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On robust adaptive control of switched linear systems

Shuai Yuan[1] and Bart De Schutter[1] and Simone Baldi[1]

Abstract— **This paper investigates robust adaptive control of uncertain switched linear systems considering disturbances. Two modifications of the adaptive law of switched linear systems [1] based on parameter projection and a leakage approach are developed to guarantee the stability of the closed-loop switched linear system: a projection law that requires knowledge of the bounds of the parameter estimates; and a leakage law based on initial conditions of the parameter estimates that does not require knowledge of the bounds of the parameter estimates. The closed-loop switched linear system is shown to be globally uniformly ultimately bounded. In addition, the ultimate bounds of both adaptive control schemes are also given. A numerical example is provided to illustrate the effectiveness of the proposed methods.**

Index Terms— **Robust adaptive control, switched linear systems**

I. Introduction

Many complex physical systems, such as automobile power trains [2], traffic light systems [3], power converters [4], and smart buildings [5], exhibit hybrid dynamics. These practical systems can be modeled as a switched system that consists of continuous-time or discrete-time subsystems and switching actions controlled by a constant piecewise signal, called switching signal.

When controlling such complex physical systems, a ubiquitous problem is how to cope with parametric uncertainties and disturbances. It is well established that robust control can be used to deal with non-switched systems subject to uncertainties and disturbances [6], [7]. To date, robust control of switched systems has been extensively investigated: a single robust controller [8], [9], and a family of robust controllers for polytopic uncertainties [10], [11]. However, a single robust controller may lead to conservative performance when considering large uncertainties [12]. As a complement to robust control, adaptive control techniques have been shown to be able to deal with large non-polytopic uncertainties and disturbances [13], [14]. Recent years have witnessed some research on adaptive control of switched systems without considering disturbances [1], [15], [16], [17]. However, to the best of the authors' knowledge, the research on robust adaptive control of switched systems considering both parametric uncertainties and disturbances is a quite open field. In [18], Qing *et al.* proposed a robust adaptive control scheme for switched linear systems that requires the existence of a common Lyapunov function. Hidetoshi and Kojiro developed a so-called adaptive robust controller without considering disturbances [19]. In light of this, the motivation for the current work stems from developing a robust adaptive control scheme to deal with switched linear systems considering non-polytopic parametric uncertainties and disturbances without requiring a common Lyapunov function.

In this paper, our recent result about adaptive control of switched linear systems without considering disturbances [1] is exploited to develop two robust adaptive control schemes for switched linear systems. With an assumption on the knowledge of the bounds of nominal parameters, a robust adaptive control scheme using parameter projection is proposed, which is an immediate extension of the result in Section 8.5.5 of [13]. Next, a robust adaptive control scheme is developed via a leakage approach involving initial conditions of the parameter estimates: this approach is different from the results established in Section 8.5.2 of [13]: the leakage terms involve the difference between the parameter estimates and the initial conditions. In addition, the closed-loop switched linear system is shown to be globally uniformly ultimately bounded based on the proposed two robust adaptive schemes, and an ultimate bounds for both cases are also given.

The paper is organized as follows: Section II presents the control problem and some preliminaries for later analysis. Section III introduces robust adaptive control schemes based on projection laws and leakage approaches, respectively. In addition, the results about global uniform ultimate boundedness of the closed-loop switched linear systems are also given. Section IV adopts a numerical example to illustrate the proposed results. The paper is concluded in Section V.

Notation: The notation used in this paper is as follows: $\mathbb{R}$ and $\mathbb{N}^+$ represent the set of real numbers and positive natural numbers, respectively. The notation $P > 0$ indicates a symmetric positive definite matrix P, and we define $\mathbf{He}\{P\} = P^T + P$, where the superscript T represents the transpose of matrix. The notation $\|\cdot\|$ represents the Euclidean norm. The function $\mathrm{sgn}[*]$ takes the sign of $*$. The identity matrix of dimension n is denoted by $I_{n\times n}$. The operators $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ return the maximum and minimum eigenvalues of the square matrix P, respectively.

II. Problem formulation and preliminaries

This paper focuses on the uncertain single-input switched linear system with a bounded disturbance defined as follows:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)}u(t) + d(t), \quad t \geq t_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ represents some piecewise continuous input, and $d(\cdot)$ is a bounded distur-

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bance with an upper bound $\overline{d}$. The switching law $\sigma(\cdot)$ is a piecewise function taking values in $\mathcal{M} := \{1,\dots,M\}$, and the capital letter M denotes the number of subsystems. The matrices $A_p \in \mathbb{R}^{n\times n}$ and vectors $b_p \in \mathbb{R}^n$ are unknown for all $p \in \mathcal{M}$.

A reference switched system representing the desired behavior of (1) is given as follows:

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}\sigma(t)}x_{\mathrm{m}}(t) + b_{\mathrm{m}\sigma(t)}r(t), \quad t \geq t_0 \tag{2}$$

where $x_{\mathrm{m}} \in \mathbb{R}^n$ is the desired state vector, and $r \in \mathbb{R}$ is a bounded reference input. The matrices $A_{\mathrm{m}p} \in \mathbb{R}^{n\times n}$ and vectors $b_{\mathrm{m}p} \in \mathbb{R}^n$ are known, and $A_{\mathrm{m}p}$ are Hurwitz matrices for all $p \in \mathcal{M}$. Suppose that the pair $(A_{\mathrm{m}p},\ b_{\mathrm{m}p})$ is controllable and each subsystem in (1) has its own corresponding reference sub-model. We assume that the measurements of the states $x(t)$ and $x_{\mathrm{m}}(t)$ are available. Hence, the nominal state-feedback controller that makes the switched system behave like the reference model is given as $u^*(t) = k^{*T}_{\sigma(t)}x(t) + l^*_{\sigma(t)}r(t)$ for $t \geq t_0$, where the nominal parameters $k^*_p \in \mathbb{R}^n$ and $l^*_p \in \mathbb{R}$ exist under the assumption that the following matching condition holds [13], [16]:

$$A_p + b_p k^{*T}_p = A_{\mathrm{m}p}, \quad b_p l^*_p = b_{\mathrm{m}p} \tag{3}$$

