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# A Novel Lyapunov Function for a Non-weighted $\mathcal{L}_2$ Gain of Asynchronously Switched Linear Systems

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**Abstract**

In this paper, a novel Lyapunov function is proposed to study switched linear systems with a switching delay between activation of system modes and activation of candidate controller modes. The novelty consists in continuity of the Lyapunov function at the switching instants and discontinuity when the system modes and controller modes are matched. This structure is exploited to construct a time-varying Lyapunov function that is non-increasing at time instants of discontinuity. Stability criteria based on the novel Lyapunov function are developed to guarantee global asymptotic stability in the noiseless case. Most importantly, when exogenous disturbances are considered, the proposed Lyapunov function can be used to guarantee a finite non-weighted $\mathcal{L}_2$ gain for asynchronously switched systems, for which Lyapunov functions proposed in literature are inconclusive. A numerical example illustrates the effectiveness of the proposed method.

*Key words:* Lyapunov method; asynchronously switched linear systems; non-weighted $\mathcal{L}_2$ gain; dwell time.

## 1 Introduction

Switched systems are a special class of hybrid systems that consists of subsystems with continuous dynamics, a.k.a. *modes*, and a rule to regulate the switching behavior between them, called *switching law*. Switched systems can be used to model a broad range of physical systems, such as networked control systems [1], flight control systems [2], and smart energy systems [3].

Being crucial notions to understand the behavior of switched linear systems, stability and stabilization have been attracting a lot of research efforts [4, 5, 6, 7]. Typically, the focus is on synchronously switched linear systems, an ideal case in which the controller is assumed to switch synchronously with the system mode. However, due to delay between a mode change and activation of its corresponding controller, or due to the time needed to detect switching of system mode, nonzero time intervals, called *unmatched intervals*, are present during which system modes and controller modes are mismatched. A typical example in engineering practice can be seen in the teleoperation, e.g. [8]. This special family of switched linear systems is called *asynchronously switched linear systems*. Most of the research on ideal switched linear systems has been carried out based on the famous Lyapunov function proposed by Branicky [4] that is discontinuous at the switching instants and continuous during the switching intervals between two consecutive switching instants. Two properties of the Lyapunov function have been exploited to develop switching strategies based on dwell time (DT) and average dwell time (ADT) [5, 9]: an exponential decreasing rate during the switching interval between two consecutive switching instants, and a bounded increment of the Lyapunov function at switching instants. For asynchronously switched linear systems, several studies have appeared on stability and stabilization problems [10, 11, 12, 13, 14, 15]. In particular, a seminal work on stability of asynchronously switched linear systems [10] introduces a new Lyapunov function for asynchronously switched systems based on the classical Lyapunov function for ideal switched systems. This new Lyapunov function uses the additional property that the Lyapunov function is allowed to increase during the unmatched interval.

Another fundamental topic, the $\mathcal{L}_2$ gain of switched lin-

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ear systems, has been extensively investigated [16, 17, 18]. A weighted $\mathcal{L}_2$ gain for ideal switched linear systems based on ADT switching was introduced initially in [16]. Subsequently, a non-weighted $\mathcal{L}_2$ gain for ideal switched linear systems was obtained in [19] via DT switching laws and in [20] via switching laws using persistent dwell time. However, to the best of the authors' knowledge, only a weighted $\mathcal{L}_2$ gain has been obtained for asynchronously switched linear systems [11, 21, 22], which is based on the Lyapunov function in [11] via ADT switching laws; considering the narrower class of switching laws based on DT does not help in achieving non-weighted $\mathcal{L}_2$ gain [1]. In view of this, an important question automatically arises: how to achieve a non-weighted $\mathcal{L}_2$ gain for asynchronously switched linear systems? In other words, what characteristics should the Lyapunov function have in order to cover the gap between ideal and asynchronously switched linear systems?

In this paper, a novel Lyapunov function is proposed to study asynchronously switched linear systems; this Lyapunov function is continuous at switching instants and discontinuous at the instant when the controller and the system mode is matched. This is in contrast with the well-known multiple Lyapunov functions proposed by Branicky [4], which are discontinuous at switching instants and continuous during the switching intervals. The major idea behind the novel Lyapunov function is the consistency with the switching mechanism of asynchronously switched systems, since the same controller is used during the matched interval of the previous subsystem and the unmatched interval of the current subsystem. The structure of the Lyapunov function is exploited to develop novel stability criteria that can be combined with the interpolation technique in [19, 24, 25] such that global asymptotic stability of asynchronously switched linear systems is guaranteed. The contribution of this paper is twofold: (i) a new Lyapunov function is proposed which is consistent with the controller design of asynchronously switched linear systems; (ii) a non-weighted $\mathcal{L}_2$ gain is guaranteed for the first time for asynchronously switched linear systems.

This paper is organized as follows: Section 2 introduces the problem formulation and some preliminaries. Section 3 gives a condition in the form of linear matrix inequalities (LMIs) to guarantee global asymptotic stability for asynchronously switched linear systems. Section 4 derives the LMI conditions for non-weighted $\mathcal{L}_2$ gain and $H_\infty$ control of asynchronously switched linear systems. A numerical example is adopted to illustrate the theoretical results in Section 5. The paper is concluded in Section 6.

*Notation*: The space of real numbers is denoted by R. Matrix transpose is represented by the superscript " $'$ ". The notation $\mathcal{M} = \{1, 2, \ldots, M\}$ represents the set of subsystems indices and $M$ is the number of subsystems. Moreover, $\mathcal{L}_2^r$ denotes the set of square integrable functions with values on $\mathrm{R}^r$ defined on $[0, \infty)$. The set of non-negative integers and positive integers are denoted by N and $\mathrm{N}_+$ , respectively. A positive definite matrix $P$ is denoted by $P > 0$. We define $\Delta P^p_{l+1,l} = P_{p,l+1} - P_{p,l}$, $p \in \mathcal{M}$, $l \in \mathrm{N}$. The identity matrix of compatible dimensions is denoted by $I$. We use $*$ as an ellipsis for the terms that are induced by symmetry.

[1] This can be verified by setting $N_0 \equiv 1$ to the derivation in [21] according to the definition of DT [23].

