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Nonlinear Systems with Uncertain Periodically Disturbed Control Gain Functions: Adaptive Fuzzy Control with Invariance Properties

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***Abstract*—This paper proposes a novel adaptive fuzzy dynamic surface control (DSC) method for an extended class of periodically disturbed strict-feedback nonlinear systems. The peculiarity of this extended class is that the control gain functions are not bounded a priori but simply taken to be continuous and with a known sign. In contrast with existing strategies, controllability must be guaranteed by constructing appropriate compact sets ensuring that all trajectories in the closed-loop system never leave these sets. We manage to do this by means of invariant set theory in combination with Lyapunov theory. In other words, boundedness is achieved a posteriori as a result of stability analysis. The approximator composed of fuzzy logic systems (FLSs) and Fourier series expansion (FSE) is constructed to deal with the unknown periodic disturbance terms.**

***Index Terms*—Dynamic surface control (DSC); adaptive fuzzy control; periodic disturbances; invariant set theory.**

I. Introduction

During the past several years, considerable attention has been paid to approximation-based adaptive control, which has emerged as a promising way to handle control problems for uncertain nonlinear systems [1-10]. Many significant results have been obtained by utilizing fuzzy logic systems (FLSs) [2-6] or neural networks (NNs) [11] as approximators. Backstepping first [12-17], and dynamic surface control (DSC) later [18-26] have become standard schemes for controlling various classes of nonlinear systems. For example, in [16], a backstepping-based adaptive robust output-feedback control scheme is proposed for a class of uncertain non-triangular stochastic systems. A robust fuzzy adaptive backstepping control strategy is designed for strict-feedback nonlinear systems in [17]. As compared to backstepping technology, DSC has the appealing feature of sensibly simplifying the control law. In [18], a robust adaptive tracking control method is presented for a class of strict-feedback nonlinear systems by using DSC. An adaptive fuzzy hierarchical sliding-mode control algorithm has been proposed in [19] for unknown nonlinear time-delay systems with saturation, while [20] proposed a fuzzy DSC method for large-scale interconnected strict-feedback nonlinear system with constrained tracking error. An adaptive neural DSC design is developed for uncertain strict-feedback nonlinear systems in [21]. In [22], a fuzzy adaptive tracking control method is studied for a class of stochastic systems with input constraints. More studies can be found in [23-26] and references therein. However, it is crucial to mention that, for all the existing DSC schemes [18-26] to work, *a priori* upper and lower bounds of the control gain functions are assumed to exist. Even if this assumption is used to guarantee controllability, it is restrictive because the control gain functions turn out to be bounded before obtaining system stability. However, it is often the case that a such a priori bounds may not exist, i.e. the control gain functions can be possibly unbounded before obtaining system stability. While this aspect has been initially studied in [27], this work has limited application because it considers ideal control gain functions not perturbed by any disturbance term.

It is well known that perturbations in the control gain functions frequently occur in many mechanical control systems (e.g. industrial robots [28] and numerical control machines [29]) and in many benchmark systems used to model electrical circuits, power systems and chemical networks (e.g. van der Pol oscillator [30] and controlled Brusselator model [31]). The main obstacle in dealing with such perturbations is that they affect the unknown system functions in a nonlinear and unknown fashion. To counteract this obstacle, FLSs and NNs have been utilized to approximate their effect [30-34]. In [31], Fourier series expansion (FSE) and multilayer neural networks (MNNs) are used as the functions approximators to model each uncertainty in periodically disturbed strict-feedback nonlinear systems. Similarly, FSE, combined with radial basis function NNs, is employed in [32]. Additionally, a combination of FLSs and FSE is used to model unknown periodically disturbed systems in [33]. However, it should be pointed out that, the aforementioned schemes [30-34] also depend on the assumption that the control gain functions are bounded *a priori*. Currently, to the authors' best knowledge, owing to the control design difficulty, no control design approach for an extended class of periodically disturbed strict-

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feedback nonlinear systems with possibly unbounded control gain functions has been reported, which motivates us to explore new approaches to deal with this challenging problem.

The main contributions of this work are highlighted as follows:

1) In contrast with the existing strategies [18-26], we remove the a priori boundedness assumption on the control gain functions. Therefore, to still ensure controllability, stability must include some well-posedness analysis, which requires a new design not available in literature.

2) Compared to the standard approximation-based adaptive methods with periodically disturbed systems [30-34], based on a priori boundedness of the control gain functions, here the approximator (namely, an FSE-FLSs approximator) is constructively combined with invariant set theory so as to guarantee well-posedness of the problem (bounded control gain functions on a compact set, i.e. controllability).

3) Differently from the standard DSC technique, the relaxed assumption on the control gain functions requires the combination of invariant set theory and Lyapunov theory stability in the form of semi-globally uniformly ultimately boundedness (SGUUB).

The rest of this paper is organized as follows. Section 2 presents the problem statement and preliminaries. The adaptive fuzzy controller is designed in Section 3. Section 4 is devoted to stability analysis. In Section 5, two simulation examples are presented to demonstrate the effectiveness of the proposed scheme, followed by conclusions in Section 6.

II. Problem statement and preliminaries

A. Problem formulation

This work considers the following class of uncertain periodically time-varying strict-feedback nonlinear systems [31-33]:

$$\begin{cases} \dot{x}_i = g_i\left(\bar{x}_i, \theta_i\left(t\right)\right) x_{i+1} + \varphi_i\left(\bar{x}_i, \theta_i\left(t\right)\right), \ 1 \le i \le n-1 \\ \dot{x}_n = g_n\left(x, \theta_n\left(t\right)\right) u + \varphi_n\left(x, \theta_n\left(t\right)\right) \\ y = x_1 \end{cases} \tag{1}$$

where $\bar{x}_i = [x_1, x_2, \dots, x_i]^T \in \mathbb{R}^i$ and $x = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ are the system state variables, $u, y \in \mathbb{R}$ are the system input and output, $\varphi_i(\cdot,\cdot)$ are unknown continuous functions with $\varphi_i(0,\theta) = 0$, $\forall\theta$, $g_i(\cdot,\cdot)$ are unknown continuous control gain functions, and $\theta_i(t) : [0, +\infty) \to \mathbb{R}^{m_i}$ are unknown and continuously time-varying perturbations with known periods T_i, namely, $\theta_i(t + T_i) = \theta_i(t)$. For the sake of brevity, $\theta_i(t)$ will be denoted by θ_i throughout this paper.

The control objective of this study is to design an adaptive control law for system (1) such that the output tracking error can be made arbitrarily small and all the signals in the closed-loop system to be SGUUB.

In contrast with existing literature [30-34], we aim to solve this problem in the presence of the following relaxed assumption.

Assumption 1: The unknown continuous control gain functions satisfy $|g_i(\bar{x}_i, \theta_i)| > 0$. Their signs are known, and without loss of generality, it is further assumed that $g_i(\bar{x}_i, \theta_i) > 0$ for $i = 1, 2, ..., n$.

Remark 1: Despite some efforts such as [27] and [35] have been made to relax the boundedness assumption of control gain functions, the considered works have limited application because they consider ideal control gain functions not perturbed by any disturbance term.

Remark 2: In [30-34] periodically disturbed control gain functions are considered but the standard assumption from [18-26] is used: that is, the control gain functions are assumed to satisfy $0 < \underline{g}_i \le |g_i(\bar{x}_i, \theta_i)| \le \bar{g}_i$ (a priori boundedness). In fact, this assumption is sufficient for controllability of the system (1). However, in practice a priori boundedness is too restrictive. For example, $g_i(\bar{x}_i, \theta_i) = x_1^2 + e^{x_i\theta_i}$ does not satisfy a priori boundedness. In other words, Assumption 1 allows the functions $g_i(\bar{x}_i, \theta_i)$ to be possibly unbounded functions of the states.

Remark 3: Clearly, the states x_i cannot be assumed to be bounded *a priori* before obtaining the system stability. Therefore, in view of Assumption 1, the control gains cannot be taken bounded *a priori* before obtaining system stability. The absence of a priori bounds requires a new control design going beyond the existing literature [30-34].

The following assumption on the trajectory r to be tracked is standard in most approximation-based designs [2-4]:

Assumption 2: The signals r, $\dot{r}$, and $\ddot{r}$ are bounded, i.e., there exists a positive constant B_0 such that

$$\Omega_0 = \left\{ (r, \dot{r}, \ddot{r}) \middle| r^2 + (\dot{r})^2 + (\ddot{r})^2 \le B_0^2 \right\}$$

Let us recall the following lemmas used for stability analysis.

Lemma 1 [8]: Consider the first-order dynamical system

$$\dot{\chi}(t) = -a\chi(t) + p\upsilon(t) \tag{2}$$

with $a > 0$, $p > 0$ and $\upsilon(t)$ a positive function. Then, for any bounded initial condition $\chi(0) \ge 0$, the inequality $\chi(t) \ge 0$ for $\forall t \ge 0$ holds.

Lemma 2 [9]: The hyperbolic tangent function $\tanh(\cdot)$ satisfies for any $\rho \in R$ and $\forall\varsigma > 0$

$$0 \le |\rho| - \rho \tanh\left(\frac{\rho}{\varsigma}\right) \le 0.2785\vartheta, \quad 0 \le \rho \tanh\left(\frac{\rho}{\varsigma}\right) \tag{3}$$

Lemma 3 [24]: (Young's inequality with ε) For any $(x, y) \in R^2$, the following inequality holds:

$$\hbar\ell \le \frac{\varepsilon^2}{\varrho} |\hbar|^2 + \frac{1}{\beta\varepsilon^2} |\ell|^2 \tag{4}$$

where $\varrho > 1$, $\beta > 1$, $\varepsilon > 0$ and $(\varrho - 1)(\beta - 1) = 1$.

B. FSE-FLSs-based approximator

The main idea behind FSE-FLSs-based approximator [33] is the following. We first employ FSE to estimate θ_i, and then we utilize the estimate as one of the FLSs inputs to approximate unknown functions in the form $\chi_i(\Theta_i, \theta_i)$.

