\delta_k)$ are assumed to be irreducible [25, p. 17], and are measurable functions with respect to \mathcal{D} . Then each entry of the system matrices is an independent and identically distributed random variable with respect to $k \in \mathbb{N}$. It is assumed that each random variable has a fixed support [25, Def. 1.4.1]. The uncertain realizations (scenarios) $\{\delta_0, \delta_1, \dots\}$ capture the complete event-varying aspect of the system and are independent and identically distributed (i.i.d.) random variables on a probability space Δ , where a sufficient number of i.i.d. samples from δ_k can be obtained either empirically or by a random process. In fact for this work, all we need is only a finite number of scenarios, and we do not require the probability space Δ and the probability measure \mathbb{P} to be known explicitly. For such technical details the reader is referred to [26, Sec. 3.3].

B. Stochastic Control Problem

Consider the discrete event nonlinear dynamics formulation of the aforementioned uncertain system (3) in a compact format as

$$\begin{cases} x_{k+1} = \mathcal{M}_x(x_k, u_k, \delta_k) \\ y_k = \mathcal{M}_y(x_k, \delta_k) \end{cases}, \quad (4)$$

where $\mathcal{M}_x, \mathcal{M}_y$ are the stochastic MPL system functions. x_k, u_k, y_k , and the random variable (uncertainty) δ_k are defined as before that appear in the stochastic MPL system equations. It is assumed that the entire state vector of the aforementioned nonlinear dynamical system can be computed due to the fact that all future state variables can be eliminated recursively given the initial state of system (4).

Let us consider output events that are desired to occur before a given deadline while delaying the input events. In this work, we consider an N step finite-horizon stochastic control problem to determine the latest time of input events such that the output events occur before a given deadline with a high confidence level. Therefore, only the input variables remain as free variables and we are interested in constructing a feedback control law

$$u_k := \kappa(x_k) \quad \text{where } \kappa : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad (5)$$

that generates an input sequence $\{u_0, u_1, \dots\}$ to control the stochastic MPL system (4), and to be chosen from a set of feasible inputs $\mathbb{U} \subseteq \mathbb{R}^m$ based only on the current state of the system. The set \mathbb{U} can be characterized via the set of constraints on the input control variables.

Define now a full prediction horizon that contains N steps into the future, and a subscript ‘ k ’ in our notation is introduced to characterize the occurrence index $k \in \mathbb{N}_N$ within the horizon. The initial value of the states is denoted by x_0 , whereas x_k and u_k are defined to be the state and input vector at occurrence index k of the horizon, respectively. The minimization of the objective function is subject to keeping the state inside a feasible set $\mathbb{X} \subseteq \mathbb{R}^n$, that can be determined using the state variables constraints, for a given prediction horizon steps which maybe too conservative, and results in a poor performance. In particular, this is the case when the best performance is achieved close to the boundary of \mathbb{X} , and thus, constraint violations will be unavoidable due to the fact that the MPL models are imperfect and uncertain. To tackle such a problem, we define a chance constraint on the state trajectories to avoid violation of the state variables constraints probabilistically even if the disturbance has unbounded support. Notice that a robust problem formulation [24] cannot cope with problems having an unbounded disturbance set.

Definition 1 (Probabilistically Feasible). *Given $\alpha \in (0, 1)$ as an admissible constraint violation parameter, the state variables are called probabilistically feasible if*

$$\mathbb{P}_\delta [x_{k+i|k} \in \mathbb{X}, i \in \mathbb{N}_N] \geq 1 - \alpha. \quad (6)$$

Note that the index of \mathbb{P}_δ denotes the dependency of $x_{k+i|k}$ on the string of random scenarios $\{\delta_0, \delta_1, \dots, \delta_{N-1}\}$.

