Delft University of Technology

Delft Center for Systems and Control

Technical report 18-022

Stabilizing controller design for state-based switching bilinear systems*

S. Lin, B. De Schutter, and D. Li

If you want to cite this report, please use the following reference instead:
S. Lin, B. De Schutter, and D. Li, "Stabilizing controller design for state-based switching bilinear systems," *Proceedings of the 57th IEEE Conference on Decision and Control*, Miami Beach, Florida, pp. 6452–6457, Dec. 2018. doi:10.1109/CDC.2018.8618956

Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

Stabilizing Controller Design for State-based Switching Bilinear Systems

Shu Lin, Bart De Schutter, and Dewei Li

Abstract— **In this paper, controllers are designed for stabilizing state-based switching bilinear systems. The controller is designed based on the special features of bilinear systems comparing with linear systems, and is carried out through three steps: first, deriving the state-based switching linear system corresponding to the switching bilinear system; then, state-feedback controllers are designed to asymptotically stabilize the corresponding switching linear system; finally, stabilizing controllers are obtained for the original system, the switching bilinear system. The stability of the controller is proved step by step through the decreasing of the overall Lyapunov function.**

I. INTRODUCTION

Bilinear systems are a kind of special nonlinear system that have been investigated a lot since 1960s [1], [2], [3], [4], [5], [6]. They are very simple and close to linear systems, which makes it possible to extend some of the theory results for linear systems to be used in nonlinear systems. A bilinear system is actually the addition of a linear term and a bilinear term. Due to the existence of the bilinear term, the structure of the system can be changed compared with the linear systems. Because of the variable structure of the bilinear systems, some uncontrollable linear systems may become controllable by simply adding the bilinear term. It has been proved that the bilinear system had a better performance than the linear system in optimal control [7]. In practice, there are systems that naturally have the term with the states multiplying the control inputs, such as the field of sociology, biology, power systems, etc.[1], [8]. Usually, the reason for the existence of the term is that the influence of the control input on the system depends on the current system state, which is normal in reality.

In practice, some complex nonlinear systems can be approximated by dividing into multiple state-based bilinear subsystems [9]. On each state region, a bilinear subsystem is activated, and the bilinear subsystems switch between each other according to the switching of the state regions. This results in a state-based switching bilinear system [10]. Developing the theory on stabilizing controllers for the state-based switching bilinear systems provides a methodology to design controllers for the systems with complex nonlinear features in reality.

S. Lin is with School of Computer and Control Engineering, University of Chinese Academy of Sciences, Beijing, China, and Key Laboratory of System Control and Information Processing, Ministry of Education, Shanghai slin@ucas.ac.cn

B. De Schutter is with Delft Center for Systems and Control, Delft University of Technology, Delft, Netherlands b.deschutter@tudelft.nl

D. Li is with Department of Automation, Shanghai Jiao Tong University dwli@sjtu.edu.cn

In this paper, a method is proposed for designing controllers that stabilize the state-based switching bilinear systems. Considering about the special features of bilinear systems, a state-based switching linear system is obtained corresponding to the switching bilinear system. Instead of designing controllers for the switching bilinear system directly, we first proposed a theorem to asymptotically stabilize the corresponding switching linear system based on multiple Lyapunov functions, and the theorem is further relaxed through allowing the Lyapunov functions jumping on the boundary of the neighboring state regions. Then, the stabilizing controller for switching bilinear systems can be obtained by checking the feasibilities of the switchings on some of the boundaries, based on the results yielding from the corresponding switching linear systems.

II. PROBLEM STATEMENT

Consider a Switching Bilinear System (SBLS)

$$\dot{x} = A_i x + \sum_{j=1}^{m_i} (G_{i,j} x + b_{i,j}) u_{i,j}, \text{ if } x \in \Omega_i, i \in \Lambda, \tag{1}$$

where A_i and $G_{i,j}$ are $[n \times n]$ matrices, $b_{i,j}$ is a $[n \times 1]$ vector, Ω_i is the corresponding state space polyhedron with $i \in \Lambda$ the state space partitions of $\Omega \subset \mathbb{R}^n$ $(\cup_{i\in\Lambda}\Omega_i = \Omega, \Omega_i \neq \emptyset, \forall i \in \Lambda)$, $j \in M_i = \{1, \ldots, m_i\}$, and $U_i = [u_{i,1}\ u_{i,2} \cdots u_{i,m_i}] \in \mathbb{R}^{m_i}$ is an m_i-dimensional control input.

We can also write (1) in the form of

$$\dot{x} = A_i x + \sum_{j=1}^{m_i} b_{i,j} (c_{i,j}^{\mathrm{T}} x + 1) u_{i,j}, \text{ if } x \in \Omega_i, i \in \Lambda, \tag{2}$$

if matrix $G_{i,j}$ can be expressed as the inner product of two vectors. In particular, if the rank of matrix $G_{i,j}$ satisfies

$$\text{rank}(G_{i,j}) = 1, \tag{3}$$

then matrix $G_{i,j}$ can be written as

$$G_{i,j} = b_{i,j} c_{i,j}^{\mathrm{T}}, \text{ if } i \in \Lambda, j \in M_i. \tag{4}$$

In practice, it is easy to find systems with this property, e.g. if each of the control inputs $u_{i,j} (j \in M_i, i \in \Lambda)$ is only related with a single element x_i $(i = \{1, \ldots, n\})$ of the system state, which means the bilinear terms in (1) only exist for the local control inputs and states having direct correlations (indirect correlations are taken into consideration through the dynamic model). This assumption holds for flow dynamic systems, such as water networks, traffic networks, power networks, etc.