Let $\Theta_i = \left[\bar{x}_i^T, a_i^T\right]^T$ be composed of two measured signals defined on a compact set $\Omega_i \times \Omega_0$, and let $\theta_i =$

$[\theta_{i,1},\theta_{i,2},\ldots,\theta_{i,m}]^T$ be unknown continuous vectors of known period T defined on a compact set

$$\Omega_\theta=\left\{(\theta_1,\theta_2,\ldots,\theta_n)\,\Big|\,\sum_{j=1}^{n}\theta_j^T\theta_j\le M_\theta^2\right\}$$

with M_θ a positive constant. The vector θ_i can be expressed by a linearly parameterized FSE as follows:

$$\theta_i=H_i^T\phi_i(t)+\delta_{\theta_i}(t) \tag{5}$$

where $H_i=[H_{i,1},\ldots,H_{i,m}]\in\mathbb{R}^{q\times m}$ is a constant matrix with $H_{i,j}\in R^q$ a vector consisting of the first q coefficients of the FSE of $\theta_{i,j}$, δ_{θ_i} is the truncation error with upper bound $\delta_{\theta_i}^*>0$, which can be made arbitrarily small by increasing q, and $\phi_i(t)=[\phi_{i,1}(t),\ldots,\phi_{i,q}(t)]^T$ with $\phi_{i,1}(t)=1$, $\phi_{i,2j}(t)=\sqrt{2}\sin(2\pi jt/T)$, $\phi_{i,2j+1}(t)=\sqrt{2}\cos(2\pi jt/T)\left(j=1,\ldots,\frac{q-1}{2}\right)$, whose derivatives up to the nth-order are smooth and bounded.

At this points, the FLSs will be employed to approximate the unknown continuous function $\chi_i(\Theta_i,\theta_i)$ as

$$\chi_i(\Theta_i,\theta_i)=\Xi_i^T\Psi_i\left(A_i^TZ_i\right)+\delta_{\chi_i}(\Theta_i,\theta_i) \tag{6}$$

where $\Psi_i\left(A_i^TZ_i\right)=\left[\Psi_{i,1}\left(A_i^TZ_i\right),\ldots,\Psi_{i,m}\left(A_i^TZ_i\right)\right]^T$ is a known smooth vector-valued function with component $\Psi_{i,l}\left(A_i^TZ_i\right)=\prod_{j=1}^{n}\mu_{F_{i,j}^l}(x_{i,j})\Big/\sum_{l=1}^{m}\left[\prod_{j=1}^{n}\mu_{F_{i,j}^l}(x_{i,j})\right]$, $(1\le l\le m)$, where A_i^T is a matrix of adjustable parameters, $Z_i=\left[\Theta_i^T,\theta_i^T,1\right]^T\in R^{l+m+1}$ is a vector-valued function, $\mu_{F_{i,j}^l}(x_{i,j})$ are fuzzy membership functions chosen as the Gaussian functions $\mu_{F_{i,j}^l}(x_{i,j})=\exp\left[-\left(\left(x_{i,j}-a_{i,j}^l\right)/b_{i,j}^l\right)^2\right]$ with $a_{i,j}^l$ and $b_{i,j}^l$ adjustable parameters, $\Xi_i=\left[\bar{y}_i^1,\bar{y}_i^2,\ldots,\bar{y}_i^m\right]^T$ a vector of adjustable parameters, $\bar{y}_i^l$ the point such that $\mu_{G^l}\left(\bar{y}_i^l\right)=1$, and $\delta_{\chi_i}(\Theta_i,\theta_i)$ the approximation error whose upper bound $\delta_{\chi_i}^*>0$ can be decreased by increasing the number of fuzzy rules l.

From (5) and $A_i^TZ_i=A_{\Theta_i}^T\Theta_i+A_{\theta_i}^T\theta_i+A_{0_i}$, we have

$$\begin{aligned}A_i^TZ_i&=A_{\Theta_i}^T\Theta_i+A_{\theta_i}^TH_i^T\phi_i(t)+A_{0_i}+A_{\theta_i}^T\delta_{\theta_i}(t)\\&=\psi_i^T\bar{Z}_i(\Theta_i,\phi_i)+A_{\theta_i}^T\delta_{\theta_i}(t)\end{aligned} \tag{7}$$

where $\psi_i^T=\left[A_{\Theta_i}^T,A_{\theta_i}^TH_i,A_{0_i}\right]$ and $\bar{Z}_i(\Theta_i,\phi_i)=\left[\Theta_i^T,\phi_i^T(t),\,1\right]^T$.

Substituting (7) into (6) leads to

$$\begin{aligned}\chi_i(\Theta_i,\theta_i)&=\Xi_i^T\Psi_i\left(\psi_i^T\bar{Z}_i(\Theta_i,\phi_i)+A_{\theta_i}^T\delta_{\theta_i}(t)\right)+\delta_{\chi_i}\\&=\Xi_i^T\Psi_i\left(\psi_i^T\bar{Z}_i(\Theta_i,\phi_i)\right)+\varepsilon_i(\Theta_i,t)\end{aligned} \tag{8}$$

where $\varepsilon_i(\Theta_i,t)=\delta_{\chi_i}+\Xi_i^T\Psi_i\left(\psi_i^T\bar{Z}_i(\Theta_i,\phi_i)+A_{\theta_i}^T\delta_{\theta_i}(t)\right)-\Xi_i^T\Psi_i\left(\psi_i^T\bar{Z}_i(\Theta_i,\phi_i)\right)$ satisfy the following lemma:

Lemma 4 [33]: For $(\Theta_i,\theta_i)\in\Omega_i\times\Omega_0\times\Omega_\theta\ (i=1,\ldots,n)$, there exist unknown positive constants ε_i^* such that

$$|\varepsilon_i(\Theta_i,t)|\le\varepsilon_i^* \tag{9}$$

where ε_i^* can be made arbitrarily small by increasing l and q.

To facilitate the control system design, we can rewrite the estimation errors as

$$\begin{aligned}&\Xi_i^T\Psi_i\left(\psi_i^T\bar{Z}_i(\Theta_i,\phi_i)\right)-\hat{\Xi}_i^T\Psi_i\left(\hat{\psi}_i^T\bar{Z}_i(\Theta_i,\phi_i)\right)=\\&\tilde{\Xi}_i^T\left(\hat{\Psi}_i-\hat{\Psi}_i'\hat{\psi}_i^T\bar{Z}_i(\Theta_i,\phi_i)\right)+\hat{\Xi}_i^T\hat{\Psi}_i'\tilde{\psi}_i^T\bar{Z}_i(\Theta_i,\phi_i)+e_i\end{aligned} \tag{10}$$

where $\hat{\Psi}_i'=\left[\hat{\Psi}_{i,1}',\hat{\Psi}_{i,2}',\ldots,\hat{\Psi}_{i,l}'\right]^T\in R^{m\times l}$ with $\hat{\Psi}_{i,j}'=\frac{\partial\Psi_{i,j}(\Theta_i,\theta_i)}{\partial\theta_i}\Big|_{\theta_i=\hat{\psi}_i^T\bar{Z}_i(\Theta_i,\phi_i)}$ $(j=1,\ldots,l)$, $\hat{\Psi}_i=\Psi_i\left(\hat{\psi}_i^T\bar{Z}_i(\Theta_i,\phi_i)\right)$ and the residual terms e_i satisfy

$$\begin{aligned}|e_i|\le e_i^*=\|\psi_i\|_F\left\|\bar{Z}_i(\Theta_i,\phi_i)\hat{\Xi}_i^T\hat{\Psi}_i'\right\|_F+|\Xi_i|_1\\+\|\Xi_i\|\left\|\hat{\Psi}_i'\hat{\psi}_i^T\bar{Z}_i(\Theta_i,\phi_i)\right\|.\end{aligned} \tag{11}$$

Remark 4: Note that the standard FLSs in [2-7] fail to approximate disturbed system functions because such perturbations appear nonlinearly in unknown system functions and can destroy the universal approximation property of FLSs. On the contrary, [31-33] showed that FSE-FLSs-based approximator in [31-33] can compensate for the nonlinearly parameterized perturbations.

For compactness, let $\|\cdot\|$ denote the Euclidean norm of a vector, $\|\cdot\|_F$ denote the Frobenius norm of a matrix, $\lambda_{\max}(\Upsilon)$, $\lambda_{\min}(\Upsilon)$ denote the largest and smallest eigenvalues of a square matrix Υ and $|\Upsilon|_1=\sum_{i=1}^{m}|\upsilon_i|$ with $\Upsilon=[\upsilon_1,\upsilon_2,\cdots,\upsilon_m]^T\in\mathbb{R}^m$, respectively.

III. Fuzzy adaptive DSC design

The DSC technique and invariant set theory are now employed to construct an adaptive fuzzy control law for (1). According to the DSC iterative procedure, let us proceed along the following steps:

Step 1: Define the output tracking error $z_1=x_1-r$. From (1), the time derivative of z_1 is

$$\dot{z}_1=\varphi_1(x_1,\theta_1)+g_1(x_1,\theta_1)x_2-\dot{r}\,. \tag{12}$$

Define the compact set $\Omega_1:=\left\{z_1\,\middle|\,z_1^2\le2\xi\right\}$, with $\xi>0$ being a positive constant. For the compact set $\Omega_1\times\Omega_0\times\Omega_\theta$, the following lemma holds:

Lemma 5: The unknown continuous control-gain function $g_1(x_1,\theta_1)$ has a maximum and a minimum in $\Omega_1\times\Omega_0\times\Omega_\theta$, namely, there exist positive constants $\underline{g}_1$ and $\bar{g}_1$ such that $\underline{g}_1=\min\limits_{\Omega_1\times\Omega_0\times\Omega_\theta}g_1(x_1,\theta_1)$ and $\bar{g}_1=\max\limits_{\Omega_1\times\Omega_0\times\Omega_\theta}g_1(x_1,\theta_1)$.