C. Stochastic Model Predictive Control

According to (4) the predicted time of output events for i occurrences into the future is denoted by $y_{k+i|k} = \varphi(x_k, \tilde{\mathbf{u}}, \tilde{\boldsymbol{\delta}})$, where x_k is assumed to be the current state. We define the augmented vector of planned input control and random scenarios by $\tilde{\mathbf{u}} := (u_0, \dots, u_i)$ and $\tilde{\boldsymbol{\delta}} := (\delta_0, \dots, \delta_i)$ to capture an incomplete sequence of them until event step i . Consider now a complete sequence of random scenarios to be $\boldsymbol{\delta} := (\delta_0, \delta_1, \dots, \delta_{N-1}) \in \Delta^N$, and a complete sequence of the planned input control to be $\mathbf{u} := (u_0, u_1, \dots, u_{N-1}) \in \mathbb{U}^N$. The main objective function for a stochastic MPL control problem is to penalize the delayed deliveries for the output elements at event step k , and furthermore, to promote input feeding as late as possible at each event step k over a finite horizon while satisfying state and input constraints, and taking into account that the uncertainty manifests itself in the form of a random variable in the system characteristics. Moreover, we define a desired behavior to be the output events occur before the deadline while trying to delay the input events as late as possible. We define

$$\|\max\{y_{k+i|k} - r_{k+i}, 0\}\|_1 - \gamma \|u_{k+i|k}\|_1 := \mathcal{J}(x_k, \tilde{\mathbf{u}}, \tilde{\boldsymbol{\delta}}),$$

where r_{k+i} is the deadline for the $(k+i)$ -th occurrence of the output events and γ is a cost coefficient term for the input variables. $\mathcal{J}(\cdot)$ is a stage cost function that reflects our control purpose, i.e., desired set-point tracking for all elements of the output sequence and maximizing (as late as possible) the occurrence of input events. Consider the following stochastic objective function:

$$J(x_k, \mathbf{u}, \boldsymbol{\delta}) := \sum_{i=1}^N \mathcal{J}(x_k, \tilde{\mathbf{u}}, \tilde{\boldsymbol{\delta}}), \quad (7)$$

where J is a random variable. Note that the index of summation appears in the augmented vectors $\tilde{\mathbf{u}}, \tilde{\boldsymbol{\delta}}$. We consider $\mathbb{E}[J(x_k, \mathbf{u}, \boldsymbol{\delta})]$ to obtain a deterministic objective function.

Now we can formulate a chance-constrained finite-horizon optimal control problem for each event step k :

$$\min_{\mathbf{u} \in \mathbb{U}} \mathbb{E}[J(x_k, \mathbf{u}, \boldsymbol{\delta})] \quad (8a)$$

$$\text{s.t.} \quad u_{k+(i+1)|k} - u_{k+i|k} \geq 0, \quad (8b)$$

$$\mathbb{P}_\delta \left[\begin{cases} y_{k+i|k} \in \mathbb{Y} \\ \mathbf{E} u_{k+i|k} + \mathbf{F} y_{k+i|k} \leq \mathbf{H} \\ i \in \{1, \dots, N\} \end{cases} \right] \geq 1 - \alpha, \quad (8c)$$

where $\mathbb{Y} \subseteq \mathbb{R}^\ell$ represents a desired convex bound on the predicted output events at each horizon step $i \in \mathbb{N}_N$. The constraint (8b) corresponds to the fact that the time between consecutive input events should be given by a nonnegative value. The constraints (8c) represent chance constraints on the inputs and the value of the output events, where $\mathbf{E}, \mathbf{F}, \mathbf{H}$ are matrices of appropriate dimensions. The solution of (8) is the optimal planned input sequence $\mathbf{u}^* := \{u_0^*, u_1^*, \dots, u_{N-1}^*\}$. The proposed optimization problem (8) is solved at each event step ‘ k ’ by using the current measurement of the state x_k , and based on the model predictive

control algorithm: the current input is set to $u_k := u_0^*$ and continue in a receding horizon fashion. Due to the presence of chance constraints, the feasible set is, in general, nonconvex and hard to determine explicitly. In the following section, we propose a tractable formulation to solve (8).