III. PRELIMINARIES

Based on the features of bilinear systems, specific controllers were designed for bilinear systems [1]. When $\Lambda = \{1\}$ in (2), the system turns into a typical bilinear system. The mark i for distinguishing subsystems is removed to represent single bilinear system. According to [3], it is possible to design a division controller to stabilize the bilinear system. Define $e_j(x) = c_j^{\mathrm{T}}x + 1 (j \in M)$, and divide the state space into the following sets:

$$\begin{cases} S_j^+ = \{x | e_j(x) \geq \varepsilon\} \\ S_j^0 = \{x | -\varepsilon \leq e_j(x) \leq \varepsilon\} \\ S_j^- = \{x | e_j(x) \leq -\varepsilon\}, \end{cases} \tag{5}$$

with ε a small positive value. The division controller can be designed for the regions as:

$$\begin{cases} u_j^+ = \frac{k_j^+ x + u_j^{\mathrm{ref}}}{e_j(x)}, & x \in S_j^+ \\ u_j^0 = 0, & x \in S_j^0 \\ u_j^- = \frac{k_j^- x + u_j^{\mathrm{ref}}}{e_j(x)}, & x \in S_j^-, \end{cases} \tag{6}$$

where k_j^+ and k_j^- are of $[1 \times n]$ state-feedback gain vectors, and u_j^{ref} is the input reference to change the equilibrium point.

The division controller stabilizes the bilinear system to the predefined equilibrium state.

IV. STABILIZING CONTROLLER DESIGN FOR SBLS

Consider the switching bilinear system in (2), it is possible to design stabilizing controller for the SBLS based on the division controllers above. The controller design can be carried out through multiple steps:

1) Obtain the corresponding Switching Linear System (SLS) for the SBLS;
2) Design a stabilizing state-feedback controller for the derived SLS;
3) Design a stabilizing controller for the SBLS.

A. Corresponding Switching Linear System

For switching bilinear systems, in order to design stabilizing switching division controllers for each bilinear subsystem $i \in \Lambda$, we need to partition the state-space polyhedron Ω_i into more subregions. If for sub-bilinear system $i \in \Lambda$, the control input is $u_i^j (i \in \Lambda, j \in M_i)$ where $M_i = \{1, \ldots, m_i\}$, then for each control input u_i^j two state-feedback controllers should be designed separately for two regions as

$$\begin{cases} S_{i,j}^+ = \{x | e_{i,j}(x) \geq 0\} \\ S_{i,j}^- = \{x | e_{i,j}(x) \leq 0\}, \end{cases} \tag{7}$$

with $e_{i,j}(x) = c_{i,j}^{\mathrm{T}}x + 1$ $(j \in M)$, and thus there will be at most 2^{m_i} state subspace partitions in Ω_i. The polyhedral partition of Ω_i $(i \in \Lambda)$ for bilinear subsystem i can be defined as $\{\Omega_{i,l}\}_{i \in \Lambda, l \in \Gamma_i}$, where $\cup_{l \in \Gamma_i} \Omega_{i,l} = \Omega_i, \Omega_{i,l} \neq \emptyset, \Omega_{i,l_1} \cap \Omega_{i,l_2} \neq \emptyset, \forall l_1 \neq l_2, l_1, l_2 \in \Gamma_i$.

Based on the polyhedral partition of the state space and defining the equilibrium as the origin, the controller is designed for each polyhedron $\Omega_{i,l}$ as

$$U_{i,l} = [u_{i,l}^1 \ u_{i,l}^2 \cdots u_{i,l}^{m_i}], \quad i \in \Lambda, l \in \Gamma_i, M_i = \{m_i\}, \tag{8}$$

where

$$u_{i,l}^j = \frac{k_{i,l}^j x}{e_{i,j}(x)}, \quad \text{if } x \in \Omega_{i,l}, j \in M_i, l \in \Gamma_i, i \in \Lambda. \tag{9}$$

If we substitute (9) into (2), then the bilinear terms are eliminated, and the bilinear system in (2) becomes a switching linear system, which is the corresponding SLS of the SBLS. In order to control the SBLS, we can first consider design a stabilizing state-feedback controller for the following corresponding SLS

$$\dot{x} = (A_i + \sum_{j=1}^{m_i} b_{i,j} k_{i,l}^j) x, \quad \text{if } x \in \Omega_{i,l}, j \in M_i, l \in \Gamma_i, i \in \Lambda. \tag{10}$$

Define

$$\begin{aligned} B_i &= [b_{i,1} \ b_{i,2} \cdots b_{i,m_i}], \\ K_{i,l} &= [(k_{i,l}^1)^{\mathrm{T}} \ (k_{i,l}^2)^{\mathrm{T}} \cdots (k_{i,l}^{m_i})^{\mathrm{T}}]^{\mathrm{T}}, \end{aligned} \tag{11}$$

then the corresponding SLS system can be written as

$$\dot{x} = (A_i + B_i K_{i,l}) x, \quad \text{if } x \in \Omega_{i,l}, j \in M_i, l \in \Gamma_i, i \in \Lambda. \tag{12}$$

B. Stabilizing State-feedback Control for SLS

Now, stabilizing switching state-feedback control laws will be designed to asymptotically steer the state to the origin for the SLS in (12) [11], [12], [9]. The method in this paper follows the theory in [9], but has a larger feasible region by relaxing the LMI constraints. In addition, this paper aims at designing stabilizing controller for SBL systems which are common in flow systems.