However, since A_p and b_p are unknown, we cannot calculate k^*_p and l^*_p from (3). In light of this, the state-feedback controller is developed as:

$$u(t) = k^T_{\sigma(t)}(t)x(t) + l_{\sigma(t)}(t)r(t), \quad t \geq t_0 \tag{4}$$

where k_p and l_p are the estimates of k^*_p and l^*_p respectively. In addition, we define the tracking error as $e(t) = x(t) - x_{\mathrm{m}}(t)$. Substituting (4) into (1), and subtracting (2), the dynamics of the tracking error are given as follows:

$$\dot{e}(t) = A_{\mathrm{m}\sigma(t)}e(t) + b_{\sigma(t)}\left(\tilde{k}^T_{\sigma(t)}(t)x(t) + \tilde{l}_{\sigma(t)}(t)r(t)\right) + d(t) \tag{5}$$

where $\tilde{k}_p = k_p - k^*_p$, $\tilde{l}_p = l_p - l^*_p$ are the parameter estimation errors.

The following definitions will be used in this work:

Definition 1: (**Dwell-time switching**) Switching laws with the switching sequence $\mathcal{S} := \{t_1, t_2, \dots\}$ are said to be *dwell-time admissible* if there exists a number $\tau_{\mathrm{d}} > 0$ such that $t_{i+1} - t_i \geq \tau_{\mathrm{d}}$ holds for all $i \in \mathbb{N}^+$. Any positive number τ_{d}, for which these constraints hold for all $i \in \mathbb{N}^+$, is called *dwell time*, and the set of dwell-time admissible switched laws is denoted by $\mathcal{D}(\tau_{\mathrm{d}})$.

Definition 2: (**Global uniform ultimate boundedness**) The uncertain switched system (1) under switching law $\sigma(\cdot)$ is *globally uniformly ultimately bounded* (GUUB) if there exists a convex and compact set $\mathcal{S}$ such that for any initial condition $x(0) = x_0$, there exists a finite $T(x_0)$ such that $x(t)$ is attracted into S and $x(t) \in \mathcal{S}$ for all $t \geq T(x_0)$.

Definition 3: (**Ultimate bound**) A signal $\phi(\cdot)$ is said to be globally uniformly ultimately bounded with *ultimate bound* b if there exists a positive constant b, and for any $a \geq 0$, there exists $T = T(a,b)$, where b and T are independent of t_0, such that $\|\phi(t_0)\| \leq a \Rightarrow \|\phi(t)\| \leq b, \quad \forall t \geq t_0 + T$.

Thus, the problem addressed in this paper is presented as follows:

Problem 1: Develop a switching law $\sigma(\cdot)$ based on dwell time and an adaptive law for k_p and l_p such that the switched system (1) with the state-feedback controller (4) is stable, and the tracking error is globally uniformly ultimately bounded.

III. MAIN RESULTS

The following lemma is given to derive the main results in this section.

Lemma 1: [20] Let $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$, and Φ, Ψ be appropriately dimensioned matrices, then for any positive constant ε and for any appropriately dimensioned matrix $X(t)$ satisfying $X^T(t)X(t) \leq I$, it holds that

$$2y^T\Phi X\Psi z \leq \varepsilon y^T\Phi\Phi^T y + \varepsilon^{-1}z^T\Psi^T\Psi z.$$

Let K be a given integer. Let us define a time sequence $\{t_{i,0},\dots,t_{i,K}\}$ with $t_{i,k+1} - t_{i,k} = h$, $k = 0,\dots,\ K-1$. We define that $t_{i,0} = t_i$ and $t_{i,K} - t_{i,0} = \tau_{\mathrm{d}}$, as shown in Fig. 1. Suppose

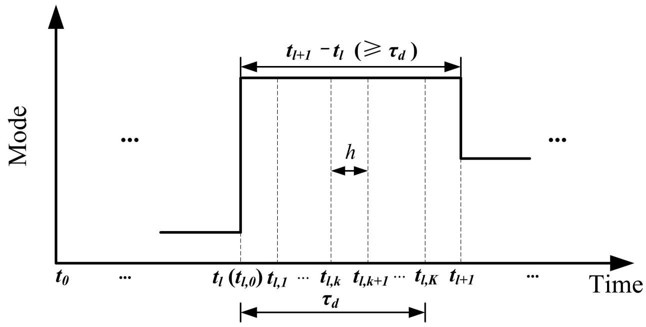

Fig. 1. The time sequence between two consecutive switching instants

that there exists a family of matrices $P_{p,k} > 0$, $p \in \mathcal{M}$, $k = 0,\dots,K$, a family of positive constants κ_p, $p \in \mathcal{M}$, and a positive constant h such that the following conditions hold:

$$\Delta P^p_{k+1,k}/h + \mathbf{He}\{P_{p,\mathscr{K}}A_{\mathrm{m}p}\} + (1+\kappa_p)P_{p,\mathscr{K}} < 0 \tag{6a}$$
$$\text{for } \mathscr{K} = k, k+1; k = 0,\dots,K-1$$
$$\mathbf{He}\{P_{p,K}A_{\mathrm{m}p}\} + (1+\kappa_p)P_{p,K} < 0 \tag{6b}$$
$$P_{p,K} - P_{q,0} \geq 0 \tag{6c}$$

where $\Delta P^p_{k+1,k} = P_{p,k+1} - P_{p,k}$, for $q \neq p \in \mathcal{M}$. Then, the adaptive laws based on parameter projections and leakage approach will be developed based on the family of the matrices $P_{p,k} > 0$, $p \in \mathcal{M}$, $k = 0,\dots,K$, and a switching law is proposed based on the following dwell time:

$$\tau_{\mathrm{d}} = K \cdot h. \tag{7}$$

Remark 1: The selection of K is dependent on rule proposed in [21]: as K grows, smaller (less conservative) h can be found by solving the LMIs (6). *In addition, there exists an integer K^* such that no less conservative h can be obtained by choosing a sufficiently large $K \geq K^*$.*

A. Adaptive control via projection laws

Before introducing the adaptive law, the following assumptions are made:

Assumption 1: The sign of l_p^*, $\forall p \in \mathcal{M}$, is known;

Assumption 2: Upper and lower bounds of k_p^* and l_p^* are known, i.e., $k_p^* \in [\underline{k}_p,\ \bar{k}_p]$ and $l_p^* \in [\underline{l}_p,\ \bar{l}_p]$, $\forall p \in \mathcal{M}$.

Remark 2: Assumption 1 and 2 are widely used in adaptive control problem based on parameter projections [16], [17] to ensure the boundedness of signals in closed-loop systems [13].

The adaptive law with the following projection laws is proposed

$$
\begin{aligned}
\dot{k}_p(t) &= -\operatorname{sgn}[l_p^*]\Gamma_p x(t) e^T(t) P_p(t) b_{\mathrm{mp}} + f_{k,p}(t) \\
\dot{l}_p(t) &= -\operatorname{sgn}[l_p^*]\gamma_p r(t) e^T(t) P_p(t) b_{\mathrm{mp}} + f_{l,p}(t)
\end{aligned} \tag{8}
$$

where $\Gamma_p \in \mathbb{R}^{n\times n}$ and $\gamma_p \in \mathbb{R}$ are given positive adaptive gains for all $p \in \mathcal{M}$ and the time-varying matrix $P_p(t)$ is defined as:

$$
P_p(t) = \begin{cases} P_{p,k} + \Delta P^p_{k+1,k} \cdot \rho(t), & \text{for } t_{i,k} \le t < t_{i,k+1} \\ P_{p,K}, & \text{for } t_{i,K} \le t < t_{i+1} \end{cases} \tag{9}
$$

where $\rho(t) = (t - t_{i,k+1})/h$. The functions $f_{k,p}(\cdot)$ and $f_{l,p}(\cdot)$ are the projection laws, which are used to prevent *parameter drift* of the parameter estimates [22]. Next, the definitions of $f_{k,p}$ and $f_{l,p}$ are given [23]: Let $k_p = [k_{p1}, \ldots, k_{pn}]$; $f_{k,p} = [f_{k1,p}, \ldots, f_{kn,p}]$; $\phi_{k,p} = -\operatorname{sgn}[l_p^*]\Gamma_p x e^T P_p b_{\mathrm{m}} = [\phi_{k1,p}, \ldots, \phi_{kn,p}]$, $\phi_{l,p} = -\operatorname{sgn}[l_p^*]\gamma_p r e^T P_p b_{\mathrm{mp}}$. Then, we have the projection terms as follows, for $s \in \{1, \cdots, n\}$, $t \ge t_0$

$$
\begin{aligned}
f_{ks,p}(t) &= \begin{cases} -\phi_{ks,p}(t) & \text{if } k_{ps}(t) \le \underline{k}_{ps}\ \&\ \phi_{ks,p}(t) \le 0, \\ & \text{or if } k_{ps}(t) \ge \bar{k}_{ps}\ \&\ \phi_{ks,p}(t) \ge 0 \\ 0 & \text{otherwise} \end{cases} \\
f_{l,p}(t) &= \begin{cases} -\phi_{l,p}(t) & \text{if } l_p(t) \le \underline{l}_p\ \&\ \phi_{l,p}(t) \le 0, \\ & \text{or if } l_p(t) \ge \bar{k}_p\ \&\ \phi_{l,p}(t) \ge 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{10}
$$

The sequence of switch-in instants of subsystem p is represented by $\{t_{p_1},\ t_{p_2},\ t_{p_3}, \ldots\}$, and the sequence of its switch-out instants is represented by $\{t_{p_1+1},\ t_{p_2+1},\ t_{p_3+1}, \ldots\}$. Note that the proposed adaptive law (8) is to be implemented as follows: at a switch-in instant of subsystem p the initial conditions of (8) are taken from the estimates available at the previous switch-out instant of the same subsystem, i.e., $k_p(t_{p_l}) = k_p(t_{p_{(l-1)}+1})$, and $l_p(t_{p_l}) = l_p(t_{p_{(l-1)}+1})$ for any $l \in \mathbb{N}^+$. Therefore, $k_p(t)$ and $l_p(t)$ update continuously. The adaptive control scheme for switched systems is illustrated in Fig. 2. Now, we are ready to introduce the following stability results.