III. PROPOSED FRAMEWORK

Our proposed framework is based on the approximation of chance-constrained optimization problems (8) using a randomized approach to avoid introducing arbitrary assumptions on \mathbb{P} and its moments. The randomized approach is a tool to approximate chance constraints and substitute the chance constraints with a finite number of pointwise constraints at independently generated scenarios of the uncertain parameters. However, the number of scenarios is a crucial parameter to achieve the desired level of approximation and has to be selected carefully. The authors of [27] have developed an elegant approach based on the randomization technique, namely the scenario approach, that provides a lower bound for the number of extracted scenarios to establish the desired probabilistic guarantees with high confidence for feasibility of the optimization problem.

The main requirement of this approach is that the theoretical bound only holds for convex problems, i.e., when the cost and the constraint functions are convex in the decision variables for each realization of uncertainties. As it is shown in [28], this property is not met for general nonlinear systems. Therefore, we need to first show convexity of the proposed optimization problem (8) to employ the scenario approach, and then develop a tractable framework for such a problem. In order to show convexity of the cost function and constraint function of the proposed chance-constrained optimization problem (8), we have to prove that future output events belong to the set of max-plus-nonnegative-scaling functions $\mathcal{S}_{\text{mpns}}$.

Proposition 2. *Given a stochastic MPL system in the form of (4), the future output events $y_{k+i|k}$ belong to the set of max-plus-nonnegative-scaling functions $\mathcal{S}_{\text{mpns}}$ for all $i \in \mathbb{N}_N$.*

Proof. Since $\mathcal{M}_x(\cdot)$ and $\mathcal{M}_y(\cdot)$ are convex functions, by definition both functions belong to $\mathcal{S}_{\text{mpns}}$. Note that the composition of two convex functions is a convex function. Thus $\varphi(\cdot) := \mathcal{M}_y(\mathcal{M}_x(\cdot))$ is a convex function. \square

Notice that the cost function is a convex function since the expected operator $\mathbb{E}[\cdot]$ is a linear operator, and any linear and $\max\{\cdot, \cdot\}$ operator over two convex functions yields another convex function. We are now able to employ the scenario approach to determine the number of required uncertain scenarios to approximate the proposed chance-constrained (8c). Consider the sets $\mathbf{W}_0 := \{\delta^{(1)}, \dots, \delta^{(S_0)}\}$ and $\mathbf{W}_1 := \{\delta^{(S_0+1)}, \dots, \delta^{(S_0+S_1)}\}$ to be a set of ‘ S_0 ’ scenarios that is used to empirically approximate the cost function J and a set of ‘ S_1 ’ scenarios to probabilistically enforce the state constraints for the full predicted stages. Consider now the following tractable formulation of (8),

called randomized MPC:

$$\min_{\mathbf{u} \in \mathbb{U}} \sum_{\delta^{(k)} \in \mathbf{W}_0} J(x_k, \mathbf{u}, \delta^{(k)}), \quad (9a)$$

$$\text{s.t. } u_{k+(i+1)|k} - u_{k+i|k} \geq 0, \quad (9b)$$

$$\begin{cases} y_{k+i|k} = \varphi(x_k, \mathbf{u}, \delta^{(l)}) \in \mathbb{Y} \\ \mathbb{E} u_{k+i|k} + \mathbb{F} y_{k+i|k} \leq \mathbb{H} \end{cases}, \begin{cases} \forall i \in \mathbb{N}_N \\ \forall \delta^{(l)} \in \mathbf{W}_1 \end{cases}, \quad (9c)$$

where (S_0, S_1) are nonnegative integers and full horizon uncertainty scenarios $S = (S_0 + S_1)$ are drawn independently with respect to Δ^N . We assume a feasible solution is admitted for every realization of uncertainties. In the case of having an infeasible solution, we have to generate a new set of S random scenarios. We state the following result that descends from [27, Th. 1].