## 2 Problem formulation and preliminaries

Consider the following switched linear system:

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u_{\sigma(t)}(t) + E_{\sigma(t)}w(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u_{\sigma(t)}(t) + F_{\sigma(t)}w(t) \end{aligned} \tag{1}$$

where $t \geq 0$, $x \in \mathrm{R}^n$ is the state vector, $u \in \mathrm{R}^m$ is the input, $y \in \mathrm{R}^g$ is the output, $w \in \mathrm{R}^r$ is an exogenous disturbance, and $\sigma(\cdot)$ is a piecewise function, taking values from the set $\mathcal{M}$. In this paper, a mode-dependent state-feedback controller is adopted, *i.e.*, $u_{\sigma(t)}(t) = G_{\sigma(t)}x(t)$. Define the switching instant sequence $\mathcal{S}$ , $\{t_i, i \in \mathrm{N}\}$. Let $\Delta\tau_i$ be the delay before switching to a new subsystem and the activation of the corresponding controller after the switching instant $t_i$. Then, the switched linear system (1) becomes an asynchronously switched linear system as follows:

$$\begin{aligned} \dot{x}(t) &= (A_{\sigma(t)} + B_{\sigma(t)}G_{\sigma(t-\Delta\tau_i)})x(t) + E_{\sigma(t)}w(t) \\ &= \begin{cases} \overline{A}_{p,q}x(t) + E_pw(t), & t \in [t_i,\, t_i + \Delta\tau_i) \\ \overline{A}_px(t) + E_pw(t), & t \in [t_i + \Delta\tau_i,\, t_{i+1}) \end{cases} \\ y(t) &= (C_{\sigma(t)} + D_{\sigma(t)}G_{\sigma(t-\Delta\tau_i)})x(t) + F_{\sigma(t)}w(t) \\ &= \begin{cases} \overline{C}_{p,q}x(t) + F_pw(t), & t \in [t_i,\, t_i + \Delta\tau_i) \\ \overline{C}_px(t) + F_pw(t), & t \in [t_i + \Delta\tau_i,\, t_{i+1}) \end{cases} \end{aligned} \tag{2}$$

where $\overline{A}_p$ is Hurwitz matrix, and $\overline{A}_{p,q}$, $p \neq q \in \mathcal{M}$, may be an unstable matrix. To keep the notation concise, we denote the unmatched interval $[t_i,\, t_i + \Delta\tau_i)$ by $\mathcal{T}_{\uparrow}(t_i, t_{i+1})$, and the matched interval $[t_i + \Delta\tau_i,\, t_{i+1})$ by $\mathcal{T}_{\downarrow}(t_i, t_{i+1})$.

The following definitions are provided for later analysis.

**Definition 1 (Class $\mathcal{K}$, $\mathcal{KL}$, $\mathcal{K}_\infty$)** *[23] We say that a function $\alpha : [0,\infty) \to [0,\infty)$ is of class $\mathcal{K}$, and write $\alpha \in \mathcal{K}$, when $\alpha$ is continuous, strictly increasing, and $\alpha(0) = 0$. We say that a function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\mathcal{KL}$, and write $\beta \in \mathcal{KL}$ when $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$. We say that a function $\zeta$: $[0,\infty) \to [0,\infty)$ is of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, unbounded, and $\zeta(0) = 0$.*

**Definition 2 (Dwell time)** *[23] A switching signal is said to be admissible with dwell time if there exists a number $\tau_{\mathrm{d}} > 0$ such that the constraint $t_{i+1} - t_i \geq \tau_{\mathrm{d}}$ holds for all $i \in \mathrm{N}_+$. Any positive number $\tau_{\mathrm{d}}$ for which this constraint holds is called dwell time.*

**Definition 3 (Global asymptotic stability)** *[26] A switched system is said to be globally asymptotically stable if there exists a class $\mathcal{KL}$ function $\beta(\cdot)$ such that for all switching signals $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$ and for any initial condition $x(0)$ the following inequality is satisfied: $|x(t)| \leq \beta(|x(0)|, t), \ \forall t \geq 0$.*

**Definition 4 (Non-weighted $\mathcal{L}_2$ gain)** *[20] The switched system (2) is said to have a non-weighted $\mathcal{L}_2$ gain $\overline{\gamma} > 0$, if under zero initial conditions, the following inequality holds $\int_0^\infty y'(t)y(t)dt \leq \int_0^\infty \overline{\gamma}^2 w'(t)w(t)dt$ for all $t \geq 0$, and all $w(t) \in \mathcal{L}_2^r$.*

The following lemma will be used to analyze the $\mathcal{L}_2$ gain.

**Lemma 1** *[23] All admissible switching laws with dwell time satisfy the following inequality:*

$$N(t_{\mathrm{s}},\ t_{\mathrm{f}}) \leq 1 + \frac{t_{\mathrm{s}} - t_{\mathrm{f}}}{\tau_{\mathrm{d}}}, \quad \forall t_{\mathrm{s}} \geq t_{\mathrm{f}} \tag{3}$$

*where $N(t_{\mathrm{s}},\ t_{\mathrm{f}})$ denotes the number of switchings over the interval $[t_{\mathrm{s}}, t_{\mathrm{f}})$.*

Define the maximum switching delay $\Delta\tau := \max_{i\in\mathrm{N}} \Delta\tau_i$, which is assumed to be known. The set of admissible switching laws with dwell time is denoted by $\mathcal{D}(\tau_{\mathrm{d}})$. Then, the problem to be solved in this work is formulated as follows:

**Problem 1** *Design an admissible switching law with dwell time such that: (i) the system (2) with the knowledge of $\Delta\tau$ is globally asymptotically stable for $w(t) \equiv 0$; and (ii) the system (2) with the knowledge of $\Delta\tau$ has a non-weighted $\mathcal{L}_2$ gain. Furthermore, design a dwell time admissible switching law and a family of mode-dependent state-feedback controllers such that: (iii) the closed-loop system has a non-weighted $\mathcal{L}_2$ gain.*

## 3 Stability analysis

In this section, a novel Lyapunov function is introduced to study the asymptotic stability of (2) with $w(t) \equiv 0$. In addition, the LMIs derived from the resulting Lyapunov stability criterion are provided.

### *3.1 A novel Lyapunov function*

It is well known [5, 7, 23] that the Lyapunov function most widely used to study the stability of switched linear systems has the form,

$$V(t) = x'(t)P_{\sigma(t)}x(t),\ V(t_i) \leq \mu V(t_i^-),\ \mu \geq 1 \tag{4}$$

for $t \in [t_i,\ t_{i+1})$, where $V(t_i^-)$ represents the left-limit of $V(t)$ at $t = t_i$. This function is continuous between two consecutive switching instants and discontinuous at switching instants. For asynchronously switched linear systems, a revised version of (4) has been developed [10, 11, 12] as follows:

$$\begin{aligned} &V(t) = x'(t)P_{\sigma(t)}x(t),\ V(t_i) \leq \mu V(t_i^-),\ \mu \geq 1 \\ &\dot{V}(t) \leq \begin{cases} -\lambda_1 V(t),\ \lambda_1 > 0,\ \forall t \in \mathcal{T}_\uparrow(t_i,\ t_{i+1}) \\ \lambda_2 V(t), \quad \lambda_2 > 0,\ \forall t \in \mathcal{T}_\downarrow(t_i,\ t_{i+1}) \end{cases} \end{aligned} \tag{5}$$

which has the following property that is different with respect to (4): (5) might increase during the unmatched intervals and it decreases during the matched intervals, as shown in Fig. 1. However, the following asymmetry can be noted in state-of-the-art results for stability of asynchronously switched linear systems via (5) (c.f. Theorem 1 in [10]): During $t \in \mathcal{T}_\uparrow(t_i,\ t_{i+1})$ the Lyapunov function corresponding to a different subsystem rather than $x'(t)P_{\sigma(t_i)}x(t)$ should be used, i.e., $x'(t)P_{\sigma(t_i^-)}x(t)$. In view of this, to reflect the key feature behind unmatched and matched intervals, a new Lyapunov function is proposed for asynchronously switched linear systems:

$$V(t) = \begin{cases} x'(t)P_{\sigma(t_i^-)}(t)x(t),\ \forall t \in \mathcal{T}_\uparrow(t_i,\ t_{i+1}) \\ x'(t)P_{\sigma(t_i)}(t)x(t), \quad \forall t \in \mathcal{T}_\downarrow(t_i,\ t_{i+1}) \end{cases} \tag{6}$$

which is *continuous at the switching instants* and *discontinuous at the instants when the modes are matched*, as shown in Fig. 2.