Each polyhedral region $\Omega_{i,l}$ can be described as a system of linear inequalities:

$$\underbrace{[F_{i,l} \quad f_{i,l}]}_{\bar{F}_{i,l}} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0, \quad \text{if } x \in \Omega_{i,l}, \tag{13}$$

and the boundary hyperplane for two neighboring regions $\Omega_{i,l}$ and $\Omega_{i',l'}$ is characterized by an equality and inequality as

$$\underbrace{[\Psi_{ii',ll'} \quad \psi_{ii',ll'}]}_{\bar{\Psi}_{ii',ll'}} \begin{bmatrix} x \\ 1 \end{bmatrix} = 0, \text{ and } \underbrace{[\Phi_{ii',ll'} \quad \phi_{i',ll'}]}_{\bar{\Phi}_{ii',ll'}} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0,$$
$$\forall \ x \in \Omega_{i,l} \cap \Omega_{i',l'}. \tag{14}$$

Lyapunov functions are defined for each polyhedral region $\Omega_{i,l}$ $(l \in \Gamma_i, i \in \Lambda)$ with the following format

$$V_{i,l}(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix} P_{i,l} & \star \\ s_{i,l}^{\mathrm{T}} & r_{i,l} \end{bmatrix}}_{\bar{P}_{i,l}} \underbrace{\begin{bmatrix} x \\ 1 \end{bmatrix}}_{\bar{x}}, \quad \forall \, l \in \Gamma_i, i \in \Lambda, x \in \Omega_{i,l}, \tag{15}$$

with $\bar{x} = [x\ 1]^{\mathrm{T}}$, $P_{i,l} \in \mathbb{R}^{n\times n}$ a symmetric matrix, $s_{i,l}$ an $[n\times 1]$ dimensional vector, and $r_{i,l} \in \mathbb{R}$.

The following theorem presents a sufficient condition to design switched state-feedback control laws for the SLS in (12) that, to asymptotically bring the state to the origin, which is the equilibrium for at least one of the subsystems.

Note that in the following theorems we use the augmented system matrices defined as follows:

$$\bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix},\ \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}. \quad (16)$$

Theorem 1: Assume there exist positive definite matrices $\bar{Q}_{i,l}$, $Q_{i,l}$, $R_{i,l}$, and $M_{i,l}$, and matrix $\Theta_{i,l}$ and scalar $\lambda_{i,l}$ that satisfy (21)-(26), taking

$$\bar{P}_{i,l} = \bar{Q}_{i,l}^{-1} \quad \forall\ (i,l) \in \{(i,l) \mid i\in\Lambda, l\in\Gamma_i, 0\notin\Omega_{i,l}\}, \quad (17)$$

and

$$P_{i,l} = Q_{i,l}^{-1} \quad \forall\ (i,l) \in \{(i,l) \mid i\in\Lambda, l\in\Gamma_i, 0\in\Omega_{i,l}\}, \quad (18)$$

for the state-based Lyapunov functions $V_{i,l} = \bar{x}^{\mathrm{T}}\bar{P}_{i,l}\bar{x}$ and $V_{i,l} = x^{\mathrm{T}}P_{i,l}x$ respectively, then the state feedback control laws with gains

$$\bar{K}_{i,l} = N_{i,l}\bar{Q}_{i,l}^{-1} \quad \forall\ (i,l) \in \{(i,l) \mid i\in\Lambda, l\in\Gamma_i, 0\notin\Omega_{i,l}\}, \quad (19)$$

and

$$K_{i,l} = N_{i,l}Q_{i,l}^{-1} \quad \forall\ (i,l) \in \{(i,l) \mid i\in\Lambda, l\in\Gamma_i, 0\in\Omega_{i,l}\}, \quad (20)$$

asymptotically stabilize the SLS system in (12).

