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Model Predictive Control for Stochastic Max-Plus Linear Systems with Chance Constraints

Jia Xu, Ton van den Boom, and Bart De Schutter

***Abstract*—The topic of this paper is model predictive control (MPC) for max-plus linear systems with stochastic uncertainties the distribution of which is supposed to be known. We consider linear constraints on the inputs and the outputs. Due to the uncertainties, these linear constraints are formulated as probabilistic or chance constraints, i.e., the constraints are required to be satisfied with a predefined probability level. The proposed chance constraints can be equivalently rewritten into a max-affine (i.e., the maximum of affine terms) form if the linear constraints are monotonically nondecreasing as a function of the outputs. Based on the resulting max-affine form, two methods are developed for solving the chance-constrained MPC problem for stochastic max-plus linear systems. Method 1 uses Boole's inequality to convert the multivariate chance constraint into univariate chance constraints for which the probability can be computed more efficiently. Method 2 employs Chebyshev's inequality and transforms the chance constraint into linear constraints on the inputs. The simulation results for a production system example show that the two proposed methods are faster than the Monte Carlo simulation method and yield lower closed-loop costs than the nominal MPC method.**

I. Introduction

Model predictive control (MPC) [1] is an advanced control strategy for the control of multivariate systems in the presence of input and state/output constraints towards achieving a high performance. At every sampling instant, an open-loop constrained optimal control problem over a finite horizon is solved to compute a sequence of control inputs. The first element of the resulting optimal control sequence is applied to the system and the optimization problem is repeated at the next sampling instant based on new measurements.

Due to model mismatch or disturbances, uncertainties are often considered in the prediction model of MPC. Many results have been achieved in the area of robust MPC dealing with the situation that the uncertainties are assumed to be deterministic and bounded, e.g., [2, 3] and the references therein. On the other hand, for the situation that the uncertainties are characterized as random variables, stochastic MPC [4, 5] has emerged as a useful control design method where usually the expected value of a cost criterion is optimized subject to input, state, or output constraints. Due to the probabilistic nature of uncertainties, those constraints are usually formulated as chance constraints, i.e., the probability of constraint violation is limited to a predefined probability level. Stochastic MPC takes advantage of the probability distributions of the uncertainties and is based on stochastic programming and chance-constrained programming [6–9].

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Max-plus linear (MPL) systems are a subclass of discrete-event systems. Maximization and addition are the basic operations in the models of MPL systems [10]. In contrast to conventional linear systems, where uncertainties are usually modeled by adding an extra term in the system equations, uncertainties in MPL systems are included in the system matrices [10]. The MPC framework has been extended to stochastic max-plus linear (SMPL) systems in [11]. The expected value of the outputs is used in the objective criterion and in the constraint. Some results about MPC for SMPL systems can be found in [12–15]. To the authors' best knowledge currently [15] is the only paper in literature that has considered the chance-constrained MPC problem for SMPL systems. In [15], the chance constraints are approximated and substituted with a finite number of pointwise constraints at independently generated scenarios of the uncertainties. The approach in [15] is different from the methods developed in this paper as we transform the chance constraints into reduced forms based on some probabilistic inequalities.

In particular, in this paper we develop two approaches for solving the chance-constrained MPC problem based on probabilistic inequalities and natural properties of SMPL systems. More specifically, if the chance constraints are monotonically nondecreasing as a function of the outputs (i.e., the coefficients of the outputs in the linear constraints are nonnegative), we rewrite the chance constraints into an equivalent max-affine form, namely, the maximum of some correlated random variables. Those correlated random variables are affine functions of the uncertainties of the SMPL system. Based on the resulting max-affine form, we develop two methods for transforming the chance constraints into reduced forms. In the first method, based on Boole's inequality, the probability of the maximum of correlated random variables is decomposed into the sum of probabilities of a single random variable. In the second method, we provide sufficient conditions for applying the multidimensional Chebyshev inequality to transform the chance constraints into constraints that are linear in the control inputs. The approaches developed in this paper are assessed with a production system example and compared with the Monte Carlo (MC) simulation method and the nominal MPC method. The results show that the two methods proposed in this paper generally take less computation time than the MC simulation method to achieve a similar performance. The nominal MPC method is faster than the other methods, but it yields a worse performance.