**Remark 1** *The main difference between synchronously switched linear systems and asynchronously switched linear systems consists in the switching delay between activation of system modes and activation of candidate controller modes. This gives rise to the key feature of asynchronously switched linear systems: the same controller is connected to the previous system mode during the matched interval and to the current system mode during the unmatched interval. In view of this key feature, to solve the stabilization problem using a Lyapunov method, the same positive definite matrix should be adopted in these two intervals to construct the Lyapunov function. This implies that the Lyapunov function is continuous at the switching instants and discontinuous at the instants when the modes are matched, as shown in Fig. 2. Note that when the switching delay $\Delta\tau$ is zero, the proposed Lyapunov function (6) reduces to the classic Lyapunov function (4). In view of this, the Lyapunov function (4) can be regarded as a special case of (6).*

Moreover, a time-varying Lyapunov function based on (6) can now be constructed by revising the so-called in-

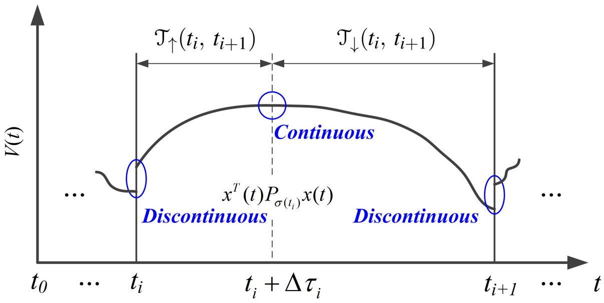


Fig. 1. The Lyapunov function (5).

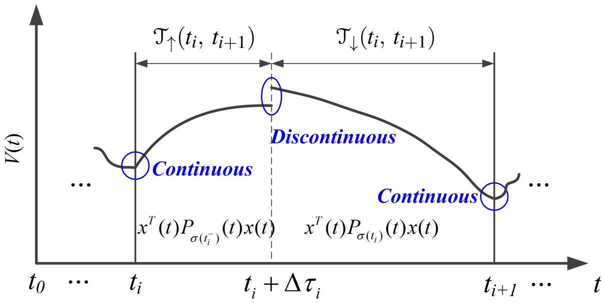


Fig. 2. The proposed Lyapunov function (6).

terpolation approach in [19, 24, 25] and by extending it to asynchronously switched linear systems. This gives rise to the new stability criteria as presented below.

### *3.2 Stability results*

**Theorem 1** *Let* $\overline{\lambda}$ *and* $\{\lambda_l\}_{l=0}^{L}$ *be given positive constants, where* $L$ *is a given integer. Suppose there exists a family of positive definite matrices* $P_{p,l}$, $p \in \mathcal{M}$, $l = 0, \ldots, L$, *and a number* $h > 0$ *such that*

$$\overline{A}'_{p,q}P_{q,L} + P_{q,L}\overline{A}_{p,q} - \overline{\lambda}P_{q,L} < 0 \tag{7a}$$

$$\Delta P^{p}_{l+1,l}/h + \overline{A}'_{p}P_{p,\ell} + P_{p,\ell}\overline{A}_{p} + \lambda_{p,\ell}P_{p,\ell} < 0 \tag{7b}$$

$$\lambda_{p,l}P_{p,l} - \lambda_{l+1}P_{p,l+1} \geq 0 \tag{7c}$$

$$\Delta P^{p}_{l+1,l} > 0 \tag{7d}$$

$$\overline{A}'_{p}P_{p,L} + P_{p,L}\overline{A}_{p} < 0 \tag{7e}$$

$$P_{q,L} - P_{p,0} \geq 0 \tag{7f}$$

$$\overline{\lambda}\Delta\tau - \sum_{l=1}^{L}\lambda_l h \leq 0 \tag{7g}$$

*for* $\ell = l,\ l+1$; $l = 0, \ldots, L-1$; $q \neq p \in \mathcal{M}$. *Then, the system* (2) *with* $w(t) \equiv 0$ *is globally asymptotically stable for any switching law* $\sigma(\cdot) \in \mathcal{D}(\tau_{\rm d})$ *with* $\tau_{\rm d} > Lh + \Delta\tau$.

**PROOF.** A well-known stability condition for switched systems proposed by Branicky in [4] is: the value of the Lyapunov function when the considered system entering into a mode is no larger than the value of the Lyapunov function at the previous entering instant. Below we will show in three steps that this stability condition can be guaranteed by the LMIs in (7a)–(7g): ($a$) we construct a quadratic Lyapunov function $V(t)$ in the fashion of (6) by interpolating a discrete set of positive definite matrices (obtained from (7a)–(7g)); ($b$) within a switching interval $[t_i,\, t_{i+1})$, we show that the increase of the Lyapunov function during unmatched intervals is compensated by a decrease part during matched intervals, i.e., $V(t_i) \geq V(t^-_{i+1})$; ($c$) we exploit continuity of the Lyapunov function at switching instants, i.e., $V(t^-_{i+1}) = V(t_{i+1})$.

($a$) Without loss of generality, we assume that subsystem $p$ is active for $t \in [t_i,\, t_{i+1})$, $i \in \mathbb{N}$, and subsystem $q$ is active for $t \in [t_{i-1},\, t_i)$. Let us define a time sequence $\{t_{i,0}, \ldots, t_{i,L}\}$, where $t_{i,l+1} - t_{i,l} = h$, $l = 0, \ldots,\ L-1$, $t_{i,0} = t_i + \Delta\tau$, and $t_{i,L} - t_i = \tau_{\rm d}$, as shown in Fig. 3.

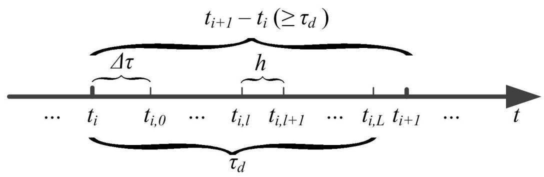


Fig. 3. The time sequence between two consecutive switching instants.