Theorem 1: With any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$ and the adaptive law (8)–(10), the uncertain switched system (1) with state-feedback controller (4) is GUUB. In addition, the ultimate bound of the tracking error is given as

$$
\mathscr{B}_{\mathrm{proj}} = \sqrt{\frac{\max_{p\in\mathcal{M}}\left\{\lambda_{\max}\left(P_p(t)\right)\right\}}{\min|_{p\in\mathcal{M}}\left\{\kappa_p\right\}\min_{p\in\mathcal{M}}\left\{\lambda_{\min}\left(P_p(t)\right)\right\}}}\|\bar{d}\|. \tag{11}
$$

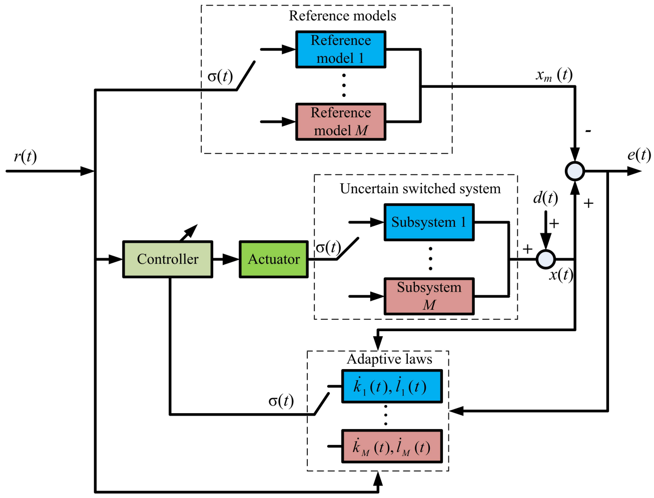

Fig. 2. The adaptive control scheme for switched linear systems

Proof: Consider the following Lyapunov function:

$$
\begin{aligned}
V(t) = {} & e^T(t) P_{\sigma(t)}(t) e(t) + \sum_{p=1}^{M} \frac{1}{|l_p^*|}\left(\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t)\right) \\
& + \sum_{p=1}^{M} \frac{1}{|l_p^*|}\left(\tilde{l}_p^2(t)\gamma_p^{-1}\right).
\end{aligned} \tag{12}
$$

Without loss of generality, we assume that subsystem p is active for $t \in [t_i,\ t_{i+})$, $i \in \mathbb{N}^+$. Therefore, using (8) and (9), the derivative of $V(t)$ with respect to time is, for $t \in [t_i,\ t_{i+1})$

$$
\begin{aligned}
\dot{V}(t) = {} & e^T(t) Q_p(t) e(t) + d^T(t) P_p(t) e(t) + e^T P_p(t) d(t) \\
& + \frac{1}{|l_p^*|}\tilde{k}_p^T \Gamma_p^{-1} f_{k,p}(t) + \frac{1}{|l_p^*|}\tilde{l}_p^T \gamma_p^{-1} f_{l,p}(t)
\end{aligned} \tag{13}
$$

with $Q_p(t) = A_{\mathrm{mp}}^T P_p(t) + \dot{P}_p(t) + P_p(t) A_{\mathrm{mp}}$. According to (10), we have $\tilde{k}_p^T \Gamma_p^{-1} f_{k,p} \le 0$, and $\tilde{l}_p \gamma_{1p}^{-1} f_{l,p} \le 0$. Since $P_p(\cdot)$ is a positive definite matrix, there exists a non-singular matrix $H_p(\cdot)$ such that $P_p(\cdot) = H_p(\cdot) H_p^T(\cdot)$. Then, substituting $P_p(\cdot) = H_p(\cdot) H_p^T(\cdot)$ into (13), according to Lemma 1, it follows that

$$
\begin{aligned}
\dot{V}(t) &\le e^T(t) Q_p(t) e(t) + e^T(t) P_p(t) e(t) + d^T(t) P_p(t) d(t) \\
&\le e^T(t) \Xi_p(t) e(t) + d^T(t) P_p(t) d(t)
\end{aligned} \tag{14}
$$

where $\Xi_p(t) = Q_p(t) + P_p(t)$. To analyze the properties of $V(t)$ for $t \in [t_i,\ t_{i+1})$, first we consider a sub-interval, i.e., $t \in [t_{i,k},\ t_{i,k+1})$, $k = 0, \ldots, K-1$. Note that

$$
\begin{aligned}
\Xi_p(t) = {} & \mathbf{He}\left\{P_p(t) A_{\mathrm{mp}}\right\} + \Delta P^p_{k+1,k}/h \\
& + \left(\eta_1 P_{p,k} + \eta_2 P_{p,k+1}\right) \\
= {} & \eta_1 \left[\Delta P^p_{k+1,k}/h + \mathbf{He}\left\{P_{p,k} A_{\mathrm{mp}}\right\} + P_{p,k}\right] \\
& + \eta_2 \left[\Delta P^p_{k+1,k}/h + \mathbf{He}\left\{P_{p,k+1} A_{\mathrm{mp}}\right\} + P_{p,k+1}\right]
\end{aligned} \tag{15}
$$

where $\eta_1 = 1 - \left(t - t_{i,k}\right)/h$, and $\eta_2 = 1 - \eta_1$. According to (6a)–(6b), it follows that

$$
\Xi_p(t) + \kappa_p P_p(t) < 0, \quad t \in [t_{i,k},\ t_{i,k+1}). \tag{16}
$$