This paper is organized as follows. Section II provides preliminaries about p-norms, probabilistic inequalities, and max-plus algebra. A brief introduction for SMPL systems is given in Section III. The MPC problem formulation with chance constraints for SMPL systems is presented in Section IV. Two approaches for solving the proposed problem are developed in Section V and illustrated with a production system example in Section VI. Finally, Section VII concludes the paper.

II. Preliminaries

A. Norms

For any $x \in \mathbb{R}^n$ and for $p \geq 1$, the p-norm of x is defined as: $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$. More specifically, $\|x\|_1 = |x_1| + \cdots + |x_n|$, $\|x\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$, $\|x\|_\infty = \max(|x_1|, \ldots, |x_n|)$. Some important norm properties are

$$\|x\|_\infty \leq \|x\|_{p+1} \leq \|x\|_p \text{ for } p \geq 1 \ . \tag{1}$$

B. Probabilistic Inequalities

This section is based on [16].

Theorem 1 (Jensen's inequality): Let φ be an integrable, concave function of a random variable v. Then $\mathbb{E}[\varphi(v)] \leq \varphi(\mathbb{E}[v])$.

Theorem 2 (Multidimensional Chebyshev inequality): Let $X = [X_1, \ldots, X_n]^T$ be a random vector with mean $\mu_X = \mathbb{E}[X]$ and covariance matrix $\Sigma_X = \mathbb{E}[(X-\mu)(X-\mu)^T]$. If Σ_X is positive definite, then for any $a > 0$ we have

$$\Pr\left\{(X-\mu_X)^T \Sigma_X^{-1} (X-\mu_X) \leq a\right\} \geq 1 - \frac{n}{a} \ . \tag{2}$$

Theorem 3: [17] Let X be a random vector with mean μ_X and covariance matrix Σ_X. Let $B \in \mathbb{R}^{m\times n}$ be a real matrix. Then the linear combination $Y = BX$ satisfies $\mu_Y = \mathbb{E}[Y] = \mathbb{E}[BX] = B\mu_X$, $\Sigma_Y = \mathrm{Cov}(Y) = \mathrm{Cov}(BX) = B\Sigma_X B^T$.

C. Max-Plus Algebra

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as [10]: $x \oplus y = \max(x,y), x \otimes y = x + y$ for any $x, y \in \mathbb{R}_\varepsilon$. The corresponding max-plus matrix operations are defined as

$$\left[A \oplus B\right]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij}),$$
$$\left[A \otimes C\right]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,n}(a_{ik} + c_{kj})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times l}$.

Definition 4 (Max-affine function): A max-affine function f of $x \in \mathbb{R}_\varepsilon^n$ is a function of the form $f(x) = \max_{i=1,\ldots,n}(\alpha_i^T x + \xi_i)$ with constant coefficients $\alpha_i \in \mathbb{R}^n$ and $\xi_i \in \mathbb{R}$.

III. Stochastic Max-Plus Linear Systems

Consider a stochastic max-plus linear (SMPL) system [11] of the form

$$x(k) = A(w(k)) \otimes x(k-1) \oplus B(w(k)) \otimes u(k), \tag{3}$$
$$y(k) = C(w(k)) \otimes x(k) \tag{4}$$

where k is the event counter, $u(k) \in \mathbb{R}_\varepsilon^{n_u}$ and $y(k) \in \mathbb{R}_\varepsilon^{n_y}$ are the input and output of the system consisting of the time instants at which the input and output events occur for the k-th cycle, and $x(k) \in \mathbb{R}_\varepsilon^{n_x}$ is the state of the system representing the time instants at which the internal processes of the system start for the k-th cycle. The random vector $w(k) \in \mathbb{R}^{n_w}$ collects uncertainties at event step k caused by disturbances or model mismatch. Just as in [11] we adopt the following assumption in this paper:

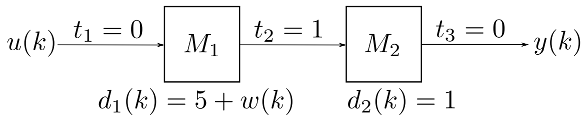

Fig. 1. A production system

Assumption 5: At any event step k, the components of $w(k)$ are independent and identically distributed random variables with a given probability distribution. In addition, the uncertainties at different event steps are independent, i.e., $w(0), w(1) \ldots$ are statistically independent.