Proof: First, the Schur complement is performed on (21) with respect to the second row and column. The result is multiplied from both sides by $\bar{Q}_{i,l}^{-1} = \bar{P}_{i,l}$, yielding

$$\bar{P}_{i,l} - \bar{F}_{i,l}^{\mathrm{T}}R_{i,l}^{-1}\bar{F}_{i,l} > 0, \qquad \forall\ (i,l) \in \{(i,l) \mid i\in\Lambda, l\in\Gamma_i, 0\notin\Omega_{i,l}\}, \quad (27)$$

which guarantees that the Lyapunov function on each state polyhedron is positive, i.e.

$$V_{i,l} > 0, \quad \text{if } \bar{F}_{i,l}\bar{x}_{i,l} \geq 0 \text{ and } \bar{x}_{i,l} \neq 0, \qquad \forall\ (i,l) \in \{(i,l) \mid i\in\Lambda, l\in\Gamma_i, 0\notin\Omega_{i,l}\}. \quad (28)$$

Second, the Schur complement is performed on (23) with respect to the second row and column. The result is multiplied from both sides by $\bar{Q}_{i,l}^{-1} = \bar{P}_{i,l}$, and (19) is used, then we obtain

$$\bar{P}_{i,l}(\bar{A}_i + \bar{B}_i\bar{K}_{i,l}) + (\bar{A}_i + \bar{B}_i\bar{K}_{i,l})^{\mathrm{T}}\bar{P}_{i,l} + \bar{F}_{i,l}^{\mathrm{T}}M_{i,l}^{-1}\bar{F}_{i,l} < 0, \qquad \forall\ (i,l) \in \{(i,l) \mid i\in\Lambda, l\in\Gamma_i, 0\notin\Omega_{i,l}\}, \quad (29)$$

which guarantees that the derivative of Lyapunov function on each state polyhedron is negative, i.e.

$$\dot{V}_{i,l} < 0, \quad \text{if } \bar{F}_{i,l}\bar{x}_{i,l} \geq 0 \text{ and } \bar{x}_{i,l} \neq 0, \qquad \forall\ (i,l) \in \{(i,l) \mid i\in\Lambda, l\in\Gamma_i, 0\notin\Omega_{i,l}\}. \quad (30)$$

Note that in the case for the subsystem containing the origin, i.e. for the polyhedron with $0\in\Omega_{i,l}$, the LMIs in (22) and (24) are applied to make sure a positive Lyapunov function and a negative derivative of Lyapunov function on the region. The same conditions are considered as in (21) and (23), but only the row and column corresponding to the augmented variable are removed here, to guarantee that the derivative of the Lyapunov function $\dot{V}_{i,l}$ would be zero only when the state x is zero. The augmented $\bar{Q}_{i,l}$ is defined for polyhedron $\Omega_{i,l}$ with the origin in (22) to make it comparable on the boundary conditions.

At last, we perform the Schur complement on (25) for 3 times, each time with respect to the last row and column. Similarly, we multiply the result from both sides by $\bar{Q}_{i,l}^{-1} = \bar{P}_{i,l}$, and use (19), then we obtain

$$\bar{P}_{i,l} - \bar{P}_{i',l'} + \lambda_{ii',ll'}^{-1}\bar{\Psi}_{ii',ll'}^{\mathrm{T}}\bar{\Psi}_{ii',ll'} + \bar{\Phi}_{ii',ll'}^{\mathrm{T}}\Theta_{ii',ll'}^{-1}\bar{\Phi}_{ii',ll'} \geq 0, \qquad \forall\ i,i'\in\Lambda, l\in\Gamma_i, l'\in\Gamma_{i'} : \Omega_{i,l}\cap\Omega_{i',l'} \neq \emptyset. \quad (31)$$

Define the states on the boundary of polyhedra $\Omega_{i,l}$ and $\Omega_{i',l'}$ can be described by an equality and inequality as

$$\mathscr{S}_{ii',ll'} = \{\bar{x} \mid \bar{\Psi}_{ii',ll'}\bar{x} = 0 \wedge \bar{\Phi}_{ii',ll'}\bar{x} \geq 0\}. \quad (32)$$

By multiplying the augmented states on both sides, and applying Finsler's Lemma [13], we have

$$V_{i,l} \geq V_{i',l'}, \quad \text{if } \bar{x} \in \mathscr{S}_{ii',ll'}, \qquad \forall\ i,i'\in\Lambda, l\in\Gamma_i, l'\in\Gamma_{i'} : \Omega_{i,l}\cap\Omega_{i',l'} \neq \emptyset. \quad (33)$$

The same procedure is applied to (26), which yields

$$\bar{P}_{i,l} - \bar{P}_{i',l'} + \lambda_{ii',ll'}^{-1}\bar{\Psi}_{ii',ll'}^{\mathrm{T}}\bar{\Psi}_{ii',ll'} + \bar{\Phi}_{ii',ll'}^{\mathrm{T}}\Theta_{ii',ll'}^{-1}\bar{\Phi}_{ii',ll'} \leq 0, \qquad \forall\ i,i'\in\Lambda, l\in\Gamma_i, l'\in\Gamma_{i'} : \Omega_{i,l}\cap\Omega_{i',l'} \neq \emptyset. \quad (34)$$

By multiplying the augmented states on both sides, and applying Finsler's Lemma, we obtain

$$V_{i,l} \leq V_{i',l'}, \quad \text{if } \bar{x} \in \mathscr{S}_{ii',ll'}, \qquad \forall\ i,i'\in\Lambda, l\in\Gamma_i, l'\in\Gamma_{i'} : \Omega_{i,l}\cap\Omega_{i',l'} \neq \emptyset. \quad (35)$$

According to (33) and (35), the values of the Lyapunov functions are equal to each other ($V_{i,l} = V_{i',l'}$) on the boundary of neighboring polyhedra $\Omega_{i,l}$ and $\Omega_{i',l'}$.