Then, let us consider $t \in [t_{i,K}, t_{i+1})$ for the case that $t_{i+1} - t_i > \tau_{\rm d}$. We have $P_p(t) = P_{p,K}$ according to (9), which indicates by (6c) that

$$\Xi_p(t) + \kappa_p P_{p,K} < 0, \quad t \in [t_{i,K},\, t_{i+1}). \tag{17}$$

Therefore, it follows from (16)–(17) that $\Xi_p(t) < -\kappa_p P_p(t)$ for $t \in [t_i,\, t_{i+1})$. In light of this, according to (14), we have

$$\dot{V}(t) \leq -\kappa_p e^T P_p(t) e(t) + d^T(t) P_p(t) d(t),\ t \in [t_i,\, t_{i+1}) \tag{18}$$

Since the signals $e(\cdot)$, $\tilde{k}_{\sigma(\cdot)}(\cdot)$, and $\tilde{l}_{\sigma(\cdot)}(\cdot)$ are continuous according to (5) and (8), we have, at switching instant t_{i+1},

$$\begin{aligned}
&V_{\sigma(t_{i+1})}(t_{i+1}) - V_{\sigma(t_{i+1}^-)}(t_{i+1}^-)\\
&= e^T(t_{i+1}) P_{\sigma(t_{i+1})}(t_{i+1}) e(t_{i+1}) - e^T(t_{i+1}^-) P_{\sigma(t_{i+1}^-)}(t_{i+1}^-) e(t_{i+1}^-)\\
&= e^T(t_{i+1}) (P_{\sigma(t_{i+1})} - P_{\sigma(t_{i+1}^-)}) e(t_{i+1})\\
&= e^T(t_{i+1}) \left(P_{q,0} - P_{p,K}\right) e(t_{i+1})
\end{aligned} \tag{19}$$

which indicates that $V(\cdot)$ is non-increasing at switching instant t_{i+1} considering $P_{p,0} - P_{q,K} \leq 0$ for $p, q \in \mathcal{M}$. Therefore, according to (18)-(19), it can be shown that there exists a ball centered at the origin with the following radius:

$$\mathscr{B}_{\rm proj} = \sqrt{\frac{\max_{p\in\mathcal{M}} \left\{\lambda_{\max}\left(P_p(t)\right)\right\}}{\min p \in \mathcal{M} \left\{\kappa_p\right\} \min_{p\in\mathcal{M}} \left\{\lambda_{\min}\left(P_p(t)\right)\right\}}} \|\bar{d}\|$$

such that when $\|e(t)\| \geq \mathscr{B}_{\rm proj}$, we have $\dot{V}(t) < 0$. Furthermore, since the parameter estimates are bounded due to the projection laws (10), $V(\cdot)$ is GUUB, and the tracking error $e(\cdot)$ is attracted into the ball centered in the origin with radius $\mathscr{B}_{\rm proj}$. This completes the proof. ■

B. Adaptive control via leakage approach

Now, the leakage approach in [22] is extended to switched linear systems to prevent parameter drift, which does not require *Assumption 2*. The resulting adaptive law is given in the following:

$$\dot{k}_p(t) = -\operatorname{sgn}[l_p^*] \Gamma_p x(t) e^T(t) P_p(t) b_{\rm mp} - \delta_p^k \Gamma_p \hat{k}_p(t) \tag{20a}$$

$$\dot{l}_p(t) = -\operatorname{sgn}[l_p^*] \gamma_p r(t) e^T(t) P_p(t) b_{\rm mp} - \delta_p^l \gamma_p \hat{l}_p(t) \tag{20b}$$

$$\dot{k}_q(t) = -\delta_q^k \Gamma_q \hat{k}_p(t) \tag{20c}$$

$$\dot{l}_q(t) = -\delta_q^l \gamma_q \hat{l}_p(t) \tag{20d}$$

for $q = 1, \ldots, p-1, p+1, \ldots, M$, where $\hat{k}_p(t) = k_p(t) - k_p(t_0)$, $\hat{l}_p(t) = l_p(t) - l_p(t_0)$, $P_p(t)$ is defined in (9), Γ_p, γ_p are positive adaptive gains, and δ_p^k, δ_p^l are positive leakage rates satisfying

$$\delta_p^k \geq \max_{p\in\mathcal{M}} \left\{\kappa_p\right\} \lambda_{\max}\left(\Gamma_p^{-1}\right), \quad \delta_p^l \geq \max_{p\in\mathcal{M}} \left\{\kappa_p\right\} \gamma_p^{-1}. \tag{21}$$

Remark 3: Different from the adaptive law (8)*, the parameter estimates of all subsystems are updated during the whole time horizon. To be more precise, the updating rules* (20a)—(20b) *of the parameter estimates are adopted when the subsystem is active. The updating rules switch to* (20c)—(20d) *when the subsystem is inactive. In addition, different from the leakage approach in [13] for adaptive control of non-switched systems, when a subsystem is inactive, the adaptive rule will bring the parameter estimates to initial conditions of* (20) *to guarantee the stability of the switched systems.*

The following stability result is given based on the adaptive law (20)–(21), and the switching signals with dwell time $\tau_{\rm d}$ defined in (7).

Theorem 2: With any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_{\rm d})$ and the adaptive law (20)–(21), the uncertain switched system (1) with state-feedback controller (4) is GUUB. In addition, the ultimate bound of the tracking error is given as

$$\mathscr{B}_{\rm Leak} = \sqrt{\frac{\Sigma + \max_{p\in\mathcal{M}} \left\{\lambda_{\max}\left(P_p(t)\right)\right\} \|\bar{d}\|^2}{\min_{p\in\mathcal{M}} \left\{\kappa_p\right\} \min_{p\in\mathcal{M}} \left\{\lambda_{\min}\left(P_p(t)\right)\right\}}} \tag{22}$$

with $\Sigma = \sum_{p=1}^{M} \frac{1}{|l_p^*|} \left(\delta_p^k \|k_p^* - k_p(t_0)\|^2 + \delta_p^l \left(l_p^* - l_p(t_0)\right)^2\right)$.