Typically, the entries of the uncertain system matrices $A(w(k)), B(w(k)), C(w(k))$ consist of sums or maximization of internal process times and transportation times [10]. In general, the components of $w(k)$ correspond to perturbations in these duration times. So instead of modeling uncertainties by adding an extra max-plus-algebraic term in the system (3) and (4), uncertainties should rather be modeled as an additive term to the system matrices. Then, the entries of the uncertain system matrices are max-affine functions of $w(k)$.

As an example, we consider the production system presented in [11] (see Figure 1). This system consists of two machines M_1 and M_2 where raw materials are fed into M_1, afterwards intermediate products are fed into M_2, and finally the finished goods leave the production system. Just as in [11] we assume that the transportation times are constant (i.e., $t_1 = t_3 = 0, t_2 = 1$) and so is the processing time of M_2 (i.e., $d_2(k) = 1$). Here $x_i(k)$ represents the time instant at which machine i starts for the k-th time. The system matrices of the corresponding SMPL model are given as follows:

$$A = \begin{bmatrix} d_1(k-1) & \varepsilon \\ d_1(k-1) + d_1(k) + t_2(k) & d_2(k-1) \end{bmatrix},$$
$$B = \begin{bmatrix} t_1(k) \\ d_1(k) + t_1(k) + t_2(k) \end{bmatrix}, \quad C = \begin{bmatrix} \varepsilon & d_2(k) + t_3(k) \end{bmatrix}.$$

IV. MPC for SMPL Systems

In this section, we formulate the MPC problem for SMPL systems of [11] using chance constraints instead of using expected value of the outputs in the constraints.

A. Cost Criterion

Define a cost criterion J that reflects the input and output cost functions from event step k to $k + N_\mathrm{p} - 1$:

$$J(k) = J_\mathrm{out}(k) + \lambda J_\mathrm{in}(k)$$

with the scalar $\lambda \geq 0$ the trade-off between J_out and J_in. In MPC, one aims to design an optimal control sequence $u(k), \ldots, u(k + N_\mathrm{p} - 1)$ that minimizes $J(k)$ subject to constraints on the inputs and the outputs. MPC uses a receding horizon scheme, i.e., only the first control input $u(k)$ of the computed optimal control sequence is applied to the system; subsequently the horizon is shifted one event step and the optimization is restarted based on new measurements of the states. Different choices for J_{out} and J_{in} are given in [18]. In the current paper J_{out} and J_{in} are chosen as

$$J_{\text{out}}(k) = \sum_{j=0}^{N_{\text{p}}-1} \sum_{i=1}^{n_y} \eta_i(k+j),$$

$$J_{\text{in}}(k) = -\sum_{j=0}^{N_{\text{p}}-1} \sum_{l=1}^{n_u} u_l(k+j)$$

where $\eta_i(k) = \max(y_i(k) - r_i(k), 0)$ reflects the delay between the completion time y and the due-date signal r. The selected J_{in} corresponds to the just-in-time rule. Note that the results in this paper can be easily extended to other cases such as 1-norm and ∞-norm cost criteria used in [18].

Note that J_{out} is random. To obtain a deterministic cost criterion, we use the expected value of $J(k)$ as the cost criterion in our MPC optimization problem. Moreover, J is actually the maximum of some correlated random variables and it is difficult to get an analytic expression for the distribution of J. So the expected value $\mathbb{E}[J(k)]$ cannot be computed analytically. In this paper, $\mathbb{E}[J(k)]$ will be approximated or computed by different methods, namely, MC simulation and MC integration. We will combine each method for $\mathbb{E}[J(k)]$ with the methods for chance constraints developed in the next section and compare the efficiency and performance of every combination for a production system example (see Section VI).

B. Chance Constraints

Define $\tilde{u}(k) = \begin{bmatrix} u^T(k) & \cdots & u^T(k+N_{\text{p}}-1) \end{bmatrix}^T$ where N_{p} is the prediction horizon ($\tilde{y}(k), \tilde{w}(k), \tilde{r}(k)$ are defined in the same way). From [11], the components of $\tilde{y}(k)$ are max-affine functions of $\tilde{w}(k)$ and $\tilde{u}(k)$. So the following linear constraints

$$G\tilde{u}(k) + H\tilde{y}(k) \leq h(k) \tag{5}$$

are also random, where $G \in \mathbb{R}^{c \times N_{\text{p}} n_u}$ and $H \in \mathbb{R}^{c \times N_{\text{p}} n_y}$ are constant matrices and $h(k) \in \mathbb{R}^c$ is a vector depending on the known information at event step k, i.e., $x(k-1)$ and $\tilde{r}(k)$.

