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T.J.J. van den Boom, J. Xu, and B. De Schutter

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Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

Corrections to "Model predictive control for stochastic max-plus linear systems with chance constraints", [IEEE Trans. on Aut. Control, 64(1): 337–342, 2019]

Ton J.J. van den Boom*, Jia Xu*, and Bart De Schutter*

***Abstract*—This paper discusses two issues in connection with [1], namely an error in Proposition 7 and the assumption that the covariance matrix in the chance constraint is positive definite. First we will discuss and correct the error in Proposition 7. Subsequently, we will consider a relaxation of the assumption in Proposition 7 and give a less restrictive and less conservative reformulation of the proposition.**

I. Introduction

There are two issues in *Proposition 7* of [1]:

- In [1] we claim in (17) the following:

$$\lambda_{\min}(\Sigma_z^{-1})\|z-\mu_z\|_2 \leq (z-\mu_z)^T\Sigma_z^{-1}(z-\mu_z)$$

  This is based on the erroneous equality $x^Tx = \|x\|_2$. The correct equality is given by $x^Tx = \|x\|_2^2$.

- In the proposition we assume that $\Sigma_z = \Lambda\Sigma_w\Lambda^T$ is positive definite. Note that there holds: rank($\Lambda\Sigma_w\Lambda^T$) $\leq$ min(rank(Λ),rank(Σ_w)). However, often the matrix Λ will be a tall matrix, which means that Σ_z will not have full rank (so Σ_z is positive *semi-definite* instead of positive definite), and then for the smallest eigenvalue of Σ_z we have $\lambda_{\min}(\Sigma_z)=0$, which means that the inverse of Σ_z is not defined.

This note is organized as follows. In Section II we give a correction on Proposition 7 in the case where Σ_z is positive definite. In Section III we present an adapted Proposition 7-bis, which is an extension of Proposition 7 and handles the case where Σ_z is positive semi-definite. We also show that the newly derived bound is tighter than the bound derived in Proposition 7 of [1] in the case where Σ_z is positive definite.

II. Correction of Proposition 7 of [1]

In the formulation of Proposition 7, the equation

$$\frac{m}{-\bar{\mu}_z\lambda_{\min}(\Sigma_z^{-1})} \leq \epsilon$$

should be replaced by

$$\frac{m}{\bar{\mu}_z^2\lambda_{\min}(\Sigma_z^{-1})} \leq \epsilon$$

so the corrected version of Proposition 7 reads as follows:

Proposition 7: If Σ_z is a positive definite matrix, let $\lambda_{\min}(\Sigma_z^{-1}) > 0$ be the smallest eigenvalue of the matrix Σ_z^{-1}. Let $\bar{\mu}_z(k) = \max_{i=1,\ldots,m}\mu_{z,i}(k)$. If $\bar{\mu}(k) < 0$ and

$$\frac{m}{\bar{\mu}_z^2\lambda_{\min}(\Sigma_z^{-1})} \leq \epsilon$$

then

$$\Pr\{\max_{i=1,\ldots,m}(z_i(k)) \leq 0\} \geq 1-\epsilon$$

In the proof of Proposition 7 of [1], equation (17) must be replaced by

$$\lambda_{\min}(\Sigma^{-1})\|z-\mu_z\|_2^2 \leq (z-\mu_z)^T\Sigma_z^{-1}(z-\mu_z)$$

while equation (18) must be replaced by

$$\begin{aligned}
&\Pr\{\max(z_1,\ldots,z_m) \leq 0\}\\
&\quad\geq \Pr\{\|z-\mu_z\|_2 \leq -\bar{\mu}_z\}\\
&\quad\geq \Pr\{\|z-\mu_z\|_2^2 \leq \bar{\mu}_z^2\}\\
&\quad\geq \Pr\{\lambda_{\min}(\Sigma_z^{-1})\|z-\mu_z\|_2^2 \leq \lambda_{\min}(\Sigma_z^{-1})\bar{\mu}_z^2\}\\
&\quad\geq \Pr\{(z-\mu_z)^T\Sigma_z^{-1}(z-\mu_z) \leq \lambda_{\min}(\Sigma_z^{-1})\bar{\mu}_z^2\}
\end{aligned}$$

Moreover, equation (19) must be replaced by

$$\begin{aligned}
\Pr\{(z-\mu_z)^T\Sigma_z^{-1}(z-\mu_z) \leq \lambda_{\min}(\Sigma_z^{-1})\bar{\mu}_z^2\}&\\
\geq 1-\frac{m}{\bar{\mu}_z^2\lambda_{\min}(\Sigma_z^{-1})}&
\end{aligned}$$

The last sentence of the proof of Proposition 7 should be:

If $\frac{m}{\bar{\mu}_z^2\lambda_{\min}(\Sigma_z^{-1})} \leq \epsilon$, therefore, from (18) and (19), we have $\Pr\{\max(z_1,\ldots,z_m) \leq 0\} \geq 1-\epsilon$.