To study the properties of the Lyapunov function in (6), we partition the interval $[t_i,\, t_{i+1})$ into three sub-intervals: $[t_i, t_{i,0})$, $[t_{i,0}, t_{i,L})$, and $[t_{i,L}, t_{i+1})$. Using linear interpolation, we construct the following time-varying positive definite matrix $P_p(t)$, for $t \in [t_{i,0},\, t_{i+1})$

$$P_p(t) = \begin{cases} P_{p,l} + \rho(t)\Delta P^{p}_{l+1,l}, & t \in [t_{i,l},\, t_{i,l+1}) \\ P_{p,L}, & t \in [t_{i,L},\, t_{i+1}) \end{cases}$$

where $\rho(t) = (t - t_{i,l})/h$ with $l = 0, \ldots, L-1$. Then, the Lyapunov function (6) becomes, for $t \in [t_i,\, t_{i+1})$

$$V(t) = \begin{cases} x'(t)P_{q,L}x(t), & t \in [t_i,\, t_{i,0}) \\ x'(t)P_p(t)x(t), & t \in [t_{i,0},\, t_{i+1}) \end{cases} \tag{8}$$

which is continuous at switching instants, and discontinuous at the instant $t_{i,0}$ when the controller and subsystem are matched.

($b$) First, we consider the sub-interval $[t_i,\, t_{i,0})$. According to (7a), the derivative of $V(t)$ in (8) is $\dot{V}(t) = x'(t)(\overline{A}'_{p,q}P_{q,L} + P_{q,L}\overline{A}_{p,q})x(t) < \overline{\lambda}V_p(t)$. At the instant $t_{i,0}$, it follows from (7f) that $V(t^-_{i,0}) - V_p(t_{i,0}) = x'(t)(P_{q,L} - P_{p,0})x(t) \geq 0$, which implies that the Lyapunov function is non-increasing at time instants of discontinuity. Next, for the second sub-interval $[t_{i,0},\, t_{i,L})$, according to (7b)–(7c), we have $\dot{V}(t) = x'(t)\mathcal{P}(t)x(t)$, for $t \in [t_{i,l},\ t_{i,l+1})$, where

$$
\begin{aligned}
\mathcal{P}(t) &= \overline{A}_p' P(t) + P(t)\overline{A}_p + \Delta P^p_{l+1,l}/h \\
&= \eta_1 \left(\Delta P^p_{l+1,l}/h + P_{p,l}\overline{A}_p + \overline{A}_p' P_{p,l}\right) \\
&\quad + \eta_2 \left(\Delta P^p_{l+1,l}/h + P_{p,l+1}\overline{A}_p + \overline{A}_p' P_{p,l+1}\right) \\
&< -\left(\eta_1 \lambda_l P_{p,l} + \eta_2 \lambda_{l+1} P_{p,l+1}\right) \\
&< -\lambda_{l+1} P_{p,l+1} - \eta_1 \left(\lambda_l P_{p,l} - \lambda_{l+1} P_{p,l+1}\right) \\
&< -\lambda_{l+1} P_{p,l+1}
\end{aligned} \tag{9}
$$

with $\eta_1 \ , \ 1-(t-t_{p,l})/h$, $\eta_2 \ , \ 1-\eta_1$. Moreover, the inequality (7d) suggests that $P_{p,l+1} - P(t) \geq 0$, which combined with (9) leads to $\dot{V}(t) < -\lambda_{l+1} x'(t)P(t)x(t)$, for $t \in [t_{i,l},\ t_{i,l+1})$, $l = 0, \ldots, L-1$. This implies that (8) is decreasing exponentially with different rates $\{\lambda_l\}_{l=1}^{L}$ during different intervals $[t_{i,l},\ t_{i,l+1})$. Then, we have that $V_p(t_{i,L}) < \exp(\sum_{l=0}^{L} \lambda_l h)V_p(t_{i,0})$. Considering the properties of (8) during the first sub-interval $[t_i,\ t_{i,0})$ and at the time instant $t_{i,0}$ of discontinuity, we have $V(t_i) < \exp(-\overline{\lambda}\Delta\tau + \sum_{l=0}^{L} \lambda_l h)V_p(t_{i,L})$, which implies that $V(t_{i,L}) \leq V(t_i)$ by (7g). Finally, we consider the third sub-interval $t \in [t_{i,L},\ t_{i+1})$. Since the matrix $P_p(t)$ is constant during the third sub-interval, it holds that $\dot{V} = x'(A_p' P_{p,L} + P_{p,L}A_p)x < 0$ according to (7e), i.e., $V(t^-_{i+1}) < V(t_{i,L})$. Then, it means that the increase of the Lyapunov function (8) during the unmatched interval is compensated by the decrease during the matched interval, i.e., $V(t^-_{i+1}) < V(t_i)$.

($c$) Since the Lyapunov function (8) is continuous at the switching instants, i.e., $V(t^-_{i+1}) = V(t_{i+1})$, we have $V(t_{i+1}) \leq V(t_i)$, which satisfies that stability condition by Branicky [4]. This completes the proof. $\Xi$

The LMIs (7a)–(7g) might be difficult to solve due to the large number of design parameters $\lambda_l$, $l = 0, \ldots, L$ when a large integer $L$ is chosen. Therefore, a more convenient option is to use a common rate of decrease during matched intervals, i.e., $\lambda_0 = \cdots = \lambda_L = \beta$. This simplification gives rise to the following corollary, which involves only two design parameters and in return may give conservative results as compared with *Theorem 1.*

**Corollary 1** *Let $\alpha$ and $\beta$ be given positive constants. Suppose there exists a family of positive definite matrices $P_{p,l}$, $p \in \mathcal{M}$, $l = 0, \ldots, L$ with a given integer $L$, and a number $h > 0$, such that*

$$\overline{A}_{p,q}' P_{q,L} + P_{q,L}\overline{A}_{p,q} - \alpha P_{q,L} < 0 \tag{10a}$$

$$\Delta P^p_{l+1,l}/h + \overline{A}_p' P_{p,\ell} + P_{p,\ell}\overline{A}_p + \beta P_{p,\ell} < 0 \tag{10b}$$

$$\Delta P^p_{l+1,l} > 0 \tag{10c}$$

$$\overline{A}_p' P_{p,L} + P_{p,L}\overline{A}_p + \beta P_{p,L} < 0 \tag{10d}$$

$$P_{q,L} - P_{p,0} \geq 0 \tag{10e}$$

*for $\ell = l,\ l+1$; $l = 0, \ldots, L-1$; $q \neq p \in \mathcal{M}$. Then, the system (2) with $w(t) \equiv 0$ is globally asymptotically stable for any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$ with $\tau_{\mathrm{d}} > (\alpha+\beta)\Delta\tau/\beta$.*

**PROOF.** The proof has the following three steps in a similar vein to the one for *Theorem 1.*

($a$) Without loss of generality, we assume that during the switching interval $[t_i,\ t_{i+1})$, $i \in \mathrm{N}$, subsystem $p$ is active, and during the switching interval $[t_{i-1},\ t_i)$, $i \in \mathrm{N}_+$, subsystem $q$ is active. In order to enforce that the increase of the Lyapunov function over the unmatched interval is compensated by a decrease in the matched interval, we define a new positive number $\hat{h}$ as

$$\hat{h} = \begin{cases} \alpha\Delta\tau/\left(\beta L\right), & \text{if } \beta L h < \alpha\Delta\tau \\ h, & \text{otherwise.} \end{cases} \tag{11}$$

It is evident that $\hat{h} \geq h$, which implies that $\Delta P^p_{l+1,l}/h - \Delta P^p_{l+1,l}/\hat{h} > 0$. Considering that $\Delta P^p_{l+1,l}/h > 0$ due to (10c), it can be shown that if (10b) holds, then $\Delta P^p_{l+1,l}/\hat{h} + \overline{A}_p' P_{p,\ell} + P_{p,\ell}\overline{A}_p + \beta P_{p,\ell} < 0$. Let us define a time sequence $\{t_{i,0}, \ldots, t_{i,L}\}$, where $t_{i,l+1} - t_{i,l} = \hat{h}$, $l = 0, \ldots,\ L-1$, $t_{i,0} = t_i + \Delta\tau$, and $t_{i,L} - t_i = \tau_{\mathrm{d}}$, as shown in Fig. 4.