Note that $\tilde{w}(k)$ is uncertain and only its distribution is supposed to be known. To reformulate the random constraints (5), we require that (5) are satisfied for sufficiently many realizations of $\tilde{w}(k)$, namely,

$$\Pr\{G\tilde{u}(k) + H\tilde{y}(k) \leq h(k)\} \geq 1 - \epsilon \tag{6}$$

where $\epsilon \in (0, 1)$ is the probability of possible violation of (5). In other words, we require that (5) is satisfied at least with a probability $1 - \epsilon$. The probabilistic constraint (6) is usually called chance constraints.

C. Problem Formulation

Now we combine the material of previous subsections. At event step k, the chance-constrained MPC problem for SMPL systems is then defined as follows:

$$\min_{\tilde{u}(k)} \ \mathbb{E}[J(k)] \tag{7}$$

$$\text{subject to (3)-(4),} \tag{8}$$

$$u(k+j) \geq u(k+j-1), \ j = 0, \ldots, N_{\text{p}} - 1, \tag{9}$$

$$\Pr\{G\tilde{u}(k) + H\tilde{y}(k) \leq h(k)\} \geq 1 - \epsilon \ . \tag{10}$$

The constraint (9) is added since the $u(k), \ldots, u(k+N_{\text{p}}-1)$ correspond to consecutive event occurrence times.

In general, problem (7)-(10) is a nonlinear nonconvex optimization problem. For decreasing the computational burden, we aim to transform the problem into reduced forms. In this paper, $\mathbb{E}[J(k)]$ will be approximated by MC simulation [19] and MC integration [20] respectively. Moreover, MC simulation will also be used to deal with the chance constraint and compared with the two approaches developed in the next section.

V. Chance-Constrained MPC for SMPL Systems

In this section, we develop two approaches for the chance-constrained MPC problem (7)-(10).

A. Max-Affine Form of Chance Constraints

In this subsection, we rewrite the chance constraint (10) into a max-affine form. We have

$$\begin{aligned}
&\Pr\{G\tilde{u}(k) + H\tilde{y}(k) \leq h(k)\} \\
&= \Pr\{G\tilde{u}(k) + H\tilde{y}(k) - h(k) \leq 0\} \\
&= \Pr\Big\{ \max_{i=1,\ldots,c} (G\tilde{u}(k) + H\tilde{y}(k) - h(k))_i \leq 0 \Big\}.
\end{aligned}$$

Note that the vector $G\tilde{u}(k) + H\tilde{y}(k) - h(k)$ only contains affine operations on the components of $\tilde{u}(k)$ and $\tilde{y}(k)$. Recall that the components of $\tilde{y}(k)$ are max-affine functions of $\tilde{w}(k)$ and $\tilde{u}(k)$. Assume that H has nonnegative entries. Therefore, each component of $G\tilde{u}(k)+H\tilde{y}(k)-h(k)$ is also a max-affine function of $\tilde{w}(k)$ and $\tilde{u}(k)$. Let $m = \sum_{i=1}^{c} n_i$ where n_i is the number of affine expressions appearing in the maximization for the i-th component of $G\tilde{u}(k) + H\tilde{y}(k) - h(k)$. Hence,

$$\begin{aligned}
&\Pr\Big\{ \max_{i=1,\ldots,c} (G\tilde{u}(k) + H\tilde{y}(k) - h(k))_i \leq 0 \Big\} \\
&= \Pr\Big\{ \max_{i=1,\ldots,m} (z_i(k)) \leq 0 \Big\}
\end{aligned}$$

with

$$z(k) = \Lambda \begin{bmatrix} w(k-1) & \tilde{w}(k) \end{bmatrix}^T + \Gamma \tilde{u}(k) + \Xi(k) \tag{11}$$

for some appropriately defined matrices and vectors

$$\Lambda \in \mathbb{R}^{m \times (N_{\text{p}}+1) n_w}, \ \Gamma \in \mathbb{R}^{m \times N_{\text{p}} n_u}, \ \Xi(k) \in \mathbb{R}^m.$$

Therefore, the chance constraint (10) is equivalent to

$$\Pr\Big\{ \max_{i=1,\ldots,m} (z_i(k)) \leq 0 \Big\} \geq 1 - \epsilon \tag{12}$$

if H has nonnegative elements.