Theorem 1. *Define the positive constant parameters $\alpha, \beta \in (0, 1)$ to be a probability of constraint violations and a confidence level, respectively. If*

$$S_1 \geq g(\alpha, \beta, mN) := \frac{2}{\alpha} \ln \frac{1}{\beta} + 2mN + \frac{2mN}{\alpha} \ln \frac{2}{\alpha},$$

then the optimal solution of the tractable formulation (9) is a feasible solution for the chance-constrained optimization problem (8) with confidence level of $(1 - \beta)$.

Applying a receding horizon policy in the MPC framework, the problem in (9) must be solved at each event step with an updated initial state x_k and the current input $u_k := u_0^*$ is set to the first element of the feasible solution $\mathbf{u}^* := \{u_0^*, u_1^*, \dots, u_{N-1}^*\}$. Note that the user-defined scenario size S_0 can be seen as a tuning variable to approximate the cost function for the predicted stages. The proposed procedure is summarized in Algorithm 1.

Algorithm 1 Randomized MPC

- 1: Fix $S_0 \in \mathbb{N}$ to approximate the cost function and select S_1 according to Theorem 1.
 - 2: Generate $S = (S_0 + S_1)$ scenarios of $\delta \in \Delta^N$.
 - 3: Solve (9) and determine a feasible solution \mathbf{u}^* .
 - 4: **if** the problem (9) is infeasible **then** go to step 2.
 - 5: Apply the first input of solution $u_k := u_0^*$ to the uncertain real MPL system (4).
 - 6: Measure state x_k .
 - 7: Go to step 2.
-

Remark 1. *The proposed framework in Algorithm 1 provides a solution to the stochastic MPL system (4) with a probabilistic feasibility certificate and it does not necessarily lead to the optimal solution. This is due to the fact that a set of S_0 random scenarios is used as a tuning variable to empirically approximate the cost function J . We refer the reader for a performance bound on the sub-optimality of the obtained solution to [26].*

IV. SIMULATION STUDY

In this section we provide two different case studies. In the first case study, a production system is considered (cf. Section IV-A). We use a small-scale problem to illustrate detailed steps of our proposed framework. To demonstrate the applicability of our proposed scheme we then study a high-dimensional problem by using a subset of Dutch railway systems in Section IV-B, which shows that our approach is suitable for real industrial applications. In the latter case study, we also adapt our framework with the framework developed in [29] which is called ‘‘Robust Randomized MPC’’. In this framework, we first determine a probabilistic bound for the generated samples and then, we solve a robust optimization problem at each event step.

The uncertainty δ_k is assumed to be an independent and identically distributed random variable that takes values from an exponential distribution $\delta_k \sim \exp(\lambda)$, where λ is the rate parameter. In both case studies, the mean (i.e. the reciprocal of λ) is chosen to be 5% of the nominal values. The prediction horizon and simulation steps in both case studies are assumed to be 10 and 30 steps, respectively, and the cost coefficient term is $\gamma = 0.5$. We consider $\alpha = 0.05$, and $\beta = 0.00001$ as in Theorem 1. We carry out a comparison with two different deterministic approaches to illustrate the performance of Algorithm 1 (Randomized MPC). In one of the deterministic formulations, we consider the nominal system model without uncertain elements, and then, a deterministic MPC formulation is solved. Whereas in the next approach the nominal system model is formulated together with considering the mean value of the uncertain elements distribution. The simulation environment is MATLAB while using YALMIP toolbox [30] as an interface for the GUROBI solver [31] on a desktop PC with a CPU 2.6 GHz Intel Core i5 processor.