As a consequence, the first equation after the proof of Proposition 7 of [1] should be:

$$\mu_z(k) \leq \sqrt{\frac{m}{\epsilon\lambda_{\min}(\Sigma_z^{-1})}}$$

and so equation (20) becomes:

$$\Gamma\tilde{u}(k) \leq \Lambda\mu_w - \Xi(k) + \sqrt{\frac{m}{\epsilon\lambda_{\min}(\Sigma_z^{-1})}}$$

* Delft University of Technology, Delft Center for Systems and Control, Delft, The Netherlands (e-mail: {a.j.j.vandenboom, j.xu-3, b.deschutter}@tudelft.nl).

Finally, equation (21) must be replaced by:

$$\Gamma_0(k) \leq -\Xi_0(k)$$
$$\Gamma_1(k) \leq -\Lambda_1\mu_w - \Xi_1(k) + \sqrt{\frac{ms}{\epsilon\lambda_{\min}(\Sigma_{z,1}^{-1})}}$$
$$\vdots$$
$$\Gamma_s(k) \leq -\Lambda_s\mu_w - \Xi_1(k) + \sqrt{\frac{ms}{\epsilon\lambda_{\min}(\Sigma_{z,s}^{-1})}}$$

where $\Sigma_{z,l} = \Lambda_l\Sigma_w\Lambda^T$, $l = 1, \dots, s$.

III. Extension of Proposition 7 [1]

In this section we will use the same notation as in [1] and the previous section.

Note that the requirement that Σ_z is positive definite is necessary to be able to use Chebyshev's inequality in the proof of *Proposition 7* in [1]. However, by using the univariate Markov inequality instead of the multivariate Chebyshev inequality we can drop this requirement. The univariate Markov inequality is given as follows:

Proposition *([2], Proposition 2.6) : Let Y be a non-negative stochastic variable with finite mean $\mathbb{E}\{Y\}$ and consider a scalar $\alpha > 0$, then* $\Pr\{Y \geq \alpha\} \leq \dfrac{\mathbb{E}\{Y\}}{\alpha}$.

The following proposition is an extension of *Proposition 7* in [1] and handles the case where Σ_z is positive semi-definite:

Proposition 7-bis:
Let Σ_z be positive semi-definite. Let $\bar{\mu}_z(k) = \max_{i=1,\dots,m}\mu_{z,i}(k)$. If $\bar{\mu}_z(k) < 0$ and

$$\frac{\mathrm{tr}(\Sigma_z)}{\bar{\mu}_z^2} \leq \epsilon$$

where $\mathrm{tr}(\Sigma_z) = \sum_{i=1}^m [\Sigma_z]_{i,i}$ is the trace of Σ_z, then

$$\Pr\Big\{ \max(z_1(k), \dots, z_m(k)) \leq 0 \Big\} \geq 1 - \epsilon$$

Proof: For the sake of simplicity, in this proof, we will write z, μ_z instead of $z(k)$, $\mu_z(k)$. Note that $\Sigma_z \in \mathbb{R}^{m\times m}$ is symmetric and positive semi-definite (see [1]), so we can write

$$\Sigma_z = USU^T$$

where $S \in \mathbb{R}^{\kappa\times\kappa}$ is a diagonal matrix with all positive eigenvalues ($s_{ii} = \lambda_i > 0$) and $U \in \mathbb{R}^{n\times\kappa}$ contains all eigenvectors corresponding to the positive eigenvalues. Note that due to the fact that $\Sigma_z = \Lambda\Sigma_w\Lambda^T$ is symmetric, U can be selected such that all its columns are orthonormal, so $U^TU = I$.

Define the signal $z' = \mu_z + US^{1/2}v$ where $v \in \mathbb{R}^\kappa$ is a stochastic variable with $\mathbb{E}\{v\} = 0$ and $\mathbb{E}\{vv^T\} = I$. We observe that $\mathbb{E}\{z'\} = \mu_z + US^{1/2}\mathbb{E}\{v\} = \mu_z + 0 = \mu_z$ and that

$$\begin{aligned}
&\mathbb{E}\{(z' - \mu_z)(z' - \mu_z)^T\} \\
&\quad = \mathbb{E}\{(US^{1/2}v)(US^{1/2}v)^T \\
&\quad = \mathbb{E}\{US^{1/2}vv^TS^{1/2}U^T\} \\
&\quad = US^{1/2}\mathbb{E}\{vv^T\}S^{1/2}U^T \\
&\quad = US^{1/2}IS^{1/2}U^T \\
&\quad = \Sigma_z
\end{aligned}$$

Therefore, for any $i, j \in \{1, \dots, m\}$ we have $\mathbb{E}\{(z_i - \mu_{z,i})^T(z_j - \mu_{z,j})\} = \mathbb{E}\{(z_i' - \mu_{z,i})^T(z_j' - \mu_{z,j})\}$, and so for any $i \in \{1, \dots, m\}$ we have $\mathbb{E}\{(z_i - \mu_{z,i})^2\} = \mathbb{E}\{(z_i' - \mu_{z,i})^2\}$.