Proof: Consider the same Lyapunov function as in (12). Using (5), (6), and (20), the derivative of $V(t)$ w.r.t. time is, for $t \in [t_i,\, t_{i+1})$,

$$\begin{aligned}
\dot{V}(t) =\ & e^T(t) Q_{\sigma(t_i)}(t) e(t) + d^T(t) P_{\sigma(t_i)}(t) e(t)\\
&+ e^T(t) P_{\sigma(t_i)}(t) d(t) - 2\sum_{p=1}^{M} \frac{1}{|l_p^*|} \delta_p^k \tilde{k}_p^T(t) \hat{k}_p(t)\\
&- 2\sum_{p=1}^{M} \frac{1}{|l_p^*|} \delta_p^l \tilde{l}_p(t) \hat{l}_p(t)\\
\leq\ & e^T(t) \Xi_{\sigma(t_i)}(t) e(t) + d^T(t) P_{\sigma(t_i)}(t) d(t)\\
&- 2\sum_{p=1}^{M} \frac{1}{|l_p^*|} \delta_p^k \tilde{k}_p^T(t) (\tilde{k}_p(t) + k_p^* - k_p(t_0))\\
&- 2\sum_{p=1}^{M} \frac{1}{|l_p^*|} \delta_p^l \tilde{l}_p(t) (\tilde{l}_p(t) + l_p^* - l_p(t_0))\\
\leq\ & -\kappa_{\sigma(t_i)} e^T(t) P_{\sigma(t_i)}(t) e(t) + d^T(t) P_{\sigma(t_i)}(t) d(t)\\
&- \sum_{p=1}^{M} \frac{1}{|l_p^*|} \delta_p^k \left(\|\tilde{k}_p(t)\|^2 - \|k_p^* - k_p(t_0)\|^2\right)\\
&- \sum_{p=1}^{M} \frac{1}{|l_p^*|} \delta_p^l \left(\tilde{l}_p^2(t) - \left(l_p^* - l_p(t_0)\right)^2\right).
\end{aligned} \tag{23}$$

The first inequality in (23) holds by following the same steps as for (15)–(17), and the second inequality holds due to $-2\tilde{k}_p^T \tilde{k}_p - 2\tilde{k}_p^T \left(k_p^* - k_p(t_0)\right) \leq -\|\tilde{k}_p\|^2 + \|k_p^* - k_p(t_0)\|^2$, and $-2\tilde{l}^2 - 2\tilde{l}_p \left(l_p^* - l_p(t_0)\right) \leq -\tilde{l}_p^2 + \left(l_p^* - l_p(t_0)\right)^2$, $\forall p \in \mathcal{M}$. Hence, according to (12), the following holds:

$$\begin{aligned}
\dot{V}(t) \leq\ & -\kappa_{\sigma(t_i)} V(t) + d^T(t) P_{\sigma(t_i)}(t) d(t)\\
&+ \kappa_{\sigma(t_i)} \sum_{p=1}^{M} \frac{1}{|l_p^*|} \left(\tilde{k}_p^T(t) \Gamma_p^{-1} \tilde{k}_p(t) + \tilde{l}_p^2(t) \gamma_p^{-1}\right)\\
&- \sum_{p=1}^{M} \frac{1}{|l_p^*|} \left(\delta_p^k \|\tilde{k}_p(t)\|^2 - \delta_p^k \|k_p^* - k_p(t_0)\|^2\right)\\
&- \sum_{p=1}^{M} \frac{1}{|l_p^*|} \left(\delta_p^l \tilde{l}_p^2(t) - \delta_p^l \left(l_p^* - l_p(t_0)\right)^2\right)
\end{aligned} \tag{24}$$

$$\begin{aligned}
\leq\ & -\kappa_{\sigma(t_i)} V(t) + d^T(t) P_{\sigma(t_i)}(t) d(t)\\
&+ \sum_{p=1}^{M} \frac{1}{|l_p^*|} \left[\kappa_{\sigma(t_i)} \lambda_{\max}\left(\Gamma_p^{-1}\right) - \delta_p^k\right] \|\tilde{k}_p(t)\|^2
\end{aligned}$$

$$+\sum_{p=1}^{M}\frac{1}{|l_p^*|}\left[\kappa_{\sigma(t_i)}\gamma_p^{-1}-\delta_p^l\right]\tilde{l}_p^2(t)+\Sigma$$

where $\Sigma = \sum_{p=1}^{M}\frac{1}{|l_p^*|}\left(\delta_p^k\|k_p^*-k_p(t_0)\|^2+\delta_p^l\left(l_p^*-l_p(t_0)\right)^2\right)$. According to (21), (24) is recast into

$$\dot{V}(t)\leq -\kappa_{\sigma(t_i)}V(t)+d^T(t)P_{\sigma(t_i)}(t)d(t)+\Sigma. \tag{25}$$

Due to the same reason as for (19), $V(\cdot)$ is non-increasing at the switching instants. In light of this, the Lyapunov function is attracted inside a ball centered at the origin with radius

$$\mathscr{B}_{\mathrm{V}}=\frac{1}{\min_{p\in\mathcal{M}}\{\kappa_p\}}\left(\Sigma+\max_{p\in\mathcal{M}}\{\lambda_{\max}(P_p(t))\}\|\bar{d}\|^2\right).$$