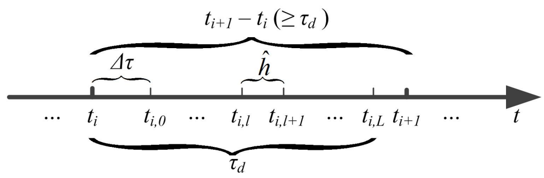


Fig. 4. The time sequence between two consecutive switching instants.

Similarly, we partition the interval $[t_i,\ t_{i+1})$ into three sub-intervals: $[t_i,\ t_{i,0})$, $[t_{i,0},\ t_{i,L})$, and $[t_{i,L},\ t_{i+1})$. The time-varying positive definite matrix $P_p(t)$ is, for $t \in [t_{i,0},\ t_{i+1})$

$$P_p(t) = \begin{cases} P_{p,l} + \hat{\rho}(t)\Delta P^p_{l+1,l}, & t \in [t_{i,l},\ t_{i,l+1}) \\ P_{p,L}, & t \in [t_{i,L},\ t_{i+1}) \end{cases} \tag{12}$$

where $\hat{\rho}(t) = (t - t_{i,l})/\hat{h}$ with $l = 0, \ldots, L-1$. Then, we construct a Lyapunov function similar with (8) using (12).

($b$) For the first sub-interval $t \in [t_i, t_{i,0})$, the derivative of the Lyapunov function is $\dot{V}(t) \leq \alpha V(t)$ due to (10a), and for $t \in [t_{i,0}, t_{i+1})$, according to (10b)–(10c), it holds that $\dot{V}(t) \leq -\beta V(t)$ using similar steps as (9) in the proof of *Theorem 1*. Since the Lyapunov function is non-increasing at the instant $t_{i,0}$, using the dwell time $\tau_{\mathrm{d}} > (\alpha+\beta)\Delta\tau/\beta$, we can guarantee that $V(t_{i+1}^-) \leq V(t_i)$.

($c$) Finally, we refer to the same reasoning as the third part in *Theorem 1*. This completes the proof. $\Xi$

**Remark 2** *As noted in [10], for a stable asynchronously switched system, one can always find $\alpha$ (characterizing the exponential rate of increase) big enough and $\beta$ (characterizing the exponential rate of decrease) small enough to certify stability; a similar situation arises also in our case. In addition, according to the results in [19, 24, 27], given $\beta$ satisfying* (10)*, there exists a lower bound $\underline{h}$ for $h$ such that feasibility occurs for any $h \geq \underline{h}$. This suggests the use of a sequence of line searches to solve* (10)*, where the scalars $\alpha, \beta, h$ are searched according to the aforementioned guidelines and* (10) *reduces to an LMI for fixed $\alpha, \beta, h$.*

## 4 $\mathcal{L}_2$ gain analysis and synthesis

In this section, a non-weighted $\mathcal{L}_2$ gain for asynchronously switched linear systems is derived from the Lyapunov function (6). Moreover, the LMIs for controller design are provided based on *Corollary 1*.

### *4.1 Non-weighted $\mathcal{L}_2$ gain analysis*

**Lemma 2** *Let $\alpha$ and $\beta$ be given positive constants. Suppose there exists a Lyapunov function $V : \mathrm{R}^n \to \mathrm{R}$, and two class-$\mathcal{K}_\infty$ functions $\kappa_1$ and $\kappa_2$ such that, for $t \in [t_i, t_{i+1})$, $\forall i \in \mathrm{N}$, we have $V((t_i+\Delta\tau_i)^-) \geq V(t_i + \Delta\tau_i)$, $V(t_i^-) = V(t_i)$, $\kappa_1(|x(t)|) \leq V(x(t)) \leq \kappa_2(|x(t)|)$, $\forall t \geq 0$, and*

$$\dot{V}(t) \leq \begin{cases} \alpha V(t) - \Gamma(t), & t \in \mathcal{T}_\uparrow(t_i,\ t_{i+1}) \\ -\beta V(t) - \Gamma(t), & t \in \mathcal{T}_\downarrow(t_i,\ t_{i+1}) \end{cases} \tag{13}$$

*where $\Gamma(t) = y'(t)y(t) - \gamma^2 w'(t)w(t)$. Then, the system* (2) *achieves a non-weighted $\mathcal{L}_2$ gain*

$$\overline{\gamma} = \sqrt{\frac{\beta\tau_{\mathrm{d}} e^{(\alpha+\beta)\Delta\tau}}{\beta\tau_{\mathrm{d}} - (\alpha+\beta)\Delta\tau}}\gamma \tag{14}$$

*for any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$ with $\tau_{\mathrm{d}} > (\alpha+\beta)\Delta\tau/\beta$.*

**PROOF.** Consider an interval $[t_i, t_{i+1})$, $i \in \mathrm{N}$. We represent the total unmatched interval and matched interval between $[t_{\mathrm{s}}, t_{\mathrm{f}})$ by $\mathcal{T}^\uparrow(t_{\mathrm{s}},\ t_{\mathrm{f}})$ and $\mathcal{T}^\downarrow(t_{\mathrm{s}},\ t_{\mathrm{f}})$, respectively. To keep the mathematical derivation concise, let us use the following notation: $\mathbf{E}(a,b) = e^{\alpha\mathcal{T}^\uparrow(a,\ b) - \beta\mathcal{T}^\downarrow(a,\ b)}$ with $a > b \geq 0$. Since $V(t_i + \Delta\tau_i) - V((t_i+\Delta\tau_i)^-) \leq 0$ for any $i \in \mathrm{N}$, it follows from (13) that

$$\begin{aligned} V(t) &\leq V(t_i)\mathbf{E}(t_i,t) - \int_{t_i}^{t} \mathbf{E}(s,t)\Gamma(s)ds \\ &\leq \left( V(t_{i-1})\mathbf{E}(t_i,t) - \int_{t_{i-1}}^{t_i} \mathbf{E}(s,t_i)\Gamma(s)ds \right) \mathbf{E}(t_i,t) \\ &\quad - \int_{t_i}^{t} \mathbf{E}(s,t)\Gamma(s)ds \\ &= V(t_{i-1})\mathbf{E}(t_{i-1},t) - \int_{t_{i-1}}^{t} \mathbf{E}(s,t)\Gamma(s)ds \\ &\quad \vdots \\ &\leq V(t_0)\mathbf{E}(t_0,t) - \int_{t_0}^{t} \mathbf{E}(s,t)\Gamma(s)ds. \end{aligned} \tag{15}$$