Consequently, by adopting the state feedback control laws to all of the polyhedral regions, with

$$U_{i,l} = \bar{K}_{i,l}\bar{x} \quad \forall\ \bar{x} \in \{\bar{x} \mid x\in\Omega_{i,l}, 0\notin\Omega_{i,l}, i\in\Lambda, l\in\Gamma_i\}, \quad (36)$$

and

$$U_{i,l} = K_{i,l}x \quad \forall\ x \in \{x \mid x\in\Omega_{i,l}, 0\in\Omega_{i,l}, i\in\Lambda, l\in\Gamma_i\}, \quad (37)$$

a positive decreasing overall Lyapunov function is guaranteed with continuous values over the boundaries between the polyhedral regions and, therefore the SLS system in (12) can be asymptotically stabilized. □

Considering (25) and (26) could be too conservative: it may result in infeasible solutions to the inequalities (21)-(26) due to the tight constraints that require the Lyapunov functions to be equal on the boundaries of the state regions. Therefore, the following theorem is proposed in which constraint (25) and (26) are removed, and a relaxed condition regarding the reduction of the Lyapunov functions on the boundaries of the state polyhedra is considered instead.

$$\begin{bmatrix} \bar{Q}_{i,l} & \star \\ \bar{F}_{i,l}\bar{Q}_{i,l} & R_{i,l} \end{bmatrix} > 0, \quad \forall\ (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{21}$$

$$\begin{bmatrix} Q_{i,l} & \star \\ F_{i,l}Q_{i,l} & R_{i,l} \end{bmatrix} > 0,\ \bar{Q}_{i,l} = \begin{bmatrix} Q_{i,l} & \star \\ 0 & q_{i,l} \end{bmatrix} > 0, \quad \forall\ (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \in \Omega_{i,l}\}, \tag{22}$$

$$\begin{bmatrix} \bar{A}_i\bar{Q}_{i,l} + \bar{Q}_{i,l}\bar{A}_i^{\mathrm{T}} + \bar{B}_iN_{i,l} + N_{i,l}^{\mathrm{T}}\bar{B}_i^{\mathrm{T}} & \star \\ \bar{F}_{i,l}\bar{Q}_{i,l} & -M_{i,l} \end{bmatrix} < 0, \quad \forall\ (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{23}$$

$$\begin{bmatrix} A_iQ_{i,l} + Q_{i,l}A_i^{\mathrm{T}} + B_iN_{i,l} + N_{i,l}^{\mathrm{T}}B_i^{\mathrm{T}} & \star \\ F_{i,l}\bar{Q}_{i,l} & -M_{i,l} \end{bmatrix} < 0, \quad \forall\ (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \in \Omega_{i,l}\}, \tag{24}$$

$$\begin{bmatrix} \bar{Q}_{i,l} & \star & \star & \star \\ \bar{Q}_{i,l} & \bar{Q}_{i',l'} & \star & \star \\ \bar{\Psi}_{ii',ll'}\bar{Q}_{i,l} & 0 & -\lambda_{ii',ll'} & \star \\ \bar{\Phi}_{ii',ll'}\bar{Q}_{i,l} & 0 & 0 & -\Theta_{ii',ll'} \end{bmatrix} \geq 0, \quad \forall\ i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} : \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset \tag{25}$$

$$\begin{bmatrix} \bar{Q}_{i',l'} & \star & \star & \star \\ \bar{Q}_{i',l'} & \bar{Q}_{i,l} & \star & \star \\ \bar{\Psi}_{ii',ll'}\bar{Q}_{i',l'} & 0 & \lambda_{ii',ll'} & \star \\ \bar{\Phi}_{ii',ll'}\bar{Q}_{i',l'} & 0 & 0 & \Theta_{ii',ll'} \end{bmatrix} \geq 0, \quad \forall\ i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} : \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset \tag{26}$$

Define $x_{i,l}$ is the point in the state polyhedron $\Omega_{i,l}$ that is closest to the origin, and $d_{i,l}$ is the distance from $x_{i,l}$ to the origin.

Theorem 2: Assume there exist positive definite matrices $\bar{Q}_{i,l}$, $Q_{i,l}$, $R_{i,l}$, and $M_{i,l}$, and matrix $\Theta_{i,l}$ and scalar $\lambda_{i,l}$ that satisfy (21)-(24), and the inequalities in (39) is satisfied, then taking (17) and (18) as the state-based Lyapunov functions, the state feedback control laws with gains given in (19) and (20) asymptotically stabilize the SLS system in (12).

Proof: To prove the conclusion, we first prove that the smallest Lyapunov function value on a state polyhedron is on the limited boundaries (i.e. inner boundaries with the neighboring polyhedra, not the unlimited boundary tending to infinity) of the polyhedron without origin; then, based on the inequality constraints on the boundaries of the polyhedra, we prove that there exists a decreasing sequence of Lyapunov functions converging to the origin, corresponding to the sequence of state polyhedral switchings.