According to (11), the components of $z(k)$ are generally not independent and it is difficult to get an analytic expression for the distribution of their maximum. Although the probability in (12) can be computed by numerical integration based on MC [20], the computational load is usually heavy. When using MC simulation, the probability in the chance constraint is estimated by the number of random vectors satisfying $\max_{i=1,\ldots,m}(z_i(k)) \le 0$ among all random vectors generated based on the given distribution. In the following subsections we will introduce two methods to transform (12) into reduced forms that can be evaluated efficiently.

B. Method 1: Based on Boole's Inequality

In this subsection, we apply Boole's inequality to convert the multivariate constraint (12) into several univariate constraints that can be evaluated efficiently.

Proposition 6: If

$$\sum_{i=1}^{m} \Pr\{z_i(k) > 0\} \le \epsilon \ , \tag{13}$$

then $\Pr\Big\{\max_{i=1,\ldots,m}(z_i(k)) \le 0\Big\} \ge 1-\epsilon$.

Proof: We have

$$\Pr\Big\{\max_{i=1,\ldots,m}(z_i(k)) \le 0\Big\} = 1 - \Pr\Big\{\max_{i=1,\ldots,m}(z_i(k)) > 0\Big\}.$$

According to the Boole's inequality, we have

$$\Pr\Big\{\max_{i=1,\ldots,m}(z_i(k)) > 0\Big\} \le \sum_{i=1}^{m} \Pr\{z_i(k) > 0\}.$$

So if $\sum_{i=1}^{m} \Pr\{z_i(k) > 0\} \le \epsilon$, then

$$\Pr\Big\{\max_{i=1,\ldots,m}(z_i(k)) \le 0\Big\} \ge 1-\epsilon \ .$$

■

Based on Proposition 6, the optimal control sequence at step k can be calculated by solving the optimization problem (7)-(9) and (13), which can be solved more efficiently than the original optimization problem (7)-(10).

C. Method 2: Based on Chebyshev's Inequality

Now we introduce an alternative method applying the multidimensional Chebyshev inequality to transform the chance constraint (12) into linear constraints on control inputs and we propose a sufficient condition for applying such method.

According to Assumption 5, the components of $\tilde{w}(k) \in \mathbb{R}^{(N_\mathrm{p}+1)n_w}$ are independent and identically distributed random variables. Let μ_w and Σ_w be the mean vector and covariance matrix of $\tilde{w}(k)$. Define

$$\mu_z(k) = \Lambda\mu_w + \Gamma\tilde{u}(k) + \Xi(k), \tag{14}$$

$$\Sigma_z = \Lambda\Sigma_w\Lambda^T. \tag{15}$$

From Theorem 3, $\mu_z(k)$ and Σ_z are the mean vector and covariance matrix of $z(k)$.

Proposition 7: If Σ_z is a positive definite matrix[1], let $\lambda_{\min}(\Sigma_z^{-1}) > 0$ be the smallest eigenvalue of the matrix Σ_z^{-1}. Let $\bar{\mu}_z(k) = \max_{i=1,\ldots,m} \mu_{z,i}(k)$. If $\bar{\mu}_z(k) < 0$ and

$$\frac{m}{-\bar{\mu}_z(k)\lambda_{\min}(\Sigma_z^{-1})} \le \epsilon \ ,$$

then

$$\Pr\{\max_{i=1,\ldots,m}(z_i(k)) \le 0\} \ge 1-\epsilon \ .$$

[1]Note that every covariance matrix is symmetric and positive semi-definite.