Proof: Observing $z_1=x_1-r$, we obtain $x_1=z_1+r$, so that the continuous function $g_1(x_1,\theta_1)$ can be expressed by

$$g_1(x_1,\theta_1)=\gamma_1(z_1,\theta_1,r) \tag{13}$$

with $\gamma_1(\cdot)$ being a continuous function of z_1, θ_1, and r. Note that $\Omega_1\times\Omega_0\times\Omega_\theta$ is a compact set since Ω_1, Ω_0, and Ω_θ are compact sets. Furthermore, it can be seen from (13) that all the variables of $\gamma_1(z_1,\theta_1,r)$ are included in the compact set $\Omega_1\times\Omega_0\times\Omega_\theta$. Thus, we have

$$0<\underline{g}_1\le g_1(x_1,\theta_1)\le\bar{g}_1,\quad(x_1,\theta_1)\in\Omega_1\times\Omega_0\times\Omega_\theta \tag{14}$$

which concludes the proof. From (14), we can rewrite (12) as

$$\dot{z}_1 = \underline{g}_1 \left(\chi_1\left(\Theta_1, \theta_1\right) + x_2 + g_{1,0} x_2\right) \tag{15}$$

with $g_{1,0} = \underline{g}_1^{-1}\big(g_1\left(x_1, \theta_1\right)\big) - 1 > 0, \chi_1\left(\Theta_1, \theta_1\right) = \underline{g}_1^{-1}\left(\varphi_1\left(x_1, \theta_1\right) - \dot{r}\right)$, and $\Theta_1 = \left[x_1, \dot{r}\right]^T$.

Remark 5: It has to be remarked that the conventional design of $\chi_1(x_1, \theta_1) = g_1^{-1}(x_1, \theta_1)\varphi_1(x_1, \theta_1)$ makes stability analysis complex due to multiple substitutions of intermediate control laws [31]. On the contrary, the choices $\chi_1(\Theta_1, \theta_1) = \underline{g}_1^{-1}\big(\varphi_1(x_1, \theta_1) - \dot{r}\big)$ and $g_{1,0} = g_1(x_1, \theta_1)/\underline{g}_1 - 1 > 0$ are able to simplify control design procedure due to the elimination of the coupling term $z_1 g_{1,0}\alpha_1 < 0$.

To address the stabilization of subsystem (12), we take the following quadratic Lyapunov function candidate

$$V_{z_1} = \frac{1}{2} z_1^2. \tag{16}$$

According to (8) and (15), the time derivative of V_{z_1} is

$$\dot{V}_{z_1} \leq z_1 \underline{g}_1 \left(\Xi_1^T \Psi_1 \left(\psi_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right) + x_2 + g_{1,0} x_2\right) + |z_1| \, \underline{g}_1 \varepsilon_1^* \tag{17}$$

Let us choose the virtual control laws α_1 and parameters adaptation laws $\hat{\Xi}_1$ and $\hat{\psi}_1$ as follows:

$$\begin{aligned} \alpha_1 &= -c_1 z_1 - \hat{\Xi}_1^T \Psi_1 \left(\hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\omega_1 \\ \omega_1 &= \tanh\left(\frac{z_1 \hat{\Xi}_1^T \Psi_1\left(\hat{\psi}_1^T \bar{Z}_1(\Theta_1, \phi_1)\right)}{\upsilon_1}\right) \end{aligned} \tag{18}$$

$$\dot{\hat{\Xi}}_1 = \Gamma_{\Xi_1}\left[z_1\left(\hat{\Psi}_1 - \hat{\Psi}_1' \hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right) - \sigma_1 \hat{\Xi}_1\right] \tag{19}$$

$$\dot{\hat{\psi}}_1 = \Gamma_{\psi_1}\left[z_1 \bar{Z}_1\left(\Theta_1, \phi_1\right) \hat{\Psi}_1^T \hat{\Psi}_1' - \sigma_1 \hat{\psi}_1\right] \tag{20}$$

where $c_1 > 0$, $\sigma_1 > 0$, and $\upsilon_1 > 0$ are design parameters, and $\Gamma_{\psi_1} = \Gamma_{\psi_1}^T > 0$ and $\Gamma_{\Xi_1} = \Gamma_{\Xi_1}^T > 0$ are adaptive gain matrices. According to Lemma 1, we have $\hat{\psi}_1(t) \geq 0$ and $\hat{\Xi}_1(t) \geq 0$ for $\forall t \geq 0$ after selecting $\hat{\Xi}_1(0) \geq 0$ and $\hat{\psi}_1(0) \geq 0$.

We can now introduce the DSC filters, which are used to avoid repeatedly differentiating α_1. Let α_1 pass through a first-order filter with time constant ι_2 to obtain α_{2f} as

$$\iota_2 \dot{\alpha}_{2f} + \alpha_{2f} = \alpha_1, \quad \alpha_{2f}(0) = \alpha_1(0)\ . \tag{21}$$

Define the output of this filter as $y_2 = \alpha_{2f} - \alpha_1$. Then, we have $\dot{\alpha}_{2f} = -\left(y_2/\iota_2\right)$ and

$$\dot{y}_2 = -\frac{y_2}{\iota_2} + \underbrace{\left(-\frac{\partial\alpha_1}{\partial z_1}\dot{z}_1 - \frac{\partial\alpha_1}{\partial\hat{\psi}_1}\dot{\hat{\psi}}_1 - \frac{\partial\alpha_1}{\partial\hat{\Xi}_1}\dot{\hat{\Xi}}_1 - \frac{\partial\alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial\alpha_1}{\partial\dot{r}}\ddot{r}\right)}_{B_2\left(z_1, z_2, y_2, \hat{\psi}_1, \hat{\Xi}_1, r, \dot{r}, \ddot{r}\right)} \tag{22}$$

where $B_2(\cdot)$ is a continuous function.

By noting that $x_2 = z_2 + \alpha_{2f}$ and $y_2 = \alpha_{2f} - \alpha_1$, one has

$$x_2 = z_2 + \alpha_1 + y_2 \tag{23}$$

Substituting (18) and (23) into (17) gives

$$\begin{aligned} \dot{V}_{z_1} \leq& \underline{g}_1\left(\left|z_1 \hat{\Xi}_1^T \Psi_1\left(\hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\right|\right) + |z_1| \, \underline{g}_1 \varepsilon_1^* \\ &- \underline{g}_1\left(z_1 \hat{\Xi}_1^T \Psi_1\left(\hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\omega_1\right) \\ &+ z_1 g_1\left(x_1, \theta_1\right)\left(z_2 + y_2\right) - c_1 \underline{g}_1 z_1^2 \\ &- \underline{g}_1\left(z_1 \hat{\Xi}_1^T \Psi_1\left(\hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\right) \\ &+ \underline{g}_1\left(z_1 \Xi_1^T \Psi_1\left(\psi_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\right) \\ &+ z_1 \underline{g}_1 g_{1,0}\alpha_1. \end{aligned} \tag{24}$$

Applying Lemma 2, we obtain

$$\begin{aligned} \dot{V}_{z_1} \leq& z_1 \underline{g}_1\left(\Xi_1^T \Psi_1\left(\psi_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\right) + z_1 \underline{g}_1 g_{1,0}\alpha_1 \\ &- z_1 \underline{g}_1\left(\hat{\Xi}_1^T \Psi_1\left(\hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\right) + |z_1| \, \underline{g}_1 \varepsilon_1^* \\ &+ z_1 g_1\left(x_1, \theta_1\right)\left(z_2 + y_2\right) + 0.2785 \underline{g}_1 \upsilon_1 \\ &- c_1 \underline{g}_1 z_1^2. \end{aligned} \tag{25}$$

In view of (14) (15) and (18), the following inequality holds:

$$\begin{aligned} z_1 \underline{g}_1 g_{1,0}\alpha_1 = \underline{g}_1 g_{1,0}\bigg[&-z_1 \hat{\Xi}_1^T \Psi_1\left(\hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\omega_1 \\ &- c_1 z_1^2\bigg] < 0 \end{aligned} \tag{26}$$

Using (10) and (26) yields

$$\begin{aligned} \dot{V}_{z_1} \leq& z_1 \underline{g}_1\left(\tilde{\Xi}_1^T\left(\hat{\Psi}_1 - \hat{\Psi}_1' \hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\right) - c_1 \underline{g}_1 z_1^2 \\ &+ z_1 \underline{g}_1\left(\hat{\Xi}_1^T \hat{\Psi}_1' \tilde{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right) + e_1\right) + |z_1| \, \underline{g}_1 \varepsilon_1^* \\ &+ z_1 g_1\left(x_1, \theta_1\right)\left(z_2 + y_2\right) + 0.2785 \underline{g}_1 \upsilon_1 \end{aligned} \tag{27}$$

with e_1 being bounded by

$$|e_1| \leq \|\psi_1\|_F \left\|\phi_1 \hat{\Xi}_1^T \hat{\Psi}_1'\right\|_F + \|\Xi_1\| \left\|\hat{\Psi}_1' \hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right\| + |\Xi_1|_1 \tag{28}$$

Substituting (28) into (27) leads to

$$\begin{aligned} \dot{V}_{z_1} \leq& z_1 \underline{g}_1\left(\tilde{\Xi}_1^T\left(\hat{\Psi}_1 - \hat{\Psi}_1' \hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\right) - c_1 \underline{g}_1 z_1^2 \\ &+ z_1 \underline{g}_1\left(\hat{\Xi}_1^T \hat{\Psi}_1' \tilde{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right) + |z_1| \, \underline{g}_1 a_1^* \\ &+ z_1 g_1\left(x_1, \theta_1\right)\left(z_2 + y_2\right) + 0.2785 \underline{g}_1 \upsilon_1 \end{aligned} \tag{29}$$

where $a_1^* = e_1^* + \varepsilon_1^*$.

We can now choose the Lyapunov function candidate

$$V_1 = V_{z_1} + tr\left\{\frac{\underline{g}_1 \tilde{\psi}_1^T \Gamma_{\psi_1}^{-1} \tilde{\psi}_1}{2}\right\} + \frac{\underline{g}_1 \tilde{\Xi}_1^T \Gamma_{\Xi_1}^{-1} \tilde{\Xi}_1}{2} + \frac{1}{2} y_2^2 \tag{30}$$

where $\tilde{\psi}_1 = \psi_1 - \hat{\psi}_1$ and $\tilde{\Xi}_1 = \Xi_1 - \hat{\Xi}_1$ are the estimation errors of ψ_1 and Ξ_1, respectively.