First, it is easy to prove that, except for the polyhedron with origin, the smallest Lyapunov function on a state polyhedron exists always on the limited boundary of the polyhedron. Suppose the smallest Lyapunov function exists inside polyhedron $\Omega_{i,l}$ without origin, then there must be a $x_{i,l}^*$ such that $\dot{V}_{i,l}(x_{i,l}^*) = 0$, which contradicts with the condition $\dot{V}_{i,l} < 0$ on $\Omega_{i,l}$. Moreover, if the smallest Lyapunov function exists on the unlimited boundary of polyhedron $\Omega_{i,l}$, since $\dot{V}_{i,l} < 0$, then $V_{i,l}(x_{i,l}^*) \mid_{x_{i,l}^* \to \infty} < 0$, which contradicts with the condition $V_{i,l} > 0$.

Second, when the boundary condition (39) is satisfied, the following inequalities are guaranteed:

$$\begin{aligned} &\bar{P}_{i,l} - \bar{P}_{i',l'} + \lambda_{ii',ll'}^{-1}\bar{\Psi}_{ii',ll'}^{\mathrm{T}}\bar{\Psi}_{ii',ll'} + \bar{\Phi}_{ii',ll'}^{\mathrm{T}}\Theta_{ii',ll'}^{-1}\bar{\Phi}_{ii',ll'} > 0, \\ &\qquad \text{if } d_{i,j} > d_{i',j'}, \text{ and } \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset, \\ &\qquad \forall i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'}, \end{aligned} \tag{40}$$

which ensures that $V_{i,l} \geq V_{i',l'}$ for all the states $\bar{x} \in \mathscr{S}_{ii',ll'}$ on the boundary of $\Omega_{i,l}$ and $\Omega_{i',l'}$.

Third, since the smallest Lyapunov function exists always on the limited boundary of a polyhedron without origin, we can suppose that the smallest Lyapunov function of polyhedron $\Omega_{i,l}$ appears on the boundary of $\Omega_{i,l}$ and $\Omega_{i',l'}$, i.e. $x_{i,l}^* \in \Omega_{i,l} \cap \Omega_{i',l'}$. In addition, since we have $V_{i,l}(x) > V_{i',l'}(x)$ on $\mathscr{S}_{ii',ll'}$, then we have $V_{i,l}(x_{i,l}^*) > V_{i',l'}(x)$ on $\mathscr{S}_{ii',ll'}$. Moreover, it is obvious that $V_{i',l'}(x) \geq V_{i',l'}(x_{i',l'}^*)$ on $\Omega_{i',l'}$. Therefore, we obtain $V_{i,l}(x_{i,l}^*) > V_{i',l'}(x_{i',l'}^*)$.

Finally, if a feasible solution exists for Theorem 2, there must exist a sequence of reducing polyhedra distances connecting all the polyhedra in Ω to form a path to the origin, that satisfies

$$d_p \geq d_{p-1} \geq \cdots \geq d_1 \geq 0, \tag{41}$$

with p as the number of polyhedron $\Omega_{i,l}$, $\forall\ i \in \Lambda, l \in \Gamma_i$ in Ω, which is corresponding to a sequence of decreasing minimal Lyapunov functions for all the polyhedra as

$$V_p(x_p^*) \geq V_{p-1}(x_{p-1}^*) \geq \cdots \geq V_1(x_1^*) \geq 0, \tag{42}$$

that guarantees to steer state asymptotically converging to the origin, from an initial state x_0 within any of the polyhedra in Ω. □

C. Stabilizing Controller Design for SBLS

For the SBLS, when $e_{i,j}(x) = 0$ on some of the boundaries of the subregions, the controller in (9) becomes infinite, which is not feasible. On the basis of the stabilizing state-feedback control for the corresponding SLS, we need to further check the feasibility on the boundaries of the SBLS to make it stabilized. Therefore, according to Sec. III, we

$$\begin{bmatrix} \bar{Q}_{i,l} & \star & \star & \star \\ \bar{Q}_{i,l} & \bar{Q}_{i',l'} & \star & \star \\ \bar{\Psi}_{ii',ll'}\bar{Q}_{i,l} & 0 & -\lambda_{ii',ll'} & \star \\ \bar{\Phi}_{ii',ll'}\bar{Q}_{i,l} & 0 & 0 & -\Theta_{ii',ll'} \end{bmatrix} > 0, \quad \text{if } d_{i,j} > d_{i',j'}, \text{ and } \{\Omega_{i,l} \cap \Omega_{i',l'}\} \neq \emptyset, \ \forall i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} \tag{39}$$

need to adjust the division of the system into

$$\begin{cases} S^{+}_{i,j} = \{x | e_{i,j}(x) \geq \varepsilon_{i,j}\} \\ S^{0}_{i,j} = \{x || e_{i,j}(x)| \leq \varepsilon_{i,j}\} \\ S^{-}_{i,j} = \{x | e_{i,j}(x) \leq \varepsilon_{i,j}\}, \end{cases}, \quad \forall i \in \Lambda, j \in M_i. \tag{43}$$

where $\varepsilon_{i,j}$ is a very small positive value. Based on the state space partition in Sec. IV-A, the controller design for the SBLS can be written as