For the expected value of the inner product we now obtain:

$$\begin{aligned}
&\mathbb{E}\{(z - \mu_z)^T(z - \mu_z)\} \\
&\quad = \sum_{i=1}^m \Big(\mathbb{E}\{(z_i - \mu_{z,i})^2\}\Big) \\
&\quad = \sum_{i=1}^m \Big(\mathbb{E}\{(z_i' - \mu_{z,i})^2\}\Big) \\
&\quad = \sum_{i=1}^m \Big(\mathbb{E}\{(U_iS^{1/2}v)^2\}\Big) \\
&\quad = \sum_{i=1}^m \Big(\mathbb{E}\{(U_iS^{1/2}v)(U_iS^{1/2}v)^T\}\Big) \\
&\quad = \sum_{i=1}^m \Big(\mathbb{E}\{U_iS^{1/2}vv^TS^{1/2}U_i^T\}\Big) \\
&\quad = \sum_{i=1}^m \Big(U_iS^{1/2}\mathbb{E}\{vv^T\}S^{1/2}U_i^T\Big) \\
&\quad = \sum_{i=1}^m \Big(U_iS^{1/2}\ I\ S^{1/2}U_i^T\Big) \\
&\quad = \sum_{i=1}^m \Big(U_iSU_i^T\Big) \\
&\quad = \sum_{i=1}^m [\Sigma_z]_{i,i} \\
&\quad = \mathrm{tr}(\Sigma_z)
\end{aligned}$$

where U_i is the ith row of U. Recall that $\bar{\mu}_z = \max_i \mu_{z,i}$. Now we derive using Markov's inequality:

$$\begin{aligned}
&\Pr\{(z - \mu_z)^T(z - \mu_z) > \bar{\mu}_z^2\} \\
&\quad \leq \frac{\mathbb{E}\{(z - \mu_z)^T(z - \mu_z)\}}{\bar{\mu}_z^2} \\
&\quad \leq \frac{\mathrm{tr}(\Sigma_z)}{\bar{\mu}_z^2}
\end{aligned}$$

so

$$\Pr\{(z - \mu_z)^T(z - \mu_z) \leq \bar{\mu}_z^2\} \geq 1 - \frac{\mathrm{tr}(\Sigma_z)}{\bar{\mu}_z^2}$$

In Proposition 7 of [1] we derived

$$\begin{aligned}&\Pr\{\max(z_1,\ldots,z_m)\le 0\}\\&\quad\ge \Pr\{\|z-\mu_z\|_2^2\le \bar{\mu}_z^2\}\end{aligned}$$

This means that we have

$$\begin{aligned}&\Pr\{\max(z_1,\ldots,z_m)\le 0\}\\&\quad\ge \Pr\{\|z-\mu_z\|_2^2\le \bar{\mu}_z^2\}\\&\quad\ge \Pr\{(z-\mu_z)^T(z-\mu_z)\le \bar{\mu}_z^2\}\\&\quad\ge 1-\frac{\operatorname{tr}(\Sigma_z)}{\bar{\mu}_z^2}\end{aligned}$$

□

Remark:
Note that in the case that Σ_z is positive definite we find that

$$\begin{aligned}\frac{\operatorname{tr}(\Sigma_z)}{\bar{\mu}_z^2}&=\frac{\sum_{i=1}^m \lambda_i(\Sigma_z)}{\bar{\mu}_z^2}\\&\le \frac{m\lambda_{\max}(\Sigma_z)}{\bar{\mu}^2}=\frac{m}{\lambda_{\min}(\Sigma_z^{-1})\bar{\mu}_z^2}\end{aligned}$$

where $\lambda_i(\Sigma_z)$ is the i-th eigenvalue of Σ_z. This means that when Σ_z is positive definite the bound in Proposition 7-bis is in general tighter than the one derived in Proposition 7 of [1].

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[1] J. Xu, T.J.J. van den Boom, and B. De Schutter. Model predictive control for stochastic max-plus linear systems with chance constraints. *IEEE Transactions on Automatic Control*, 64(1):337–342, 2019.

[2] S. Ross. *A First Course in Probability, 8th ed.* Prentice-Hall, Upper Saddle River, NJ, USA, 2010.