From (22) and (29), the time derivative of (30) is

$$\begin{aligned} \dot{V}_1 \leq& z_1 \underline{g}_1\left[\tilde{\Xi}_1^T\left(\hat{\Psi}_1 - \hat{\Psi}_1' \hat{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right)\right] + |y_2 B_2(\cdot)| \\ &+ z_1 \underline{g}_1\left(\hat{\Xi}_1^T \hat{\Psi}_1' \tilde{\psi}_1^T \bar{Z}_1\left(\Theta_1, \phi_1\right)\right) + 0.2785 \underline{g}_1 \upsilon_1 \\ &+ z_1 g_1\left(x_1, \theta_1\right)\left(z_2 + y_2\right) + |z_1| \, \underline{g}_1 a_1^* - y_2^2/\iota_2 \\ &- \mathrm{tr}\left\{\underline{\mathrm{g}}_1 \tilde{\psi}_1^{\mathrm{T}} \Gamma_{\psi_1}^{-1} \dot{\hat{\psi}}_1\right\} - \underline{\mathrm{g}}_1 \tilde{\Xi}_1^{\mathrm{T}} \Gamma_{\Xi_1}^{-1} \dot{\hat{\Xi}}_1 - \mathrm{c}_1 \underline{\mathrm{g}}_1 \mathrm{z}_1^2. \end{aligned} \tag{31}$$

Using $\hat{\Xi}_1^T \hat{\Psi}_1' \tilde{\psi}_1^T \bar{Z}_1(\Theta_1, \phi_1) = \mathrm{tr}\left\{\tilde{\psi}_1^{\mathrm{T}} \bar{Z}_1(\Theta_1, \phi_1) \hat{\Xi}_1^{\mathrm{T}} \hat{\Psi}_1'\right\}$ and substituting the parameter adaptation laws (19) and (20) into (31) results in

$$\begin{aligned}\dot{V}_1 \leq& \sigma_1 \mathrm{tr}\left\{\underline{g}_1 \tilde{\psi}_1^{\mathrm{T}} \hat{\psi}_1\right\} + \sigma_1 \underline{g}_1 \tilde{\Xi}_1^{\mathrm{T}} \hat{\Xi}_1 + 0.2785 \underline{g}_1 \upsilon_1 \\ &+ z_1 g_1(x_1, \theta_1)(z_2 + y_2) + |y_2 B_2(\cdot)| \\ &+ |z_1| \underline{g}_1 a_1^* - c_1 \underline{g}_1 z_1^2 - y_2^2/\iota_2 .\end{aligned} \tag{32}$$

Step i $(i = 2, \ldots, n-1)$: The design process for step i is similar to Step 1. From $z_i = x_i - \alpha_{if}$, one has

$$\dot{z}_i = \varphi_i(\bar{x}_i, \theta_i) + g_i(\bar{x}_i, \theta_i) x_{i+1} - \dot{\alpha}_{if} . \tag{33}$$

Choose the virtual control laws α_i and parameters adaptation laws $\hat{\psi}_i$ and $\hat{\Xi}_i$ as follows

$$\begin{aligned}\alpha_i &= -c_i z_i - \hat{\Xi}_i^T \Psi_i\left(\hat{\psi}_i^T \bar{Z}_i(\Theta_i, \phi_i)\right)\omega_i \\ \omega_i &= \tanh\left(\frac{z_i \hat{\Xi}_i^T \Psi_i\left(\hat{\psi}_i^T \bar{Z}_i(\Theta_i, \phi_i)\right)}{\upsilon_i}\right)\end{aligned} \tag{34}$$

$$\dot{\hat{\Xi}}_i = \Gamma_{\Xi_i}\left[z_i\left(\hat{\Psi}_i - \hat{\Psi}_i' \hat{\psi}_i^T \bar{Z}_i(\Theta_i, \phi_i)\right) - \sigma_i \hat{\Xi}_i\right] \tag{35}$$

$$\dot{\hat{\psi}}_i = \Gamma_{\psi_i}\left[z_i \bar{Z}_i(\Theta_i, \phi_i) \hat{\Xi}_i^T \hat{\Psi}_i' - \sigma_i \hat{\psi}_i\right] \tag{36}$$

where $c_i > 0$, $\sigma_i > 0$, and $\upsilon_i > 0\,(i = 2, 3, \ldots, n-1)$ are design parameters. Moreover, $\Gamma_{\psi_i} = \Gamma_{\psi_i}^T > 0$ and $\Gamma_{\Xi_i} = \Gamma_{\Xi_i}^T > 0$ are adaptive gain matrices. According to Lemma 1, we have $\hat{\psi}_i(t) \geq 0$ and $\hat{\Xi}_i(t) \geq 0$ for $\forall t \geq 0$.

Next, let α_i pass through a first-order filter with time constant ι_{i+1} to obtain α_{i+1f} as

$$\iota_{i+1} \dot{\alpha}_{i+1f} + \alpha_{i+1f} = \alpha_i, \;\; \alpha_{i+1f}(0) = \alpha_i(0) \tag{37}$$

Define the filter errors $y_{i+1} = \alpha_{i+1f} - \alpha_i$. We have $\dot{\alpha}_{i+1f} = -(y_{i+1}/\iota_{i+1})$ and

$$\begin{aligned}\dot{y}_{i+1} = -\tfrac{y_{i+1}}{\iota_{i+1}} + \\ \underbrace{\left(-\frac{\partial \alpha_i}{\partial z_i}\dot{z}_i - \frac{\partial \alpha_i}{\partial \hat{\psi}_i}\dot{\hat{\psi}}_i - \frac{\partial \alpha_i}{\partial \hat{\Xi}_i}\dot{\hat{\Xi}}_i - \frac{\partial \alpha_i}{\partial \bar{x}_i}\dot{\bar{x}}_i - \frac{\partial \alpha_i}{\partial y_i}\dot{y}_i\right)} \\ B_{i+1}\left(\bar{z}_{i+1}, \bar{y}_{i+1}, \bar{\hat{\psi}}_i, \bar{\hat{\Xi}}_i, r, \dot{r}, \ddot{r}\right)\end{aligned} \tag{38}$$

where $B_{i+1}(\cdot)$ is a continuous function and $\bar{z}_{i+1} = [z_1, \ldots, z_{i+1}]^T$, $\bar{y}_{i+1} = [y_2, \ldots, y_{i+1}]^T$, $\bar{\hat{\psi}}_i = \left[\hat{\psi}_1, \ldots, \hat{\psi}_i\right]^T$, $\bar{\hat{\Xi}}_i = \left[\hat{\Xi}_1, \ldots, \hat{\Xi}_i\right]^T$.

Noting that $x_{i+1} = z_{i+1} + \alpha_{i+1f}$ and $y_{i+1} = \alpha_{i+1f} - \alpha_i$, one has

$$x_{i+1} = z_{i+1} + \alpha_i + y_{i+1} . \tag{39}$$

In view of (34), we know that the virtual control signal α_{i-1} is a continuous function with respect to z_{i-1}, $\hat{\psi}_{i-1}$, y_{i-1}, and $\hat{\Xi}_{i-1}$. Therefore, x_i is a continuous function of z_i, y_i, $\hat{\psi}_{i-1}$, and $\hat{\Xi}_{i-1}$. From $x_1 = z_1 + y_d$ and (39), it follows that the control gain functions $g_i(\bar{x}_i, \theta_i)$ can be expressed in the following form:

$$g_i(\bar{x}_i, \theta_i) = \gamma_i\left(\bar{z}_i, \bar{y}_i, \bar{\hat{\psi}}_{i-1}, \bar{\hat{\Xi}}_{i-1}, r, \theta_i\right) \tag{40}$$

where $\gamma_i(\cdot)$ is a continuous function.

Define the sets Ω_i $(i = 2, \ldots, n-1)$ as

$$\begin{aligned}\Omega_i := \Bigg\{ \left[\bar{z}_i^T, \bar{y}_i^T, \bar{\hat{\psi}}_{i-1}^T, \bar{\hat{\Xi}}_{i-1}^T\right]^T \Bigg| \sum_{j=1}^{i} z_j^2 + \sum_{j=2}^{i} y_j^2 + \\ \sum_{j=1}^{i-1}\left(\underline{g}_j \tilde{\Xi}_j^T \Gamma_{\Xi_j}^{-1} \tilde{\Xi}_j + tr\left\{\underline{g}_j \tilde{\psi}_j^T \Gamma_{\psi_j}^{-1} \tilde{\psi}_j\right\}\right) \leq 2\xi \Bigg\}\end{aligned} \tag{41}$$

where ξ is the same positive design constant as that after (12). In a similar fashion as Lemma 5 was derived, we have that the unknown continuous functions $g_i(\bar{x}_i, \theta_i)$ have a maximum and a minimum in the compact set $\Omega_i \times \Omega_0 \times \Omega_\theta$, i.e. there exist positive constants $\bar{g}_i$ and $\underline{g}_i$ satisfying

$$0 < \underline{g}_i \leq g_i(\bar{x}_i, \theta_i) \leq \bar{g}_i, \quad (\bar{x}_i, \theta_i) \in \Omega_i \times \Omega_0 \times \Omega_\theta . \tag{42}$$

By using (33) and (42), one has

$$\dot{z}_i = \underline{g}_i\left(\chi_i(\Theta_i, \theta_i) + x_{i+1} + g_{i,0} x_{i+1}\right) \tag{43}$$

where $\chi_i(\Theta_i, \theta_i) = \underline{g}_i^{-1}\left(\varphi_i(\bar{x}_i, \theta_i) - \dot{\alpha}_{if}\right)$, $\Theta_i = [\bar{x}_i, \dot{\alpha}_{if}]^T$ and $g_{i,0} = \underline{g}_i^{-1}\left(g_i(\bar{x}_i, \theta_i)\right) - 1 > 0$.