This implies that the switched system (1) with state-feedback controller (4) is GUUB. Considering that $\|e(t)\|^2\leq V(t)/\min_{p\in\mathcal{M}}\{\lambda_{\min}(P_p(t))\}$, it can be shown that the tracking error is attracted inside a ball centered at the origin with the following radius:

$$\mathscr{B}_{\mathrm{Leak}}=\sqrt{\frac{\Sigma+\max_{p\in\mathcal{M}}\{\lambda_{\max}(P_p(t))\}\|\bar{d}\|^2}{\min_{p\in\mathcal{M}}\{\kappa_p\}\min_{p\in\mathcal{M}}\{\lambda_{\min}(P_p(t))\}}}.$$

This completes the proof. ■

Remark 4: Note that the ultimate bound of the tracking error depends on the initial parameter estimate errors: when the initial estimates are far away from the nominal parameters, a large tracking error is expected, and vice versa. In light of this, comparing with (11) *and* (22)*, Assumption 2 is removed for the adaptive law with leakage method* (20) *at the expense of possibly impairing the steady-state performance of the tracking error. This will be illustrated in the simulation of adaptive control of leakage approach in Section IV.*

IV. Numerical Example

In this section, a numerical example is used to demonstrate the effectiveness of the proposed robust adaptive tracking control scheme. Consider the following uncertain switched linear system:

$$A_1=\begin{bmatrix}-0.6 & 3.0 & 3.3\\ 1.0 & -0.1 & 2.1\\ -0.2 & 2.3 & 1.5\end{bmatrix},\quad B_1=\begin{bmatrix}-2.3\\ 1.8\\ 0.4\end{bmatrix}$$
$$A_2=\begin{bmatrix}2.6 & 3.6 & 1.2\\ 1.8 & -0.5 & 3.6\\ 1.2 & 1.8 & 2.0\end{bmatrix},\quad B_2=\begin{bmatrix}0.0\\ -0.4\\ -1.5\end{bmatrix}$$
$$A_3=\begin{bmatrix}3.5 & 2.4 & 1.3\\ 2.4 & 2.2 & 2.3\\ 3.9 & 2.6 & -0.9\end{bmatrix},\quad B_3=\begin{bmatrix}0.2\\ -1.3\\ -0.8\end{bmatrix}$$

and the following reference switched model:

$$A_{\mathrm{m}1}=\begin{bmatrix}7.9 & 26.1 & 32.8\\ -5.7 & -18.1 & -21.0\\ -1.7 & -1.7 & -3.6\end{bmatrix},\quad B_{\mathrm{m}1}=\begin{bmatrix}-2.3\\ 1.8\\ 0.4\end{bmatrix}$$
$$A_{\mathrm{m}2}=\begin{bmatrix}25.9 & 23.1 & 22.7\\ -7.5 & -8.3 & -5.0\\ -33.8 & -27.4 & -30.3\end{bmatrix},\quad B_{\mathrm{m}2}=\begin{bmatrix}0.0\\ -0.4\\ -1.5\end{bmatrix}$$
$$A_{\mathrm{m}3}=\begin{bmatrix}7.3 & 4.7 & 2.5\\ -22.5 & -12.7 & -5.5\\ -11.4 & -6.5 & -5.7\end{bmatrix},\quad B_{\mathrm{m}3}=\begin{bmatrix}0.2\\ -1.3\\ -0.8\end{bmatrix}$$

Let $\kappa_1=0.1$, $\kappa_2=0.15$, and $\kappa_3=0.12$, $K=1$, and $h=2$. Solving the LMIs (6) gives

$$P_{10}=\begin{bmatrix}0.02 & 0.03 & 0.03\\ 0.03 & 0.04 & 0.04\\ 0.03 & 0.04 & 0.05\end{bmatrix},P_{11}=\begin{bmatrix}0.15 & 0.17 & 0.19\\ 0.17 & 0.30 & 0.34\\ 0.19 & 0.34 & 0.44\end{bmatrix}$$
$$P_{20}=\begin{bmatrix}0.11 & 0.08 & 0.07\\ 0.08 & 0.07 & 0.05\\ 0.07 & 0.05 & 0.05\end{bmatrix},P_{21}=\begin{bmatrix}0.70 & 0.55 & 0.46\\ 0.55 & 0.49 & 0.36\\ 0.46 & 0.36 & 0.34\end{bmatrix}$$
$$P_{30}=\begin{bmatrix}0.07 & 0.03 & 0.01\\ 0.03 & 0.012 & 0.003\\ 0.009 & 0.003 & 0.006\end{bmatrix},P_{31}=\begin{bmatrix}0.60 & 0.31 & 0.17\\ 0.31 & 0.17 & 0.11\\ 0.17 & 0.11 & 0.10\end{bmatrix}$$

We select the switching interval $t_{i+1}-t_i=\tau_{\mathrm{d}}$ for $i=1,2,3$. Therefore, the time-varying positive matrices $P_p(t)$ for $p\in\{1,2,3\}$ are $P_p(t)=(t-\tau_{\mathrm{d}}\cdot\mathrm{floor}(t/\tau_{\mathrm{d}}))\cdot(P_{p,1}-P_{p,0})/\tau_{\mathrm{d}}+P_{p,0}$, where $\mathrm{floor}(t/\tau_{\mathrm{d}})$ rounds t/τ_{d} to the nearest integer less than or equal to t/τ_{d}. Before performing the adaptation process, we select a bounded disturbance $d(t)=[0.2\sin(10t)\ e^{-0.1t}\ 0.1\cos(5t)]^T$, the initial conditions $x_0=[0\ 0\ 0]^T$, $x_{\mathrm{m}0}=[3\ 1\ 0]^T$, and the adaptive gains $\Gamma_p=10I_{3\times3}$, $\gamma_p=10$, $\forall p\in\mathcal{M}$. The switching signal is designed with a dwell time $\tau_{\mathrm{d}}=2$ as shown in Fig. 3.