$$\begin{cases} u^{j}_{i,l} = \frac{k^{j}_{i,l}x}{e_{i,j}(x)}, \ \forall\ x \in \Omega_{i,l} \setminus S^{0}_{i,j}: \ l \neq l', l, l' \in \Gamma_i, \\ \qquad j \in M_i, i \in \Lambda, \\ u^{j}_{i,ll'} = 0, \ \forall\ x \in (\Omega_{i,l} \cap S^{0}_{i,j}) \cup (\Omega_{i,l'} \cap S^{0}_{i,j}): \\ \qquad j \in M_i, l \neq l', l, l' \in \Gamma_i, i \in \Lambda, \end{cases} \tag{44}$$

where the state-feedback controllers are designed the same as the controllers for the corresponding SLS within the state polyhedra, however when $e_{i,j}(x) = 0$, no control input can influence the system states, thus only an autonomous system exists on these boundaries of the bilinear subsystems. If the movement of the state trajectory for the autonomous system does not contradict with the converging sequence obtained in Theorem 2 on these boundaries, then the SBLS is stabilized.

Theorem 3: For a SBLS as in (1), assume it has a corresponding SLS with the form of (12), and the corresponding SLS system can be stabilized by the state-feedback controller designed according to Theorem 2, then the SBLS can be asymptotically stabilized by the controller in (44), if along the state-based switching sequence, at each switching from polyhedron $\Omega_{i,l}$ to $\Omega_{i,l'}$ for any bilinear subsystem i, when $u^{j}_{i,ll'} = 0, \ \forall\ j \in M_i$, it holds

$$e^{A_i t}x \in \Omega_{i,l'} \setminus S^{0}_{i,j}, \quad \exists\ x \in (\Omega_{i,l} \cap S^{0}_{i,j}) \cup (\Omega_{i,l'} \cap S^{0}_{i,j}), \tag{45}$$

which means there exist initial states on the boundary of $\Omega_{i,l}$ and $\Omega_{i,l'}$ that can reach set $\Omega_{i,l'} \setminus S^{0}_{i,j}$ without control inputs.

Proof: It is obvious that along the state-based switching sequence with asymptotically reduced Lyapunov functions designed for the corresponding SLS, if for all the switching boundaries with $e_{i,j}(x) = 0$ inside a bilinear subsystem, the system state can always transit from the previous state polyhedron to the next state polyhedron along the switching sequence to the origin even without control inputs on the boundaries, then the SBLS is asymptotically stabilized. □

V. EXAMPLE

In this section, an example is presented to evaluate the performance of the controller designed for a SBLS based on Theorem 3.

In the example, we use the conditions presented in Theorem 3 to design stabilized control laws. We directly use the SBLS model in (2) with the following vectors and matrices:

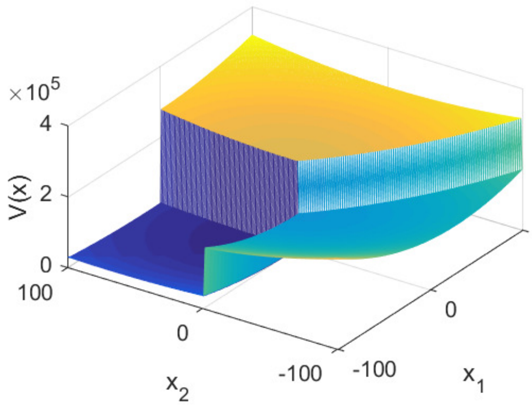

Fig. 1. Illustration for the overall Lyapunov function

$$A_1 = \begin{bmatrix} -3 & 1 \\ -5 & -8 \end{bmatrix}, \ b_{1,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ c_{1,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix}, \ b_{2,1} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \ c_{2,1} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

There are 2 bilinear subsystems separated by $x_1 - x_2 = 0$. According to Sec. IV-A, the state space is partitioned into 4 regions with $\Lambda = \{1,2\}$ and $\Gamma_1 = \{1,2\}, \Gamma_2 = \{1,2\}$. Then, the parameters for the obtained corresponding SLS with the format in (12) are with the following parameters:

$$A_1 = \begin{bmatrix} -3 & 1 \\ -5 & -8 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$F_{1,1} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \ \bar{F}_{1,2} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix},$$

$$\bar{\Psi}_{11,12} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \ \bar{\Phi}_{11,12} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$F_{2,1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \ \bar{F}_{2,2} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\bar{\Psi}_{22,12} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}, \ \bar{\Phi}_{22,12} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

Using the Yalmip toolbox (with SeDuMi solver) to solve the feasibility problem (21)-(24) and (39), an decreasing overall Lyapunov function is obtained as in Fig. 1, where the Lyapunov functions reduce from polyhedron to polyhedron, from far away to the vicinity of the origin. As a result, the controllers are obtained as

$$U_{1,1} = \frac{K_{1,1}x}{x_1 + 1}, \ U_{1,2} = \frac{\bar{K}_{1,2}\bar{x}}{x_1 + 1}, \ U_{1,12} = 0,$$

$$U_{2,1} = \frac{K_{2,1}x}{-x_2 + 1}, \ U_{1,2} = \frac{\bar{K}_{2,2}\bar{x}}{-x_2 + 1}, \ U_{2,12} = 0,$$