Consider the quadratic Lyapunov function candidate:

$$V_{z_i} = \frac{1}{2} z_i^2 . \tag{44}$$

From (8), (34), (39), and (43), the time derivative of V_{z_i} is

$$\begin{aligned}\dot{V}_{z_i} \leq& z_i \underline{g}_i\left(\Xi_i^T \Psi_i\left(\psi_i^T \bar{Z}_i(\Theta_i, \phi_i)\right)\right) + z_i \underline{g}_i g_{i,0} \alpha_i \\ &- z_i \underline{g}_i\left(\hat{\Xi}_i^T \Psi_i\left(\hat{\psi}_i^T \bar{Z}_i(\Theta_i, \phi_i)\right)\right) - c_i \underline{g}_i z_i^2 \\ &+ z_i g_i(\bar{x}_i, \theta_i)(z_{i+1} + y_{i+1}) + 0.2785 \underline{g}_i \upsilon_i \\ &+ |z_i| \underline{g}_i \varepsilon_i^* .\end{aligned} \tag{45}$$

In view of (34), (42) and (43), it holds that

$$\begin{aligned}z_i \underline{g}_i g_{i,0} \alpha_i =& \underline{g}_i g_{i,0}\Big[-c_i z_i^2 - \\ & z_i \hat{\Xi}_i^T \Psi_i\big(\hat{\psi}_i^T \bar{Z}_i(\Theta_i, \phi_i)\big)\omega_i\Big] < 0\end{aligned} \tag{46}$$

Invoking (10) and (46), we can further obtain

$$\begin{aligned}\dot{V}_{z_i} \leq& z_i \underline{g}_i\Big(\tilde{\Xi}_i^T\big(\hat{\Psi}_i - \hat{\Psi}_i' \hat{\psi}_i^T \bar{Z}_i(\Theta_i, \phi_i)\big)\Big) - c_i \underline{g}_i z_i^2 \\ &+ z_i \underline{g}_i\big(\hat{\Xi}_i^T \hat{\Psi}_i' \tilde{\psi}_i^T \bar{Z}_i(\Theta_i, \phi_i) + e_i\big) + |z_i| \underline{g}_i \epsilon_i^* \\ &+ z_i g_i\big(\bar{x}_i, \theta_i\big)\big(z_{i+1} + y_{i+1}\big) + 0.2785 \underline{g}_i \upsilon_i\end{aligned} \tag{47}$$

with e_i being bounded by

$$\begin{aligned}|e_i| \leq& \|\psi_i\|_F \|\phi_i \hat{\Xi}_i^T \hat{\Psi}_i'\|_F + \|\Xi_i\| \|\hat{\Psi}_i' \hat{\psi}_i^T \bar{Z}_i(\Theta_i, \psi_i)\| \\ &+ |\Xi_i|_1\end{aligned} \tag{48}$$

Substituting (48) into (47) yields

$$\begin{aligned}\dot{V}_{z_i} \leq& z_i \underline{g}_i\left(\tilde{\Xi}_i^T\left(\hat{\Psi}_i - \hat{\Psi}_i' \hat{\psi}_i^T \bar{Z}_i(\Theta_i, \phi_i)\right)\right) + |z_i| \underline{g}_i a_i^* \\ &+ z_i g_i(\bar{x}_i, \theta_i)(z_{i+1} + y_{i+1}) + 0.2785 \underline{g}_i \upsilon_i \\ &+ z_i \underline{g}_i\left(\hat{\Xi}_i^T \hat{\Psi}_i' \tilde{\psi}_i^T \bar{Z}_i(\Theta_i, \phi_i)\right) - c_i \underline{g}_i z_i^2\end{aligned} \tag{49}$$

where $a_i^* = e_i^* + \varepsilon_i^*$.

Choose the Lyapunov function candidate as

$$V_i = V_{z_i} + tr\left\{\frac{\underline{g}_i \tilde{\psi}_i^T \Gamma_{\psi_i}^{-1} \tilde{\psi}_i}{2}\right\} + \frac{\underline{g}_i \tilde{\Xi}_i^T \Gamma_{\Xi_i}^{-1} \tilde{\Xi}_i}{2} + \frac{1}{2} y_{i+1}^2 \tag{50}$$

where $\tilde{\psi}_i = \psi_i - \hat{\psi}_i$ and $\tilde{\Xi}_i = \Xi_i - \hat{\Xi}_i$.

It follows from (38) and (49) that the time derivative of (50) is

$$\begin{aligned}\dot{V}_i \le\ & z_i\underline{g}_i\Big[\tilde{\Xi}_i^T\big(\hat{\Psi}_i - \hat{\Psi}_i'\hat{\psi}_i^T\bar{Z}_i(\Theta_i,\phi_i)\big)\Big] + |y_{i+1}B_{i+1}(\cdot)| \\ & + z_i\underline{g}_i\Big(\hat{\Xi}_i^T\hat{\Psi}_i'\tilde{\psi}_i^T\bar{Z}_i(\Theta_i,\phi_i)\Big) + 0.2785\underline{g}_i\upsilon_i \\ & + z_i g_i(\bar{x}_i,\theta_i)(z_{i+1}+y_{i+1}) + |z_i|\underline{g}_i a_i^* - \frac{y_{i+1}^2}{\iota_{i+1}} \\ & + tr\Big\{\underline{g}_i\tilde{\psi}_i^T\Gamma_{\psi_i}^{-1}\dot{\hat{\psi}}_i\Big\} - \underline{g}_i\tilde{\Xi}_i^T\Gamma_{\Xi_i}^{-1}\dot{\hat{\Xi}}_i - c_i\underline{g}_i z_i^2\end{aligned} \tag{51}$$

Recalling (35) and (36), we can obtain the time derivative of V_i as

$$\begin{aligned}\dot{V}_i \le\ & \sigma_i tr\left\{\underline{g}_i\tilde{\psi}_i^T\hat{\psi}_i\right\} + \sigma_i\underline{g}_i\tilde{\Xi}_i^T\hat{\Xi}_i + |y_{i+1}B_{i+1}(\cdot)| \\ & + z_i g_i(\bar{x}_i,\theta_i)(z_{i+1}+y_{i+1}) - y_{i+1}^2/\iota_{i+1} \\ & + 0.2785\underline{g}_i\upsilon_i - c_i\underline{g}_i z_i^2 + |z_i|\,\underline{g}_i a_i^*.\end{aligned} \tag{52}$$

Step n : Define $z_n = x_n - \alpha_{nf}$, whose time derivative is

$$\dot{z}_n = \varphi_n(x,\theta_n) + g_n(x,\theta_n)\,u - \dot{\alpha}_{nf} \tag{53}$$

Choose the virtual control law u and the parameter adaptation laws $\hat{\psi}_n$ and $\hat{\Xi}_n$ as follows

$$\begin{aligned} u &= -c_n z_n - \hat{\Xi}_n^T\Psi_n\left(\hat{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\right)\omega_n \\ \omega_n &= \tanh\left(\frac{z_n\hat{\Xi}_n^T\Psi_n\left(\hat{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\right)}{\upsilon_n}\right)\end{aligned} \tag{54}$$

$$\dot{\hat{\Xi}}_n = \Gamma_{\Xi_n}\left[z_n\left(\hat{\Psi}_n - \hat{\Psi}_n'\hat{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\right) - \sigma_n\hat{\Xi}_n\right] \tag{55}$$

$$\dot{\hat{\psi}}_n = \Gamma_{\psi_n}\left[z_n\bar{Z}_n(\Theta_n,\phi_n)\,\hat{\Xi}_n^T\hat{\Psi}_n' - \sigma_n\hat{\psi}_n\right] \tag{56}$$

where $c_n > 0$, $\sigma_n > 0$, and $\upsilon_n > 0$ are design parameters, and $\Gamma_{\psi_n} = \Gamma_{\psi_n}^T > 0$ and $\Gamma_{\Xi_n} = \Gamma_{\Xi_n}^T > 0$ are adaptive gain matrices. According to Lemma 1, we have $\hat{\psi}_n(t) \ge 0$ and $\hat{\Xi}_n(t) \ge 0$ for $\forall t \ge 0$ after selecting $\hat{\psi}_n(0) = 0$ and $\hat{\Xi}_n(0) = 0$ for $\forall t \ge 0$.

Similarly to the former steps, the continuous function $g_n(x,\theta_n)$ can be expressed in the following form:

$$g_n(x,\theta_n) = \gamma_n\left(\bar{z}_n, \bar{y}_n, \bar{\hat{\psi}}_{n-1}, \bar{\hat{\Xi}}_{n-1}, r, \theta_n\right) \tag{57}$$

where $\gamma_n(\cdot)$ is a continuous function and $\bar{z}_n$, $\bar{y}_n$, $\bar{\hat{\psi}}_{n-1}$, $\bar{\hat{\Xi}}_{n-1}$ and θ_n are defined in a similar way as after (38).

Define the following compact set:

$$\begin{aligned}\Omega_n := \Bigg\{ & \left[\bar{z}_n^T, \bar{y}_n^T, \bar{\hat{\psi}}_{n-1}^T, \bar{\hat{\Xi}}_{n-1}^T\right]^T \,\Big|\, \sum_{j=1}^n z_j^2 + \sum_{j=2}^n y_j^2 + \\ & \sum_{j=1}^{n-1}\left(\underline{g}_j\tilde{\Xi}_j^T\Gamma_{\Xi_j}^{-1}\tilde{\Xi}_j + \mathrm{tr}\left\{\underline{\mathrm{g}}_\mathrm{j}\tilde{\psi}_\mathrm{j}^\mathrm{T}\Gamma_{\psi_\mathrm{j}}^{-1}\tilde{\psi}_\mathrm{j}\right\}\right) \le 2\xi \Bigg\}\end{aligned} \tag{58}$$

It has to be noted that all the variables of $\gamma_n(\cdot)$ are included in the compact set $\Omega_n \times \Omega_0 \times \Omega_\theta$. In other words, in line with Lemma 5, the continuous function $\gamma_n(\cdot)$ has a maximum $\bar{g}_n = \max\limits_{\Omega_n\times\Omega_0\times\Omega_\theta} g_n(x,\theta_n)$ and a minimum $\underline{g}_n = \min\limits_{\Omega_n\times\Omega_0\times\Omega_\theta} g_n(x,\,\theta_n)$ such that

$$0 < \underline{g}_n \le g_n(x,\theta_n) \le \bar{g}_n\,. \tag{59}$$

From (59), we can rewrite (53) as

$$\dot{z}_n = \underline{g}_n\left(\chi_n(\Theta_n,\theta_n) + u + g_{n,0}u\right) \tag{60}$$

where $\chi_n(\Theta_n,\theta_n) = \underline{g}_n^{-1}\left(\varphi_n(x,\theta_n) - \dot{\alpha}_{nf}\right)$, $\Theta_n = [x, \dot{\alpha}_{nf}]^T$ and $g_{n,0} = g_n(x,\theta_n)/\underline{g}_n - 1 > 0$.