Proof: For the sake of simplicity, in this proof, we will write z, μ_z instead of $z(k), \mu_z(k)$. Consider

$$\begin{aligned}
&\max(z_1,\ldots,z_m) \\
&= \max(z_1-\bar{\mu}_z,\ldots,z_m-\bar{\mu}_z) + \bar{\mu}_z \\
&\le \max(z_1-\mu_{z,1},\ldots,z_m-\mu_{z,m}) + \bar{\mu}_z \\
&\le \max(|z_1-\mu_{z,1}|,\ldots,|z_m-\mu_{z,m}|) + \bar{\mu}_z \\
&=: \|z-\mu_z\|_\infty + \bar{\mu}_z \\
&\overset{(1)}{\le} \|z-\mu_z\|_2 + \bar{\mu}_z \ .
\end{aligned} \tag{16}$$

For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, the smallest eigenvalue $\lambda_{\min}(A)$ has a property that $\lambda_{\min}(A)x^Tx \le x^TAx$ for all $x \in \mathbb{R}^n$. If Σ_z is positive definite, so is Σ_z^{-1}; then we have $\lambda_{\min}(\Sigma_z^{-1}) > 0$ and

$$\lambda_{\min}(\Sigma_z^{-1})\|z-\mu_z\|_2 \le (z-\mu_z)^T\Sigma_z^{-1}(z-\mu_z). \tag{17}$$

Combining (16) and (17), we have

$$\begin{aligned}
&\Pr\{\max(z_1,\ldots,z_m) \le 0\} \\
&\ge \Pr\{\|z-\mu_z\|_2 \le -\bar{\mu}_z\} \\
&\ge \Pr\{\lambda_{\min}(\Sigma_z^{-1})\|z-\mu_z\|_2 \le -\lambda_{\min}(\Sigma_z^{-1})\bar{\mu}_z\} \\
&\ge \Pr\{(z-\mu_z)^T\Sigma_z^{-1}(z-\mu_z) \le -\lambda_{\min}(\Sigma_z^{-1})\bar{\mu}_z\}
\end{aligned} \tag{18}$$

From the multidimensional Chebyshev inequality (2), we have

$$\begin{aligned}
&\Pr\{(z-\mu_z)^T\Sigma_z^{-1}(z-\mu_z) \le -\lambda_{\min}(\Sigma_z^{-1})\bar{\mu}_z\} \\
&\ge 1 - \frac{m}{-\bar{\mu}_z\lambda_{\min}(\Sigma_z^{-1})} \ .
\end{aligned} \tag{19}$$

If $\frac{m}{-\bar{\mu}_z\lambda_{\min}(\Sigma_z^{-1})} \le \epsilon$, therefore, from (18) and (19), we have $\Pr\{\max(z_1,\ldots,z_m) \le 0\} \ge 1-\epsilon$. ■

Based on Proposition 7, the chance constraint (12) can be transformed into the following constraint:

$$\mu_z(k) \le -\frac{m}{\epsilon\lambda_{\min}(\Sigma_z^{-1})} \ .$$

By substituting (14), we have

$$\Gamma\tilde{u}(k) \le -\Lambda\mu_w - \Xi(k) - \frac{m}{\epsilon\lambda_{\min}(\Sigma_z^{-1})} \ . \tag{20}$$

Note that this constraint is linear in $\tilde{u}(k)$. Thus the optimal control sequence at step k can be calculated by solving the optimization problem (7)-(9) with the linear constraint (20).

Remark 8: It is important to know that the sufficient condition for this transformation into linear constraints is $\Sigma_z > 0$ (i.e., Σ_z is positive definite). From Assumption 5, Σ_w is positive definite. So from (15), Σ_z is positive definite if and only if Λ is a full-row rank matrix. However, in practice, Λ is not always full-row rank and it can even have zero rows. In that case, an alternative procedure is to separate the zero rows from Λ and to divide the remaining part of Λ into several block matrices along the row dimension , i.e., $\Lambda = \begin{bmatrix}\mathbf{0} & \Lambda_1 & \cdots & \Lambda_s\end{bmatrix}^T$ such that every block matrix Λ_l is full-row rank. Then we have

$$z(k) = \begin{bmatrix} z^0(k) \\ z^1(k) \\ \vdots \\ z^s(k) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \Lambda_1 \\ \vdots \\ \Lambda_s \end{bmatrix} \tilde{w}(k) + \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_s \end{bmatrix} \tilde{u}(k) + \begin{bmatrix} \Xi_0(k) \\ \Xi_1(k) \\ \vdots \\ \Xi_s(k) \end{bmatrix} .$$