A. Production Systems Case Study

We consider the example of a production system in Figure 1, which is taken from [19]. This system comprises of two machines M_1 and M_2 , that operate in batches. Machine M_1 requires raw material and yields an intermediate product. Machine M_2 requires the intermediate product generated by M_1 and yields a final product. We assume that each machine starts working as soon as the required material is available and the machine is idle. A machine is idle when the previous batch has been processed. We define u_k as the time instant at which the k -th batch of raw materials enters the system, y_k as the time instant at which the k -th batch of final products leaves the system, x_k^i as the time instant at which machine M_i starts processing the k -th batch, t_k^i as the transportation time of the k -th batch, d_k^i as the time spent by machine M_i to process the k -th batch. The system equations are given by (3), where

$$A = \begin{bmatrix} d_{k-1}^1 & \varepsilon \\ d_{k-1}^1 + d_k^1 + t_k^2 & d_{k-1}^2 \end{bmatrix}$$

$$B = \begin{bmatrix} t_k^1 \\ d_k^1 + t_k^1 + t_k^2 \end{bmatrix}, \quad C = [\varepsilon \quad d_k^2 + t_k^3].$$

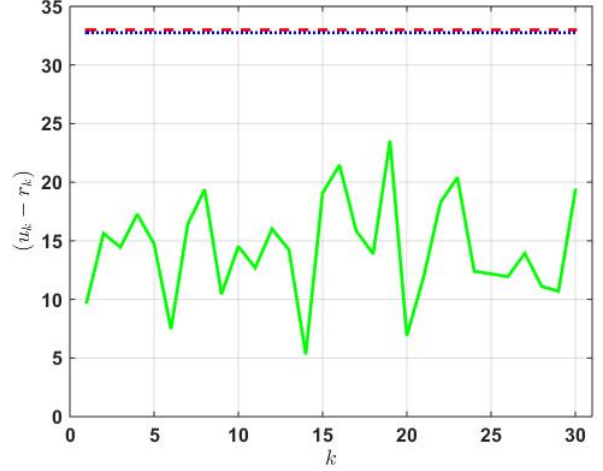


Fig. 2. Production system case study; the difference $(y_k - r_k)$ between the output signal y_k and the due date signal r_k with $\gamma = 0.5$. The ‘green’ solid line is related to the result of our developed framework (see Algorithm 1), the ‘blue’ dotted and ‘red’ dashed lines are based on the deterministic approach. The ‘red’ dashed line corresponds to the nominal system model, whereas the ‘blue’ dotted line shows the results for the nominal system model together with taking into account the mean value of the uncertain elements. The ‘black’ line is the boundary for $(y_k - r_k) \leq 10$.

We assume that the transportation times are constant: $t_k^1 = 0$, $t_k^2 = 1$, $t_k^3 = 0$. Furthermore the processing time of machines M_1 and M_2 is assumed to be $d_k^1 = 5 + \delta_k$ and $d_k^2 = 1$, respectively. The due date signal is given by $r_k = 10 + 40 \cdot k$, where the initial state is equal to $x_0 = [40 \quad 10]^T$.

Figure 2 depicts the difference $(y_k - r_k)$ between the output signal y_k and the due date signal r_k when the cost coefficient term is given by $\gamma = 0.5$. The ‘green’ solid, ‘blue’ dotted and ‘red’ dashed lines correspond respectively to the results of our developed framework (see Algorithm 1), the nominal deterministic approach for the nominal system model, and nominal deterministic approach together with taking into account the mean value of the uncertain elements’ distributions. In this simulation study, we consider to have a probabilistic constraint on the output signal as $(y_k - r_k) \leq 10$ where in Figure 2 the boundary condition is shown by the ‘black’ line. As it can be seen in Figure 2, both deterministic approaches lead to infeasible closed-loop trajectories, whereas the solution obtained via the proposed Algorithm 1 yields a feasible solution trajectory and better performance since almost all the results at each event step are negative which means that the due date signal is reached.

The difference $(u_k - r_k)$ between the input signal u_k and the due date signal r_k is shown in Figure 3. Following the same results, the obtained cost associated with the input signal u_k via the proposed Algorithm 1 is much lower than that of both deterministic approaches. In conclusion, the obtained feasible solution highlights a less conservative solution together with a better performance in terms of value function of the problem compared to the deterministic solutions.