To address the stabilization of subsystem (53), take the following quadratic Lyapunov function candidate:

$$V_{z_n} = \frac{1}{2}z_n^2\,. \tag{61}$$

From (8), (54) and (60), the time derivative of V_{z_n} is

$$\begin{aligned}\dot{V}_{z_n} \le\ & z_n\underline{g}_n\left(\Xi_n^T\Psi_n\left(\psi_n^T\bar{Z}_n(\Theta_n,\phi_n)\right)\right) + |z_n|\,\underline{g}_n\varepsilon_n^* \\ & - z_n\underline{g}_n\left(\hat{\Xi}_n^T\Psi_n\left(\hat{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\right)\right) - c_n\underline{g}_n z_n^2 \\ & + z_n\underline{g}_n g_{n,0}u + 0.2785\underline{g}_n\upsilon_n\,.\end{aligned} \tag{62}$$

According to (54), (59) and (60), the following inequality holds

$$\begin{aligned} z_n\underline{g}_n g_{n,0}u =\ & \underline{g}_n g_{n,0}\Big[-c_n z_n^2 - \\ & z_n\hat{\Xi}_n^T\Psi_n\big(\hat{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\big)\omega_n\Big] < 0\end{aligned} \tag{63}$$

Using (10) and (63) leads to

$$\begin{aligned}\dot{V}_{z_n} \le\ & z_n\underline{g}_n\Big(\tilde{\Xi}_n^T\big(\hat{\Psi}_n - \hat{\Psi}_n'\hat{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\big)\Big) - c_n\underline{g}_n z_n^2 \\ & + z_n\underline{g}_n\big(\hat{\Xi}_n^T\hat{\Psi}_n'\tilde{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n) + e_n\big) + |z_n|\underline{g}_n\varepsilon_n^* \\ & + 0.2785\underline{g}_n\upsilon_n\end{aligned} \tag{64}$$

with e_n being bounded by

$$\begin{aligned}|e_n| \le\ & \|\psi_n\|_F\|\phi_n\hat{\Xi}_n^T\hat{\Psi}_n'\|_F + \|\Xi_n\|\|\hat{\Psi}_n'\hat{\psi}_n^T\bar{Z}_n(\Theta_n,\psi_n)\| \\ & + |\Xi_n|_1\end{aligned} \tag{65}$$

Substituting (65) into (64), one arrives

$$\begin{aligned}\dot{V}_{z_n} \le\ & z_n\underline{g}_n\left(\tilde{\Xi}_n^T\left(\hat{\Psi}_n - \hat{\Psi}_n'\hat{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\right)\right) \\ & + z_n\underline{g}_n\left(\hat{\Xi}_n^T\hat{\Psi}_n'\tilde{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\right) - c_n\underline{g}_n z_n^2 \\ & + 0.2785\underline{g}_n\upsilon_n + |z_n|\,\underline{g}_n a_n^*\end{aligned} \tag{66}$$

where $a_n^* = e_n^* + \varepsilon_n^*$.

Choose the following Lyapunov function candidate:

$$V_n = V_{z_n} + tr\left\{\frac{\underline{g}_n\tilde{\psi}_n^T\Gamma_{\psi_n}^{-1}\tilde{\psi}_n}{2}\right\} + \frac{\underline{g}_n\tilde{\Xi}_n^T\Gamma_{\Xi_n}^{-1}\tilde{\Xi}_n}{2} \tag{67}$$

where $\tilde{\psi}_n = \psi_n - \hat{\psi}_n$ and $\tilde{\Xi}_n = \Xi_n - \hat{\Xi}_n$.

It follows from (66) that the time derivative of $\dot{V}_n$ is

$$\begin{aligned}\dot{V}_n \le\ & z_n\underline{g}_n\Big(\tilde{\Xi}_n^T\big(\hat{\Psi}_n - \hat{\Psi}_n'\hat{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\big)\Big) - c_n\underline{g}_n z_n^2 \\ & + z_n\underline{g}_n\big(\hat{\Xi}_n^T\hat{\Psi}_n'\tilde{\psi}_n^T\bar{Z}_n(\Theta_n,\phi_n)\big) + 0.2785\underline{g}_n\upsilon_n \\ & + |z_n|\underline{g}_n a_n^* - tr\Big\{\underline{g}_n\tilde{\psi}_n^T\Gamma_{\psi_n}^{-1}\dot{\hat{\psi}}_n\Big\} - \underline{g}_n\tilde{\Xi}_n^T\Gamma_{\Xi_n}^{-1}\dot{\hat{\Xi}}_n\end{aligned} \tag{68}$$

Using (55) and (56), one reaches

$$\begin{aligned}\dot{V}_n \le\ & \sigma_n tr\left\{\underline{g}_n\tilde{\psi}_n^T\hat{\psi}_n\right\} + \sigma_n\underline{g}_n\tilde{\Xi}_n^T\hat{\Xi}_n + |z_n|\,\underline{g}_n a_n^* \\ & + 0.2785\underline{g}_n\upsilon_n - c_n\underline{g}_n z_n^2.\end{aligned} \tag{69}$$

IV. Stability analysis

Consider the following Lyapunov function:

$$V = V_1 + V_2 + \cdots + V_n = \sum_{i=1}^{n} V_i \tag{70}$$

with $(i = 1, \ldots, n-1)$

$$V_i = \frac{1}{2}z_i^2 + \mathrm{tr}\left\{\frac{\underline{g}_i\tilde{\psi}_i^{\mathrm{T}}\Gamma_{\psi_i}^{-1}\tilde{\psi}_i}{2}\right\} + \frac{\underline{g}_i\tilde{\Xi}_i^{\mathrm{T}}\Gamma_{\Xi_i}^{-1}\tilde{\Xi}_i}{2} + \frac{y_{i+1}^2}{2}$$

$$V_n = \frac{1}{2}z_n^2 + \mathrm{tr}\left\{\frac{\underline{g}_n\tilde{\psi}_n^{\mathrm{T}}\Gamma_{\psi_n}^{-1}\tilde{\psi}_n}{2}\right\} + \frac{\underline{g}_n\tilde{\Xi}_n^{\mathrm{T}}\Gamma_{\Xi_n}^{-1}\tilde{\Xi}_n}{2}\ .$$

Theorem 1: Consider the strict-feedback nonlinear system described by (1) with Assumptions 1-2. Consider the intermediate virtual control laws (18), (34), the actual control law (54), and the adaptive laws (19), (20), (35), (36), (55), and (56). For any $\xi > 0$, and bounded initial conditions satisfying $\hat{\psi}_i(0) \geq 0$, $\hat{\Xi}_i(0) \geq 0$ and $V(0) \leq \xi$ with ξ being any given positive constant, there exist design parameters c_i, σ_i, υ_i, and τ_i such that: i) The compact set $\Omega_n \times \Omega_0 \times \Omega_\theta$ is an invariant set, namely, $V(t) \leq \xi$ for $\forall t > 0$, and hence all the signals in the closed-loop system are semi-globally uniformly ultimately bounded (SGUUB); ii) The output tracking error z_1 satisfies $\lim_{t\to\infty} |z_1| \leq \sqrt{2\Sigma}$, where $\Sigma > 0$ is a constant that can be made arbitrarily small by properly selecting the design parameters.

Proof: It follows from (32), (52), and (69) that

$$\begin{aligned}\dot{V} \leq & \sum_{i=1}^{n}\left(\sigma_i \mathrm{tr}\left\{\underline{g}_i\tilde{\psi}_i^{\mathrm{T}}\hat{\psi}_i\right\} + \sigma_i\underline{g}_i\tilde{\Xi}_i^{\mathrm{T}}\hat{\Xi}_i\right) - \sum_{i=1}^{n} c_i\underline{g}_i z_i^2 \\ & + \sum_{i=1}^{n}\left(0.2785\underline{g}_i\upsilon_i + |z_i|\,\underline{g}_i a_i^*\right) \\ & + \sum_{i=1}^{n-1} z_i g_i(\bar{x}_i,\theta_i)(z_{i+1} + y_{i+1}) \\ & + \sum_{i=1}^{n-1}\left(-\frac{y_{i+1}^2}{\iota_{i+1}} + |y_{i+1}B_{i+1}(\cdot)|\right).\end{aligned} \tag{71}$$

It is apparent from (38) that all the variables of $B_{i+1}(\cdot)$ are included in the compact set $\Omega_i \times \Omega_0 \times \Omega_\theta$. Consequently, $B_{i+1}(\cdot)$ has a maximum D_{i+1} over $\Omega_i \times \Omega_0 \times \Omega_\theta$. So, on $\Omega_i \times \Omega_0 \times \Omega_\theta$, we have $|B_{i+1}(\cdot)| \leq D_{i+1}$.

By completion of squares, it holds that

$$\tilde{\Xi}_i^T\hat{\Xi}_i \leq \frac{\|\Xi_i\|^2}{2} - \frac{\left\|\tilde{\Xi}_i\right\|^2}{2}$$

$$\mathrm{tr}\left\{\tilde{\psi}_i^{\mathrm{T}}\hat{\psi}_i\right\} \leq \frac{\|\psi_i\|_F^2}{2} - \frac{\left\|\tilde{\psi}_i\right\|_F^2}{2}$$

$$|y_{i+1}D_{i+1}(\cdot)| \leq \frac{y_{i+1}^2 D_{i+1}^2(\cdot)}{2k_1} + \frac{k_1}{2}$$

$$|z_i|\,\underline{g}_i a_i^* \leq \frac{\underline{g}_i^2 z_i^2}{2k_3} + \frac{k_3 {a_i^*}^2}{2}$$

$$z_i g_i(\bar{x}_i,\theta_i) z_{i+1} \leq \frac{z_i^2}{2} + \frac{g_i^2(\bar{x}_i,\theta_i) z_{i+1}^2}{2}$$

$$z_i g_i(\bar{x}_i,\theta_i) y_{i+1} \leq \frac{k_2 g_i^2(\bar{x}_i,\theta_i) y_{i+1}^2}{2} + \frac{z_i^2}{2k_2}$$

where $k_1 > 0$, $k_2 > 0$, and $k_3 > 0$ are unknown constants.