On the one hand, if $\Gamma_0\tilde{u}(k) + \Xi_0(k) > 0$, then $\Pr\Big\{ \max_{i=1,\ldots,m}(z_i(k)) \leq 0\Big\} = 0$. On the other hand, if $\Gamma_0\tilde{u}(k) + \Xi_0(k) \leq 0$, then

$$\Pr\Big\{ \max_{i=1,\ldots,m}(z_i(k)) \leq 0\Big\}$$
$$= \Pr\left\{ \begin{bmatrix}\Lambda_1\\ \vdots \\ \Lambda_s\end{bmatrix} \tilde{w}(k) + \begin{bmatrix}\Gamma_1\\ \vdots \\ \Gamma_s\end{bmatrix} \tilde{u}(k) + \begin{bmatrix}\Xi_1(k)\\ \vdots \\ \Xi_s(k)\end{bmatrix} \leq 0 \right\}$$

and the linear constraint (20) becomes

$$\begin{aligned} &\Gamma_0\tilde{u}(k) \leq -\Xi_0(k) \\ &\Gamma_1\tilde{u}(k) \leq -\Lambda_1\mu_w - \Xi_1(k) - \tfrac{ms}{\epsilon\lambda_{\min}(\Sigma_{z,1}^{-1})} \\ &\vdots \\ &\Gamma_s\tilde{u}(k) \leq -\Lambda_s\mu_w - \Xi_s(k) - \tfrac{ms}{\epsilon\lambda_{\min}(\Sigma_{z,s}^{-1})} \end{aligned} \tag{21}$$

with $\Sigma_{z,l} = \Lambda_l\Sigma_w\Lambda_l$, $l = 1, \ldots, s$.

The linear constraints (21) guarantee that

$$\begin{aligned} &\Pr\{z^0(k) > 0\} = 0 \\ &\Pr\{z^1(k) > 0\} \leq \epsilon/s \\ &\vdots \\ &\Pr\{z^s(k) > 0\} \leq \epsilon/s. \end{aligned} \tag{22}$$

Consequently,

$$\sum_{l=1}^{s} \Pr\{z^l(k) > 0\} \leq \epsilon. \tag{23}$$

Similarly to the proof of Proposition 6, according to Boole's inequality, then we have

$$\Pr\Big\{ \max_{i=1,\ldots,m}(z_i(k)) \leq 0\Big\} = 1 - \Pr\Big\{ \max_{i=1,\ldots,m}(z_i(k)) > 0\Big\}$$
$$\geq 1 - \sum_{l=1}^{s} \Pr\{z^l(k) > 0\} \overset{(23)}{\geq} 1 - \epsilon \ .$$

D. Discussion

For Method 1, we need to know the respective distributions of $z_1(k), \ldots, z_m(k)$ instead of the distribution of their maximum; and for Method 2, we need to know the mean vector and covariance matrix of $z(k)$. Based on (11), $z(k)$ is an affine function of $\tilde{w}(k)$. Therefore, to apply the two methods in this paper, we require $\tilde{w}(k)$ to be random variables the distribution of which is preserved or known under summation and multiplication by a scalar, such as the normal distribution, the Poisson distribution, and the gamma distribution [21].

VI. Example

In this section, we consider the production system presented in [11] (see Figure 1 in Section III). The initial state is $x(0) = \begin{bmatrix}0 & 10\end{bmatrix}^T$, $u(0) = 0$, the prediction horizon is chosen as $N_\mathrm{p} = 3$, and the trade-off between the output and input costs is selected as $\lambda = 10^{-5}$. We assume that the processing time of M_1 is perturbed by a random variable $w(k)$: $d_1(k) = 5 + w(k)$ where $w(k)$ has a normal distribution with expected value 0 and variance 2. We consider the chance constraint

$$\Pr\{y(k+j) - r(k+j) \leq h, j = 0, \ldots, N_\mathrm{p} - 1\} \geq 1 - \epsilon$$

which is equivalent to

$$\Pr\Big\{ \max_{i=1,\ldots,19}(z_i(k)) \leq 0\Big\} \geq 1 - \epsilon \ .$$

We consider two different cases: (i) $r(k) = 10 + 30 \cdot k$, $\epsilon = 0.5$, $h = 20$; (ii) $r(k) = 10 + 65 \cdot k$, $\epsilon = 0.2$, $h = 50$. The Boole method (Method 1) and the Chebyshev method (Method 2) developed in Section V are applied to deal with the chance constraint and compared with two other methods: the MC simulation method and the nominal MPC method. For each case, we solve the chance-constrained MPC problem (7)-(10) in closed loop for $k = 1, \ldots, 50$ and run the experiment 10 times, each time with a different realization of w. For each round, the same realization is used for all methods. Table I lists the mean computation time and the mean closed-loop costs over the 10 realizations. The closed-loop costs are computed as $J_\mathrm{clp} = \sum_{k=1}^{50}\big(\max(y(k) - r(k), 0) - \lambda u(k)\big)$.