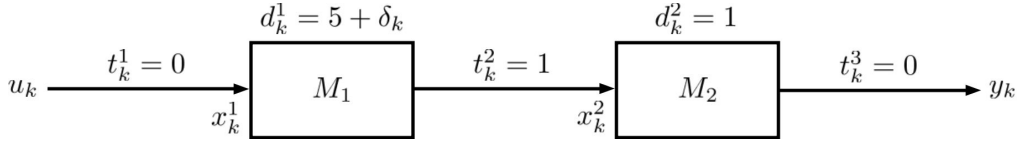


Fig. 1. An example of a production system. The system comprises of two subsystems M_1 and M_2 that operate in batches.

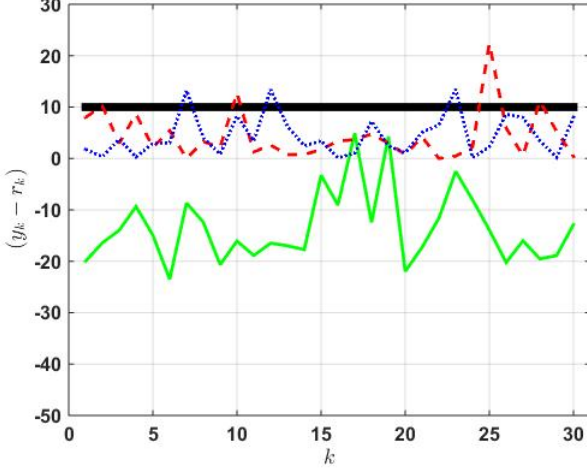


Fig. 3. Production system case study; the difference $(u_k - r_k)$ between the input signal u_k and the due date signal r_k with $\gamma = 0.5$. The ‘green’ solid line is related to the result of our developed framework (see Algorithm 1), the ‘blue’ dotted and ‘red’ dashed lines are based on the deterministic approach. The ‘red’ dashed line corresponds to the nominal system model, whereas the ‘blue’ dotted line shows the results for the nominal system model with taking into account the mean value of the uncertain elements.

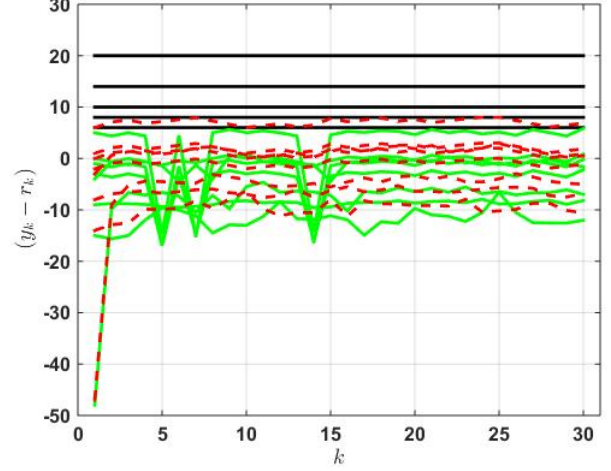


Fig. 4. Dutch railway network case study; the difference $(y_k - r_k)$ between the output signal y_k and the due date signal r_k with $\gamma = 0.5$. The ‘green’ solid line is related to the result of our developed framework (see Algorithm 1), the ‘red’ dashed line is based on the deterministic approach where the nominal system model is considered. The ‘black’ lines are the boundaries for $(y_k - r_k) \leq x_0$.