Thus, we can rewrite (71) as

$$\begin{aligned}\dot{V} \leq & \sum_{i=1}^{n-1}\left(-\frac{y_{i+1}^2}{\iota_{i+1}} + \frac{y_{i+1}^2 D_{i+1}^2(\cdot)}{2k_1} + \frac{k_2 g_i^2(\bar{x}_i,\theta_i) y_{i+1}^2}{2}\right) \\ & + \sum_{i=1}^{n-1}\left(\frac{z_i^2}{2k_2} + \frac{z_i^2}{2} + \frac{g_i^2(\bar{x}_i,\theta_i) z_{i+1}^2}{2}\right) \\ & - \sum_{i=1}^{n}\left[\frac{1}{2}\underline{g}_i\sigma_i\left(\left\|\tilde{\Xi}_i\right\|^2 + \left\|\tilde{\psi}_i\right\|_F^2\right)\right] \\ & + \sum_{i=1}^{n}\left(-c_i\underline{g}_i z_i^2 + \frac{\underline{g}_i^2 z_i^2}{2k_3}\right) + \varpi_1\end{aligned} \tag{72}$$

where $\varpi_1 = \sum_{i=1}^{n}\left[\frac{1}{2}\underline{g}_i\sigma_i\left(\|\Xi_i\|^2 + \|\psi_i\|_F^2\right) + 0.2785\underline{g}_i\upsilon_i\right] + \frac{(n-1)k_1}{2} + \sum_{i=1}^{n}\frac{k_3 {a_i^*}^2}{2}$.

From (14), (42), and (59), we can further rewrite (72) as

$$\begin{aligned}\dot{V} \leq & -\sum_{i=2}^{n-1}\left(c_i\underline{g}_i - \frac{1}{2} - \frac{1}{2k_2} - \frac{\bar{g}_{i-1}^2}{2} - \frac{\underline{g}_i^2}{2k_3}\right)z_i^2 \\ & - \sum_{i=1}^{n}\left(\frac{\sigma_i}{\lambda_{\max}\left(\Gamma_{\psi_i}^{-1}\right)}\mathrm{tr}\left\{\frac{\underline{g}_i\tilde{\psi}_i^{\mathrm{T}}\Gamma_{\psi_i}^{-1}\tilde{\psi}_i}{2}\right\}\right) \\ & - \sum_{i=1}^{n}\left(\frac{\sigma_i}{\lambda_{\max}\left(\Gamma_{\Xi_i}^{-1}\right)}\frac{\underline{g}_i\tilde{\Xi}_i^T\Gamma_{\Xi_i}^{-1}\tilde{\Xi}_i}{2}\right) + \varpi_1 \\ & - \sum_{i=1}^{n-1}\left(\frac{1}{\iota_{i+1}} - \frac{D_{i+1}^2}{2k_1} - \frac{k_2\bar{g}_i^2}{2}\right)y_{i+1}^2 \\ & - \left(c_1\underline{g}_1 - \frac{1}{2} - \frac{1}{2k_2} - \frac{\underline{g}_1^2}{2k_3}\right)z_1^2 \\ & - \left(c_n\underline{g}_n - \frac{\bar{g}_{n-1}^2}{2} - \frac{\underline{g}_n^2}{2k_3}\right)z_n^2\ .\end{aligned} \tag{73}$$

Choose $c_1 \geq \underline{g}_1^{-1}\left(\frac{1}{2} + \frac{1}{2k_2} + \frac{\underline{g}_1^2}{2k_3} + \varpi_2\right)$, $c_i \geq \underline{g}_i^{-1}\left[\frac{1}{2} + \frac{1}{2k_2} + \frac{\bar{g}_{i-1}^2}{2} + \frac{\underline{g}_i^2}{2k_3} + \varpi_2\right]$, $(i = 2, ..., n-1)$, $c_n \geq \underline{g}_n^{-1}\left(\frac{\bar{g}_{n-1}^2}{2} + \frac{\underline{g}_n^2}{2k_3} + \varpi_2\right)$ and $\frac{1}{\iota_{i+1}} \geq \frac{D_{i+1}^2}{2k_1} + \frac{k_2\bar{g}_i^2}{2} + \varpi_2$, $(i = 1, ..., n-1)$ with $\varpi_2 = \min\left\{\frac{\sigma_i}{\lambda_{\max}(\Gamma_{\psi_i}^{-1})}, \frac{\sigma_i}{\lambda_{\max}(\Gamma_{\Xi_i}^{-1})}\right\}$.

Invoking (73), one gets

$$\dot{V} \leq -\varpi_2 V + \varpi_1\ . \tag{74}$$

Multiplying (74) by $e^{\varpi_2 t}$ and integrating over $[0, t]$ yields

$$V(t) \leq (V(0) - \Sigma)\,e^{-\varpi_2 t} + \Sigma \leq V(0) + \Sigma \tag{75}$$

where $\Sigma = \varpi_1/\varpi_2$, which can be made arbitrarily small by increasing c_i, and meanwhile decreasing $\lambda_{\max}\left(\Gamma_{\Xi_i}^{-1}\right)$, $\lambda_{\max}\left(\Gamma_{\psi_i}^{-1}\right)$, σ_i, and υ_i. It is always possible to make $\varpi_1/\varpi_2 \leq \xi$ by choosing the design parameters appropriately. Then, in view of (74), we have that $\dot{V} \leq 0$ holds for $V = \xi$: consequently, the compact set $\Omega_n \times \Omega_0 \times \Omega_\omega$ is an invariant set and all signals of the closed-loop system are SGUUB. Therefore, property (i) of Theorem 1 is proved.

From (30), (50), and (67), we have $\sum_{i=1}^{n} z_i^2/2 \le V$. By using (75) and $\sum_{i=1}^{n} z_i^2/2 \le V$, the following inequality holds:

$$\lim_{t\to\infty} |z_1| \le \lim_{t\to\infty} \sqrt{2V(t)} \le \sqrt{2\Sigma}\,. \tag{76}$$

This completes the proof of Theorem 1. ■

Remark 6: The novel contribution of Theorem 1 is the adoption of invariant set theory in order to handle Assumption 1. In fact, the stability analysis in the Proof of Theorem 1 is formulated based on the condition that all the state variables stay inside the set $\Omega_n \times \Omega_0 \times \Omega_\theta$ where $\Omega_n \subset \left(\Omega_{n-1} \times R^4\right) \subset \cdots \subset \left(\Omega_2 \times R^{4(n-2)}\right) \subset \left(\Omega_1 \times R^{4(n-1)}\right)$. The fact that $\Omega_n \times \Omega_0 \times \Omega_\theta$ is an invariant set (as explained after (75)) validates the stability analysis even when the control gain functions are possibly unbounded functions of the state.

V. Simulation results

In this section, two examples are given to illustrate the effectiveness of the proposed method.

Example 1: Consider a second-order system described by

$$\begin{cases} \dot{x}_1 = \left(1.1 - 0.1\cos\left(x_1\theta_1(t)\right)\right)e^{x_1^2}x_2 + \frac{x_1^2\theta_1^2(t)+x_1\theta_1(t)}{x_1^2\theta_1^2(t)+1} \\ \dot{x}_2 = \left(0.9 - 0.1\theta_2^2(t)\sin^2(x_1x_2)\right)e^{|x_2|}u + \sin\left(x_1x_2\theta_2(t)\right) \\ \qquad \times \exp\left(-x_1^2x_2^2\theta_2^2(t)\right) \\ y = x_1 \end{cases} \tag{77}$$

where the unknown time-varying disturbances are $\theta_1(t) = |\cos(0.5t)|$, $\theta_2(t) = |\cos(0.25t)|$ with known periods $T_1 = 2\pi$ and $T_2 = 4\pi$, respectively. Note that the control gain functions $g_1 = (1.1 - 0.1\cos(x_1\theta_1(t)))e^{x_1^2}$ and $g_2 = e^{|x_2|}\left(0.9 - 0.1\theta_2^2(t)\sin^2(x_1x_2)\right)$ cannot be bounded a priori because of the exponential term. However, they satisfy Assumption 1. Therefore, our proposed scheme can be used to nonlinear system (77) while existing methods cannot be applied.

Based on Theorem 1, the actual control law and virtual control law are chosen as follows:

$$u = -8z_2 - \hat{\Xi}_2^T\Psi_2\left(\hat{\psi}_2^T\bar{Z}_2(\Theta_2,\phi_2)\right) \cdot \tanh\left[\frac{z_2\hat{\Xi}_2^T\Psi_2\left(\hat{\psi}_2^T\bar{Z}_2(\Theta_2,\phi_2)\right)}{0.25}\right]$$

$$\alpha_1 = -6z_1 - \hat{\Xi}_1^T\Psi_1\left(\hat{\psi}_1^T\bar{Z}_1(\Theta_1,\phi_1)\right) \cdot \tanh\left[\frac{z_1\hat{\Xi}_1^T\Psi_1\left(\hat{\psi}_1^T\bar{Z}_1(\Theta_1,\phi_1)\right)}{0.25}\right]$$

The adaptation parameters laws are given by

$$\dot{\hat{\Xi}}_1 = 1.5\cdot\left[z_1\left(\hat{\Psi}_1 - \hat{\Psi}_1'\hat{\psi}_1^T\bar{Z}_1(\Theta_1,\phi_1)\right) - 0.25\hat{\Xi}_1\right]$$
$$\dot{\hat{\Xi}}_2 = 1.5\cdot\left[z_2\left(\hat{\Psi}_2 - \hat{\Psi}_2'\hat{\psi}_2^T\bar{Z}_2(\Theta_2,\phi_2)\right) - 0.1\hat{\Xi}_2\right]$$
$$\dot{\hat{\psi}}_1 = 1.2\cdot\left[z_1\bar{Z}_1(\Theta_1,\phi_1)\hat{\Psi}_1^T\hat{\Psi}_1' - 0.25\hat{\psi}_1\right]$$
$$\dot{\hat{\psi}}_2 = 1.2\cdot\left[z_2\bar{Z}_2(\Theta_2,\phi_2)\hat{\Psi}_2^T\hat{\Psi}_2' - 0.1\hat{\psi}_2\right]$$

In simulation, let the initial conditions be $[x_1(0), x_2(0)]^T = [0, 0.5]^T$, $\hat{\psi}_1(0) = \hat{\psi}_2(0) = 0$ and $\hat{\Xi}_1(0) = \hat{\Xi}_2(0) = 0$. The desired reference trajectory is $r = \sin(t) + \sin(0.5t)$. The simulation results are shown in Figs. 1-4.