Considering the initial condition $V(t_0) = 0$, and $V(t) \geq 0$, and substituting $\Gamma(t) = y'(t)y(t) - \gamma^2 w'(t)w(t)$ into (15) gives $\int_{t_0}^{t} \mathbf{E}(s,t)y'(s)y(s)ds \leq \int_{t_0}^{t} \mathbf{E}(s,t)\gamma^2 w'(s)w(s)ds$, where the left-hand side is given by

$$\begin{aligned} &\int_{t_0}^{t} \mathbf{E}(s,t)y'(s)y(s)ds \\ &= \int_{t_0}^{t} e^{(\alpha+\beta)\mathcal{T}^\uparrow(s,\ t) - \beta(t-s)} y'(s)y(s)ds \\ &\geq \int_{t_0}^{t} e^{-\beta(t-s)} y'(s)y(s)ds \end{aligned} \tag{16}$$

and the right-hand side is

$$\begin{aligned} &\int_{t_0}^{t} \mathbf{E}(s,t)\gamma^2 w'(s)w(s)ds \\ &= \int_{t_0}^{t} e^{(\alpha+\beta)\mathcal{T}^\uparrow(s,\ t) - \beta(t-s)} \gamma^2 w'(s)w(s)ds \\ &\leq \int_{t_0}^{t} e^{N(s,\ t)(\alpha+\beta)\Delta\tau - \beta(t-s)} \gamma^2 w'(s)w(s)ds \\ &\leq \int_{t_0}^{t} e^{(1+\frac{t-s}{\tau_{\mathrm{d}}})(\alpha+\beta)\Delta\tau - \beta(t-s)} \gamma^2 w'(s)w(s)ds \\ &\leq \int_{t_0}^{t} e^{(\alpha+\beta)\Delta\tau} e^{\frac{(\alpha+\beta)\Delta\tau - \beta\tau_{\mathrm{d}}}{\tau_p}(t-s)} \gamma^2 w'(s)w(s)ds \end{aligned} \tag{17}$$

where the second inequality in (17) holds due to (3). Let $t_0 = 0$. Integrating (16) and (17) for $t$ going from 0 to

$\infty$, we have:

$$\begin{aligned}&\int_0^\infty \int_0^t e^{-\beta(t-s)} y'(s)y(s)dsdt \\ &= \int_0^\infty y'(s)y(s) \left( \int_s^\infty e^{-\beta(t-s)}dt \right) ds \\ &= \frac{1}{\beta} \int_0^\infty y'(s)y(s)ds\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\int_0^\infty \int_0^t e^{(\alpha+\beta)\Delta\tau} e^{-\frac{\Delta T}{\tau_\mathrm{d}}(t-s)} \gamma^2 w'(s)w(s)ds\ dt \\ &= e^{(\alpha+\beta)\Delta\tau} \int_0^\infty \left( \int_s^\infty e^{-\frac{\Delta T}{\tau_\mathrm{d}}(t-s)} dt \right) \gamma^2 w'(s)w(s)ds \\ &= \frac{\tau_\mathrm{d}}{-\Delta T} e^{(\alpha+\beta)\Delta\tau} \gamma^2 \int_0^\infty w'(s)w(s)ds\end{aligned} \tag{19}$$

due to $\Delta T$ , $(\alpha+\beta)\Delta\tau - \beta\tau_\mathrm{d} < 0$. Combining (18) and (19) leads to

$$\frac{1}{\beta} \int_0^\infty y'(s)y(s)ds \le \frac{\tau_\mathrm{d} e^{(\alpha+\beta)\Delta\tau}}{-\Delta T} \gamma^2 \int_0^\infty w'(s)w(s)ds$$

which indicates that a non-weighted $\mathcal{L}_2$ gain as (14) for the system (2) is guaranteed. $\quad \Xi$

**Theorem 2** *Let $\alpha$ and $\beta$ be given positive constants. Suppose there exists a family of positive definite matrices $P_{p,l}$, $p \in \mathcal{M}$, $l = 0, \ldots, L$, such that*

$$\begin{aligned}&\begin{bmatrix} \Phi_{p,q} & P_{q,L}E_p & \overline{C}'_{p,q} \\ * & -\gamma^2 I & F'_p \\ * & * & -I \end{bmatrix} < 0, \quad \begin{bmatrix} \Theta_p & P_{p,\ell}E_p & \overline{C}'_p \\ * & -\gamma^2 I & F'_p \\ * & * & -I \end{bmatrix} < 0 \\ &\begin{bmatrix} \Psi_p & P_{p,L}E_p & \overline{C}'_p \\ * & -\gamma^2 I & F'_p \\ * & * & -I \end{bmatrix} < 0, \quad \begin{matrix} \Delta P^p_{l+1,l} > 0 \\ P_{q,L} - P_{p,0} \ge 0 \end{matrix}\end{aligned} \tag{20}$$

*for $\ell = l,\ l+1;\ l = 1, \ldots, L-1;\ p \neq q \in \mathcal{M}$, where*

$$\begin{aligned}\Phi_{p,q} &= \overline{A}'_{p,q} P_{q,L} + P_{q,L}\overline{A}_{p,q} - \alpha P_{q,L} \\ \Theta_p &= \Delta P^p_{l+1,l}/\hat{h} + P_{p,\ell}\overline{A}_p + \overline{A}'_p P_{p,\ell} + \beta P_{p,\ell} \\ \Psi_p &= P_{p,L}\overline{A}_p + \overline{A}'_p P_{p,L} + \beta P_{p,L}.\end{aligned}$$

*Then, the asynchronously switched linear system* (2) *achieves a non-weighted $\mathcal{L}_2$ gain $\overline{\gamma}$* (14) *for any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$ with $\tau_\mathrm{d} > (\alpha+\beta)\Delta\tau/\beta$.*

**PROOF.** According to the standard derivation of the bounded real lemma for linear systems [28] and the definition of $P_p(t)$ in (12), it can be verified that (20) leads to the following: $\dot{V}(t) \le \alpha V(t) + y'(t)y(t) - \gamma^2 w'(t)w(t)$ for $t \in \mathcal{T}_\uparrow(t_i,\ t_{i+1})$; $\dot{V}(t) \le -\beta V(t) + y'(t)y(t) - \gamma^2 w'(t)w(t)$, for $t \in \mathcal{T}_\downarrow(t_i,\ t_{i+1})$, which is in the same form as (13). Furthermore, $V(t)$ is continuous at the switching instants, and non-increasing at the instants when the modes are matched. This means that *Lemma 2* holds, and we can guarantee a non-weighted $\mathcal{L}_2$ gain for asynchronously switched systems (2) via the dwell time $\tau_\mathrm{d} > (\alpha+\beta)\Delta\tau/\beta$. $\quad \Xi$