The nominal MPC method consists in computing the optimal control sequence by using the deterministic MPL model as the prediction model and considering deterministic linear constraints. The MC simulation method consists in approximating $E[J(k)]$ and the chance constraint on the basis of a large number of random samples. When using the Boole method or the Chebyshev method to deal with the chance constraint, we consider two different ways to compute the value of $E[J(k)]$, namely, MC integration and MC simulation.

From Table I, we can see that for both cases, although the nominal MPC method is faster than the other methods, it yields higher closed-loop costs. The MC simulation method generally achieves the lowest closed-loop costs, but it takes a longer computation time,

When using the Boole method or the Chebyshev method to deal with the chance constraint, using MC simulation for computing $E[J(k)]$ is better than using MC integration in terms of computation time. Moreover, given the same number of samples, compared with only using MC simulation, the computation time of the combination of the Boole method and MC simulation decreases by about 30% and the computation time of the combination of the Chebyshev method and MC simulation decreases by about 70%.

VII. Conclusions

We have considered the chance-constrained MPC problem for stochastic max-plus linear systems and developed two methods to deal with the chance constraints. Method 1 converts the chance constraint into several univariate constraints by applying Boole's inequality. Method 2 uses Chebyshev's inequality and transforms the chance constraint into linear constraints on the control inputs. The two methods are assessed with a production system and compared with two other methods: Monte Carlo simulation and nominal MPC. The results show that the two methods are faster than Monte Carlo simulation to achieve a similar performance and yield a better performance than nominal MPC.

In the future, one possible improvement of Method 2 is to find some optimal way to allocate the probability level ϵ of constraint violation to each inequalities in (21) (note that in the current paper ϵ is allocated uniformly).

TABLE I

The computation time and closed-loop costs J_{clp} using different methods (The number following MC simulation (MCsim) and MC integration (MCint) indicates the number of random samples used)

Case (i): $r(k) = 10 + 30 \cdot k$, $\epsilon = 0.5$, $h = 20$					
Methods				Time [s]	J_{clp}
Constraint		$\mathbb{E}[J(k)]$			
Nominal MPC		Nominal MPC		1.3	29.6440
MCsim	10^3	MCsim	10^3	447	-0.3196
	$5 \cdot 10^3$		$5 \cdot 10^3$	2117	-0.3817
	10^4		10^4	4245	-0.3816
Boole		MCint	$6 \cdot 10^5$	897	0.8237
			10^6	1399	-0.3667
			$2 \cdot 10^6$	2781	-0.3807
		MCsim	10^3	310	-0.3196
			$5 \cdot 10^3$	1342	-0.3817
			10^4	2829	-0.3816
Chebyshev		MCint	$6 \cdot 10^5$	1212	-0.3614
			10^6	1646	-0.3813
			$2 \cdot 10^6$	3328	-0.3809
		MCsim	10^3	146	-0.3195
			$5 \cdot 10^3$	639	-0.3817
			10^4	1228	-0.3816

Case (ii): $r(k) = 10 + 75 \cdot k$, $\epsilon = 0.2$, $h = 50$					
Methods				Time [s]	J_{clp}
Constraint		$\mathbb{E}[J(k)]$			
Nominal MPC		Nominal MPC		1.3	29.6440
MCsim	10^3	MCsim	10^3	456	-0.8933
	$5 \cdot 10^3$		$5 \cdot 10^3$	2288	-0.9554
	10^4		10^4	4222	-0.9553
Boole		MCint	$6 \cdot 10^5$	939	-0.1153
			10^6	1513	-0.4710
			$2 \cdot 10^6$	3087	-0.9539
		MCsim	10^3	313	-0.8933
			$5 \cdot 10^3$	1380	-0.9554
			10^4	2761	-0.9553
Chebyshev		MCint	$6 \cdot 10^5$	1199	-0.4124
			10^6	1789	-0.9480
			$2 \cdot 10^6$	3821	-0.9542
		MCsim	10^3	151	-0.8933
			$5 \cdot 10^3$	686	-0.9554
			10^4	1205	-0.9553

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