B. Dutch Railways Case Study

As an additional case study, we transform the deterministic MPL system developed in [32, p. 31] to the following stochastic MPL systems that take into account the uncertainties:

$$\begin{aligned}
 x_{k+1}^1 &= (38 + \delta_k^1) \otimes x_k^8 \oplus u_k^1 \\
 x_{k+1}^2 &= (36 + \delta_k^2) \otimes x_k^8 \oplus u_k^2 \\
 x_{k+1}^3 &= (55 + \delta_k^3) \otimes x_k^1 \oplus (54 + \delta_k^4) \otimes x_k^2 \oplus u_k^3 \\
 x_{k+1}^4 &= (35 + \delta_k^5) \otimes x_k^3 \oplus u_k^4 \\
 x_{k+1}^5 &= (54 + \delta_k^6) \otimes x_k^4 \oplus u_k^5 \\
 x_{k+1}^6 &= (58 + \delta_k^7) \otimes x_k^4 \oplus u_k^6 \\
 x_{k+1}^7 &= (90 + \delta_k^8) \otimes x_k^5 \oplus (93 + \delta_k^9) \otimes x_k^6 \oplus u_k^7 \\
 x_{k+1}^8 &= (16 + \delta_k^{10}) \otimes x_k^7 \oplus u_k^8
 \end{aligned}$$

$$y_k^i = x_k^i, \quad \forall i \in \mathbb{N}_8,$$

where x_k^i and y_k^i , for all $i \in \mathbb{N}_8$ represent the k -th departure time of trains at the stations Den Haag CS to Amersfoort (via Utrecht), Rotterdam CS to Amersfoort (via Utrecht), Amersfoort to Zwolle, Zwolle to Leeuwarden and to Groningen, Leeuwarden to Amersfoort (via Zwolle), Groningen to Amersfoort (via Zwolle), Amersfoort to Utrecht, Utrecht to Den Haag CS and to Rotterdam CS, respectively. Finally u_k^i for $i \in \mathbb{N}_8$ represents the lower bound on the k -th departure time of trains at the previous routes. Let us now clarify

the difference between y_k , r_k and u_k . y_k is the actual k -th departure (same with x_k). r_k is the deadline for the actual k -th departure. u_k is the earliest possible k -th departure. This means that any train cannot depart before the earliest departure time, i.e. y_k cannot be earlier than u_k . Furthermore if there is a problem, a train may depart after the earliest possible departure time, i.e. y_k may be later than u_k . Our objective is the following: given r_k , we want to find u_k (as late as possible) such that y_k is not later than r_k . We consider an exponential distribution for uncertainties. The due date signal is given by $r_k = x_0 + 55 \cdot k$, where the initial state is equal to $x_0 = [8 \ 6 \ 14 \ 6 \ 6 \ 10 \ 53 \ 20]^T$. We also introduce some constraints for the passengers transfer time. This is possible by requiring a minimum time interval between the two consecutive arrival and departure of a train. In this case, the constraints are given by $(y_{k+1} - y_k) \geq 0$ which are quite difficult to handle because it contains variables at two consecutive event steps $k+1$ and k .

Figure 4 shows the difference $(y_k - r_k)$ between the output signal y_k and the due date signal r_k with $\gamma = 0.5$. In this figure the result of our proposed Algorithm 1 is determined via the ‘green’ solid line, and the result of the nominal deterministic approach by the ‘red’ dashed line. The ‘black’ lines are the boundaries for $(y_k - r_k) \leq x_0$. This constraint implies that the difference between departure and due date for any train should be smaller than the initial state. Moreover, we

impose this constraint by considering uncertain behavior of the railway network. The obtained results via ‘green’ solid lines as it was guaranteed by our proposed framework (see Algorithm 1), yields a feasible closed-loop trajectories whereas the solution of the deterministic approach using ‘red’ lines leads to an infeasible network trajectories.

V. CONCLUSIONS

In this paper, we formulated a chance-constrained MPC control problem for stochastic MPL systems as a rich class of stochastic hybrid dynamical systems which is of wide practical interest. We then developed a computationally tractable framework for such a problem while providing theoretical guarantees on the feasibility of the solution in a probabilistic sense. The scalability of the proposed framework is demonstrated via Dutch railway case study as a large-scale stochastic MPL system. In our ongoing research, we are focusing on combining this approach with formal method techniques in order to obtain a more generalized framework for this class of systems.

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