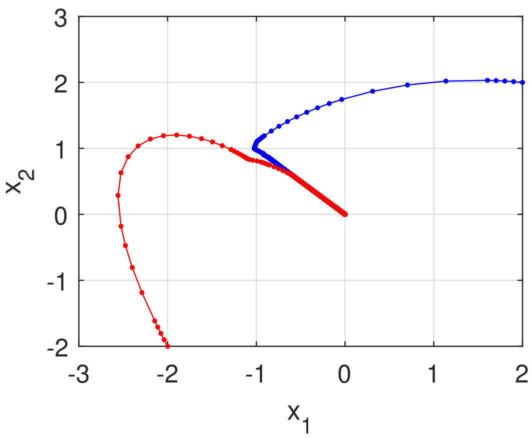

Fig. 2. The close-loop trajectories with initial states $[2, 2]$ and $[-2, -2]$

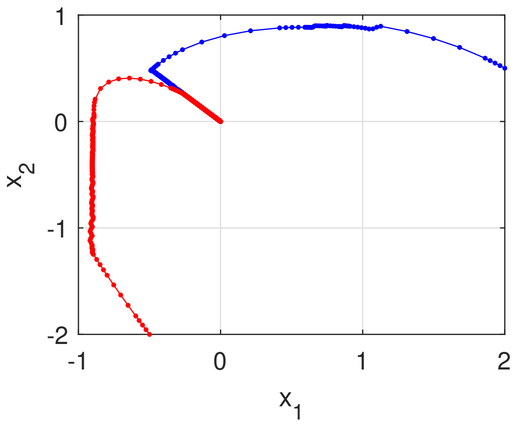

Fig. 3. The close-loop trajectories with initial states $[2, 0.5]$ and $[-0.5, -2]$

which is able to steer state to the origin for different initial conditions, as shown in Fig. 2 and 3.

VI. CONCLUSIONS

In practice, there are some complex nonlinear systems that can be approximated by switching bilinear systems. Designing stabilizing controller for switching bilinear systems makes it possible to better control these complex nonlinear systems. In this paper, a method is proposed for designing controllers that stabilize the state-based switching bilinear systems. Based on the similarity between bilinear systems and linear systems, the switching bilinear system is first written into a corresponding switching linear system. Then, state-feedback controllers are designed to asymptotically stabilize the derived corresponding switching linear system by using state-based multiple Lyapunov functions. A relaxed condition guaranteeing a decreasing overall Lyapunov function allowing jumping on the switchings of the state regions, is proved to be able to asymptotically stabilize the corresponding switching linear system with more feasibility. Finally, stabilizing controllers are derived for state-based switching bilinear systems based on the results from their corresponding switching linear systems.

VII. ACKNOWLEDGMENTS

The research is supported by the National Science Foundation of China (61673366, 61473288, 61433002, 61332016, 61620106009,61473317), the European COST Action TU1102, and Key Lab of Big Data Mining and Knowledge Management, Chinese Academy of Sciences.

REFERENCES

[1] Ronald R Mohler. *Bilinear control processes: with applications to engineering, ecology and medicine*. Academic Press, Inc., 1973.

[2] R Longchamp. Stable feedback control of bilinear systems. *IEEE Transactions on Automatic Control*, 25(2):302–306, 1980.

[3] P-O Gutman. Stabilizing controllers for bilinear systems. *IEEE Transactions on Automatic Control*, 26(4):917–922, 1981.

[4] D Koditschek and K Narendra. Stabilizability of second-order bilinear systems. *IEEE Transactions on Automatic Control*, 28(10):987–989, 1983.

[5] A Yu Khapalov and RR Mohler. Reachable sets and controllability of bilinear time-invariant systems: A qualitative approach. *IEEE transactions on automatic control*, 41(9):1342–1346, 1996.

[6] AY Khapalov and RR Mohler. Asymptotic stabilization of the bilinear time-invariant system via piecewise-constant feedback. *Systems & control letters*, 33(1):47–54, 1998.

[7] Ronald R Mohler and WJ Kolodziej. An overview of bilinear system theory and applications. *IEEE Transactions on Systems, Man and Cybernetics*, 10(10):683–688, 1980.

[8] RR Mohler and AY Khapalov. Bilinear control and application to flexible ac transmission systems. *Journal of Optimization Theory and Applications*, 105(3):621–637, 2000.

[9] Mohammad Hajiahmadi, Bart De Schutter, and Hans Hellendoorn. Design of stabilizing switching laws for mixed switched affine systems. *IEEE Transactions on Automatic Control*, 61(6):1676–1681, 2016.

[10] Daizhan Cheng. Controllability of switched bilinear systems. *IEEE Transactions on Automatic Control*, 50(4):511–515, 2005.

[11] Jose C Geromel and Patrizio Colaneri. Stability and stabilization of continuous-time switched linear systems. *SIAM Journal on Control and Optimization*, 45(5):1915–1930, 2006.

[12] Hai Lin and Panos J Antsaklis. Stability and stabilizability of switched linear systems: a survey of recent results. *IEEE Transactions on Automatic control*, 54(2):308–322, 2009.

[13] Thomas Kailath. *Linear systems*, volume 156. Prentice-Hall Englewood Cliffs, NJ, 1980.