### *4.2 State-feedback $H_\infty$ control*

**Theorem 3** *Let $\alpha$ and $\beta$ be given positive constants. Suppose there exists a family of positive definite matrices $Q_{p,l}$, a family of vectors $U_{p,l}$, $p \in \mathcal{M}$, $l = 0, \ldots, L$, and a positive number $h$ such that*

$$\begin{aligned}&\begin{bmatrix} \mathcal{H}_{p,q} & E_p & \mathcal{Z}_{p,q} \\ * & -\gamma^2 I & F'_p \\ * & * & -I \end{bmatrix} < 0, \quad \begin{bmatrix} \Xi_p & E_p & \Lambda_p \\ * & -\gamma^2 I & F'_p \\ * & * & -I \end{bmatrix} < 0 \\ &\begin{bmatrix} \Omega_p & E_p & \Upsilon_p \\ * & -\gamma^2 I & F'_p \\ * & * & -I \end{bmatrix} < 0, \quad \begin{matrix} \Delta Q^p_{l+1,l} < 0 \\ -Q_{q,L} + Q_{p,0} \ge 0 \end{matrix}\end{aligned} \tag{21}$$

*for $\ell = l,\ l+1;\ l = 0, \ldots, L-1;\ p \neq q \in \mathcal{M}$, where*

$$\begin{aligned}\mathcal{H}_{p,q} &= Q_{q,L}A'_p + A_pQ_{q,L} + U'_{q,L}B'_p + B_pU_{q,L} - \alpha Q_{q,L} \\ \mathcal{Z}_{p,q} &= Q_{q,L}C'_p + U'_{q,L}D'_p \\ \Xi_p &= \Delta Q^p_{l+1,l}/\hat{h} + Q_{p,\ell}A'_p + A_pQ_{p,\ell} + U'_{p,\ell}B'_p \\ &\quad + B_pU_{p,\ell} + \beta P_{p,\ell} \\ \Lambda_p &= Q_{p,\ell}C'_p + U'_{p,\ell}D'_p \\ \Omega_p &= Q_{p,L}A'_p + A_pQ_{p,L} + U'_{p,L}B'_p + B_pU_{p,L} + \beta_p P_{p,L} \\ \Upsilon_p &= Q_{p,L}C'_p + U'_{p,L}D'_p.\end{aligned}$$

*Then, there exists a family of mode-dependent state-feedback controllers $u(t) = G_{\sigma(t)}x(t)$ with the maximum switching delay $\Delta\tau$ such that the system* (2) *achieves a non-weighted $\mathcal{L}_2$ gain $\overline{\gamma}$* (14) *for any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$ with $\tau_\mathrm{d} > (\alpha+\beta)\Delta\tau/\beta$. Additionally, the gains of state-feedback controllers with switching delay can be obtained as*

$$G_p(t) = \begin{cases} \left[U_{p,l} + \hat{\rho}(t)\Delta U^p_{l+1,l}\right] \cdot \\ \quad \left[Q_{p,l} + \hat{\rho}(t)\Delta Q^p_{l+1,l}\right]^{-1}, & t \in [t_{i,l},\, t_{i,l+1}) \\ U_{p,L}Q^{-1}_{p,L}, & t \in [t_{i,L},\, t_{i+1,0}) \end{cases} \tag{22}$$

*for $l = 0, \ldots, L-1$, where $\Delta U^p_{l+1,l} = U_{p,l} - U_{p,l+1}$, $\hat{\rho}(t) = (t - t_{i,l})/\hat{h}$ with $t_{i,l}$ shown in Fig. 4.*

**PROOF.** Let $Q_{p,l} = P_{p,l}^{-1}$, for $l = 0, \ldots, L$. Substituting $\overline{A}_{p,q}$, $\overline{A}_p$, $\overline{C}_{p,q}$ and $\overline{C}_p$ in (2) into (20), and then pre-multiplying and post-multiplying by diag $\{Q_{p,l},\ I,\ I\}$ from both sides, the state-feedback gains $G_p(t)$ are obtained. $\Xi$

**Remark 3** *The following difference must be remarked between the results in [19, 24, 25] and the results of this work. In [19, 24, 25], it has been shown that by increasing L, a less conservative dwell time can be found. In Corollary 1, the dwell time $\tau_{\mathrm{d}} > (\alpha+\beta)\Delta\tau/\beta$ is not affected by the choice of L. However, increasing L might reduce conservativeness in terms of $\mathcal{L}_2$ gain, as shown in the example of Section 5.*

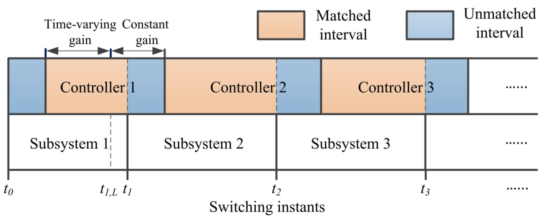


Fig. 5. The control scheme of (2).

**Remark 4** *Derived from the novel Lyapunov function* (8)*, the state-feedback gains of the mode-dependent controller are designed to be time-varying only during the matched interval according to* (22)*. This implies that after the switching, the controller associated to the subsystem has a constant gain during the unmatched interval, as shown in Fig. 5. This can simplify the analysis and design of the closed-loop system formed, during an unmatched interval whose actual length is unknown, by the currently active subsystem and the constant-gain controller.*

## 5 Numerical example

In this section, the following asynchronously switched linear system with maximum switching delay $\Delta\tau = 2$ is adopted to illustrate the proposed results:

$$A_1 = \begin{bmatrix} 0.9 & -5.8 \\ 2.75 & 0.9 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & 2 \\ 2.1 & -1.3 \end{bmatrix},\ B_1 = \begin{bmatrix} 1.5 \\ 2.2 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 1.85 \\ 1.75 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.45 \\ 0 \end{bmatrix},\ E_1 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 0.2 & 0.6 \end{bmatrix}^T,\ D_1 = D_2 = 1.5,\ F_1 = F_2 = 0.65.$$

### *5.1 Non-weighted $\mathcal{L}_2$ gain using* (6)

In this subsection, different choices for the value of $L$ are considered to illustrate the results in this paper.