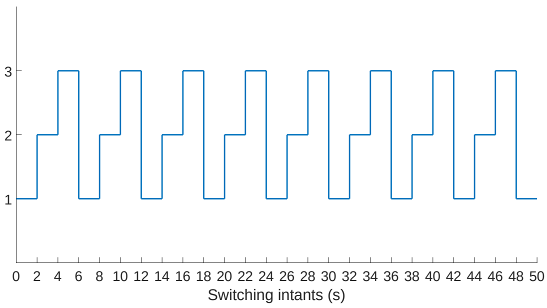

Fig. 3. The switching signal

A. Adaptive control with projection laws

We select the initial parameter estimates $k_p(0)=0.2k_p^*$, $l_p(0)=0.2l_p^*$, $\forall p\in\mathcal{M}$. Assume the parameter estimates reside in the following known bounds: $k_1(t)\in[1.2k_1^*\ 0.2k_1^*]$, $k_s(t)\in[0.2k_s^*\ 1.2k_s^*]$ with $s\in\{2,3\}$, and $l_p(t)\in\left[0.2l_p^*\ 1.2l_p^*\right]$ with $p\in\{1,2,3\}$. The resulting tracking error is given in Fig. 4, which shows that the tracking error is attracted inside a ball.

B. Adaptive control with leakage approach

The leakage rates $\delta_p^k=\delta_p^l=0.015$ are chosen to satisfy the conditions (21). To study the effect of the initial conditions of the parameter estimates on the steady-state performance of the tracking error, we select the two initial conditions of the parameter estimates $k_p(0)=0.8k_p^*$, $l_p(0)=0.8l_p^*$, and $k_p(0)=0.2k_p^*$, $l_p(0)=0.2l_p^*$, $\forall p\in\mathcal{M}$. The resulting tracking errors based on the two initial conditions of parameter estimates are given in Fig. 5–6, which show that the tracking errors are attracted inside a ball. By comparing Fig. 5 and Fig. 6, it can be observed that larger initial parameter estimation errors give rise to larger tracking errors. Moreover, it can be noticed from Fig. 4 and Fig. 6 that the ultimate bound of tracking error in Fig. 6 is bigger than that in Fig. 4, which indicates that the adaptive law with leakage may negatively impact the steady-state performance of the tracking error when improper initial conditions in (8)–(20) are selected.

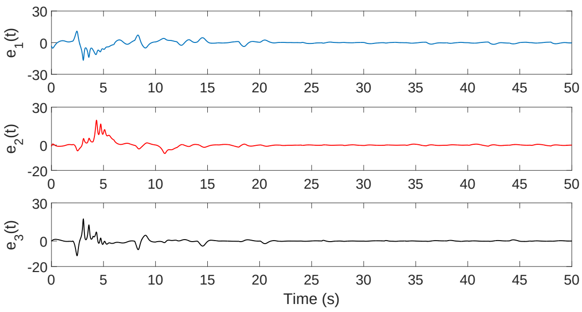

Fig. 4. The tracking error $e(t)$ via projection laws

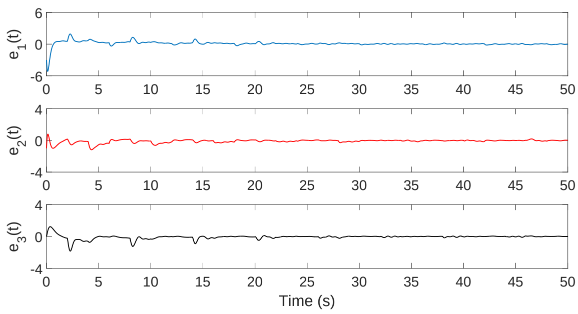

Fig. 5. The tracking error $e(t)$ with $k_p(0) = 0.8k_p^*$ and $l_p(0) = 0.8l_p^*$

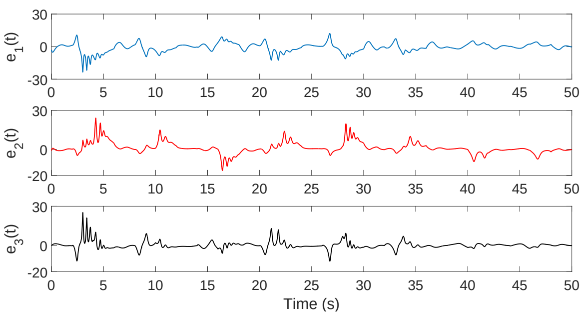

Fig. 6. The tracking error $e(t)$ with $k_p(0) = 0.2k_p^*$ and $l_p(0) = 0.2l_p^*$

V. Conclusions

Robust adaptive control problem of uncertain switched linear systems has been studied. As an extension of the results in [1], two control schemes have been introduced based on parameter projection and a leakage approach, respectively. With the robust adaptive control schemes, the closed-loop switched linear systems have been shown to be globally uniformly ultimately bounded. In addition, ultimate bounds of the tracking error of both cases have been given, which has indicated that the leakage approach might negatively impact the steady-state performance of the tracking error compared with parameter projection. Future work will focus on the improvement of the adaptive law with leakage approach that will not depend on the initial conditions.

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