(a) ($L = 1$) We select $L = 1$, $\alpha = 0.26$, $\beta = 0.2$. After solving the convex optimization problem (21), we obtain $\gamma = 1.2746$, $h = 0.06$, $\hat{h} = 2.6$, $\tau_{\mathrm{d}}^*$ , $(\alpha+\beta)\Delta\tau/\beta = 4.6$, and the following matrices and vectors:

$$Q_{1,0} = \begin{bmatrix} 4.8481 & -0.0466 \\ -0.0466 & 0.8858 \end{bmatrix}, Q_{1,1} = \begin{bmatrix} 4.8420 & -0.0613 \\ -0.0613 & 0.8503 \end{bmatrix}$$
$$Q_{2,0} = \begin{bmatrix} 6.3250 & -0.8206 \\ -0.8206 & 1.2390 \end{bmatrix}, Q_{2,1} = \begin{bmatrix} 4.2490 & -0.1715 \\ -0.1715 & 0.8598 \end{bmatrix}$$
$$U_{1,0} = \begin{bmatrix} -3.6725 & -0.7633 \end{bmatrix}, U_{1,1} = \begin{bmatrix} -3.7385 & -0.6484 \end{bmatrix}$$
$$U_{2,0} = \begin{bmatrix} -1.1715 & -1.0804 \end{bmatrix}, U_{2,1} = \begin{bmatrix} -2.8473 & -0.1361 \end{bmatrix}.$$

Selecting $t_{i+1} - t_i = \tau_{\mathrm{d}} = 5.6 > \tau_{\mathrm{d}^*}$ , $i \in \mathbb{N}$, we have the non-weighted $\mathcal{L}_2$ gain $\overline{\gamma} = 4.7865$ according to (14). Then, using (22), the controller gains for the two system modes are obtained as follows:

$$G_p(t) = \begin{cases} \left[(t - t_{i,0})\Delta U_p/\hat{h} + U_{p,0}\right] \cdot \\ \quad \left[(t - t_{i,0})\Delta Q_p/\hat{h} + Q_{p,0}\right]^{-1}, & t \in [t_{i,0},\, t_{i,1}) \\ U_{p,1}Q_{p,1}^{-1}, & t \in [t_{i,1},\, t_{i+1,0}) \end{cases} \tag{23}$$

for $p \in \{1, 2\}$, where $\Delta U_p = U_{p,1} - U_{p,0}$, and $\Delta Q_p = Q_{p,1} - Q_{p,0}$. Let the disturbance $w(t) \equiv 0$, and the initial condition $x_0 = [2\ 1]'$. The resulting Lyapunov function is given in Fig. 6, which shows that when the controller

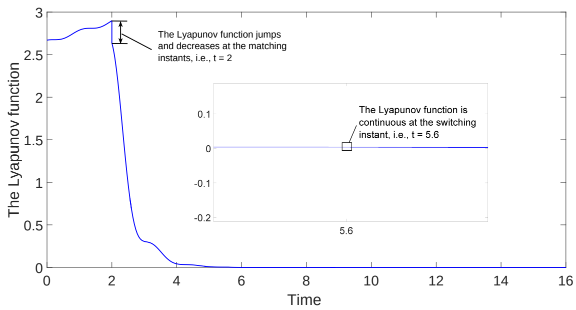


Fig. 6. The proposed Lyapunov functions with a zoomed detail around $t = 5.6$.

mode and the system mode are matched, i.e., at $t = 2$, the Lyapunov function is decreasing, and at the switching instant $t = 5.6$, the Lyapunov function is continuous. In addition, the Lyapunov function tends to zero, as predicted by the global asymptotic stability results.

For the disturbance, let us consider an example $w(t) = 0.5\exp(-0.2t)$, and let the initial condition $x_0 = [2\ 1]'$. Adopting the controllers (23) with $\mathcal{L}_2$ gain $\overline{\gamma} = 4.7865$ gives rise to the state response shown in Fig. 7, which is stable.

(b) ($L > 1$) Now we choose different values of $L$, and $\alpha = 0.26$, $\beta = 0.2$. By solving the convex optimization

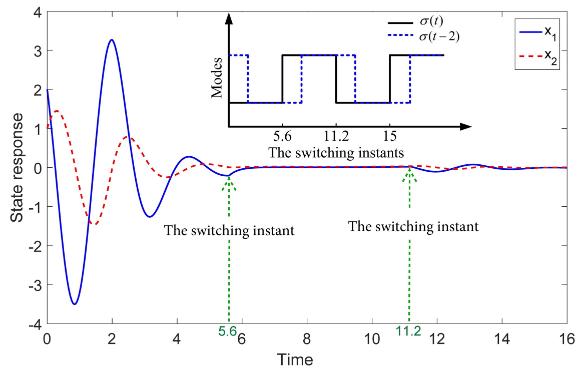


Fig. 7. State response of closed-loop systems with controllers (23).

problem (21), we get different $\mathcal{L}_2$ gain $\overline{\gamma}$ as shown in Table 1. It can be observed that a less conservative $\mathcal{L}_2$ gain is obtained as $L$ increases.

Table 1
Non-weighted $\mathcal{L}_2$ gain $\overline{\gamma}$ for different values of $L$.

| $L$ | 1 | 5 | 20 | 90 | 100 |
|---|---|---|---|---|---|
| $\gamma$ | 1.2769 | 1.0546 | 1.0474 | 1.0436 | 1.0435 |
| $\overline{\gamma}$ | 4.7865 | 3.9116 | 3.9119 | 3.9118 | 3.9118 |

### 5.2 *Comparison between* (5) *and* (6)

The key properties about continuity and discontinuity of (5) and (6) are compared herein. To facilitate understanding of the comparison between (5) and (6), we let $L = 1$. First, we adopt the same technique to develop time-varying matrices $P_p(t)$ for (5) and derive the conditions for designing the two mode-dependent controllers: substituting $Q_{p,0}$ with $Q_{q,L}$ in $\mathcal{H}_{p,q}$, and replacing $-Q_{q,L} + Q_{p,0} \geq 0$ with $-Q_{p,L} + Q_{q,0} \geq 0$ in (21), $p \neq q \in \{1, 2\}$. Therefore, the resulting controller gains are, for $p = 1, 2$

$$G_p(t) = \begin{cases} \Big[(t - t_i)\Delta U_p/\hat{h} + U_{p,0}\Big] \cdot \\ \qquad \Big[(t - t_i)\Delta Q_p/\hat{h} + Q_{p,0}\Big]^{-1}, & t \in [t_i,\, t_{i,1}) \\ U_{p,1}Q_{p,1}^{-1}, & t \in [t_{i,1},\, t_{i+1}) \end{cases}$$

which shows that the mode-dependent controllers $u_p$ are active during the interval $[t_i,\, t_{i+1})$. This implies that the mode-dependent controllers designed via (5) fails to deal with the switching delay $\Delta\tau$. Now, let us focus on the controllers in (23) designed via (6). They are active during the interval $[t_{i,0}, t_{i+1,0})$, which implies that the controllers are designed considering the switching delays based on (6). Therefore, we conclude that the proposed Lyapunov function (6) reflects the key feature of asynchronously switched linear systems (as explained more technically in *Remark 1*).

## 6 Conclusion

In this paper, a novel Lyapunov function for asynchronously switched linear systems has been proposed. In contrast with the classical Lyapunov function introduced by Branicky, this Lyapunov function is continuous at the switching instants and discontinuous when the system modes and controller modes are matched, which is consistent with the essence of asynchronously switched systems. A new stability condition via dwell time has been introduced to guarantee asymptotic stability in the noiseless case. Moreover, the proposed Lyapunov function can be used to guarantee a non-weighted $\mathcal{L}_2$ gain for asynchronously switched linear systems. Finally, a numerical example has been used to illustrate the proposed methodologies. Future work might include the adoption of the Lyapunov function (6) to study the tolerant control in the spirit of [27].

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