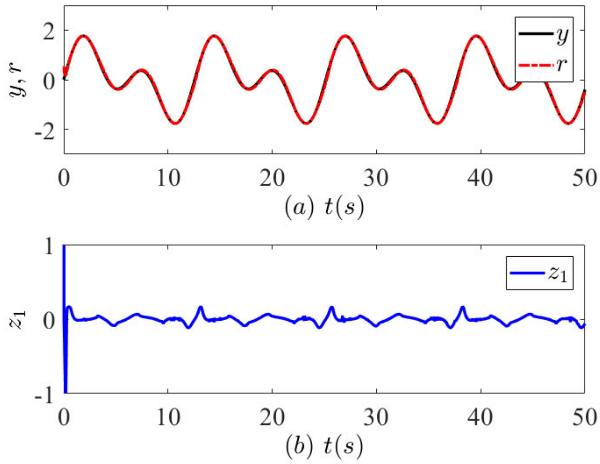

Fig. 1: (a): System output y and desired trajectory r; (b): Tracking error z_1.

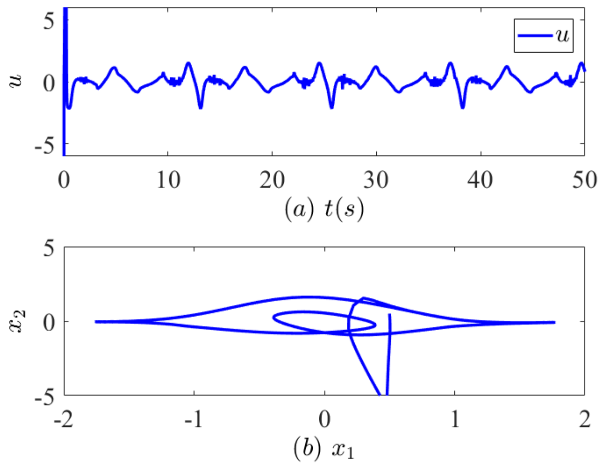

Fig. 2: (a): Control input u; (b): Phase portrait of system states x_1 and x_2.

It can be easily seen from Fig. 1 that the system output y can follow the desired reference trajectory r with bounded error. Fig. 2 (a) shows that the proposed scheme has a bounded control input and Fig. 2 (b) is presented to demonstrate the boundedness of systems states x_1 and x_2. Additionally, Fig. 3 is given to show the response curves of adaptive parameters $\hat{\psi}_1$, $\hat{\psi}_2$, $\hat{\Xi}_1$, and $\hat{\Xi}_2$.

To further demonstrate the effectiveness of the presented approach with different design parameters, three different sets of parameters are considered.

Case 1: $c_1 = 8$, $c_2 = 9$, $\sigma_1 = 0.25$, $\sigma_2 = 0.15$, $\upsilon_1 = 0.25$, $\upsilon_2 = 0.2$, $\iota_2 = 0.05$, $\Gamma_{\psi_1} = \Gamma_{\psi_2}^T = 2$, and $\Gamma_{\Xi_1} = \Gamma_{\Xi_2}^T = 2.5$.

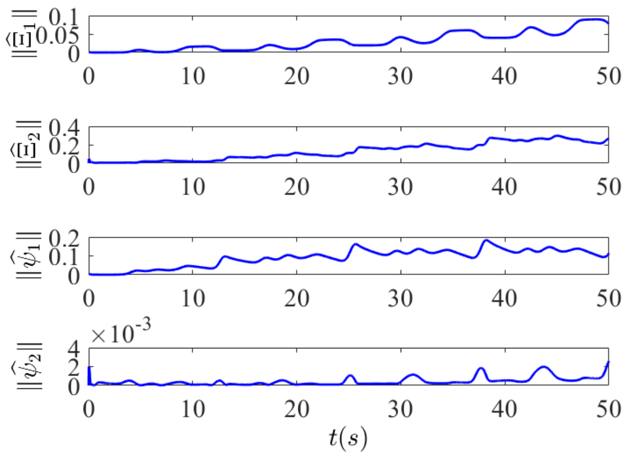

Fig. 3: Curves of $\left\|\hat{\Xi}_1\right\|$, $\left\|\hat{\Xi}_2\right\|$, $\left\|\hat{\psi}_1\right\|$ and $\left\|\hat{\psi}_2\right\|$.

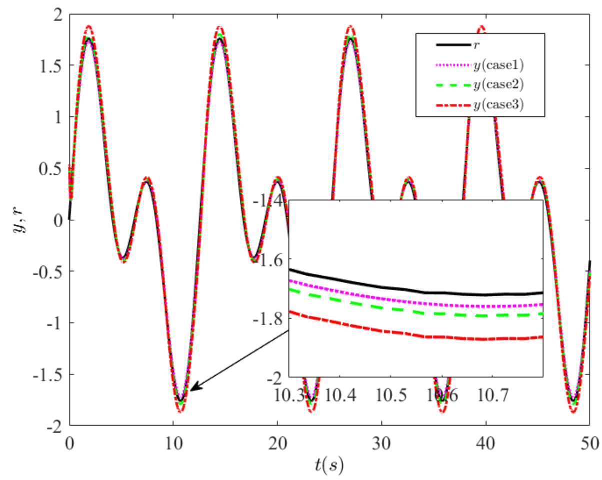

Fig. 4: System output y under 3 different sets of parameters.

Case 2: $c_1 = 6$, $c_2 = 7$, $\sigma_1 = 0.35$, $\sigma_2 = 0.25$, $\upsilon_1 = 0.35$, $\upsilon_2 = 0.3$, $\iota_2 = 0.05$, $\Gamma_{\psi_1} = \Gamma_{\psi_2}^T = 1.5$, and $\Gamma_{\Xi_1} = \Gamma_{\Xi 2}^T = 2$.

Case 3: $c_1 = 4$, $c_2 = 5$, $\sigma_1 = 0.5$, $\sigma_2 = 0.35$, $\upsilon_1 = 0.5$, $\upsilon_2 = 0.45$, $\iota_2 = 0.05$, $\Gamma_{\psi_1} = \Gamma_{\psi_2}^T = 1$, and $\Gamma_{\Xi_1} = \Gamma_{\Xi_2}^T = 1.5$.

The system output responses are presented in Fig. 4, which confirms the fact that system output tracking error can be made arbitrarily small by increasing c_i, Γ_{ψ_i} and $\Gamma_{\Xi i}$ and decreasing σ_i and υ_i (cf. discussion after (75)).

Example 2: Consider the following van der Pol oscillator:

$$\begin{cases} \dot{x}_1 = x_2 + x_1 - 1/3x_1^3 + p + F(t) \\ \dot{x}_2 = u + 0.1(x_1 + a - bx_2) \\ y = x_1 \end{cases} \tag{78}$$

where $F(t) = q\cos(\theta t)$ is a periodic exciting signal. When the system parameters are chosen as $\theta = 1$, $a = 0.7$, $b = 0.8$, $p = 0$, and $q = 0.74$, the system (78) without control will present chaotic behavior [33]. The period of the time-varying disturbances $F(t)$ is 2π.

In accordance with the Theorem 1, the actual control law is chosen as

$$\begin{aligned} u = & -c_2 z_2 - \hat{\Xi}_2^T \Psi_2\left(\hat{\psi}_2^T \bar{Z}_2(\Theta_2, \phi_2)\right) \\ & \cdot \tanh\left[\frac{z_2 \hat{\Xi}_2^T \Psi_2\left(\hat{\psi}_2^T \bar{Z}_2(\Theta_2, \phi_2)\right)}{\upsilon_2}\right] \end{aligned}$$

The virtual control law is chosen as

$$\begin{aligned} \alpha_1 = & -c_1 z_1 - \hat{\Xi}_1^T \Psi_1\left(\hat{\psi}_1^T \bar{Z}_1(\Theta_1, \phi_1)\right) \\ & \cdot \tanh\left[\frac{z_1 \hat{\Xi}_1^T \Psi_1\left(\hat{\psi}_1^T \bar{Z}_1(\Theta_1, \phi_1)\right)}{\upsilon_1}\right] \end{aligned}$$

In simulation, the adaption laws are provided by (19), (20), (55) and (56). The design parameters are chosen as $\sigma_1 = 0.1$, $\sigma_2 = 0.01, \Gamma_{\Xi_1} = \Gamma_{\Xi_2} = \mathrm{diag}[0.5]$, $\Gamma_{\psi_1} = \Gamma_{\psi_2} = \mathrm{diag}[0.1]$, $\upsilon_1 = \upsilon_2 = 0.25$, $\iota_2 = 0.05$, and $c_1 = c_2 = 8$. Let the initial conditions be $[x_1(0), x_2(0)]^T = [1, 0.5]^T$, $\hat{\psi}_1(0) = \hat{\psi}_2(0) = 0$, and $\hat{\Xi}_1(0) = \hat{\Xi}_2(0) = 0$. The desired reference trajectory is $r = 0.5(\sin(t) + \cos(0.5t))$. Because the control gain functions are bounded a priori, this system is amenable for some comparisons with existing methods. For comparison purposes, two schemes are taken into account, the method proposed here and an existing method (of reference [33]). The simulation results are shown in Figs. 5-6 for the proposed scheme, while the comparison in terms of the tracking error is given in Fig. 7.

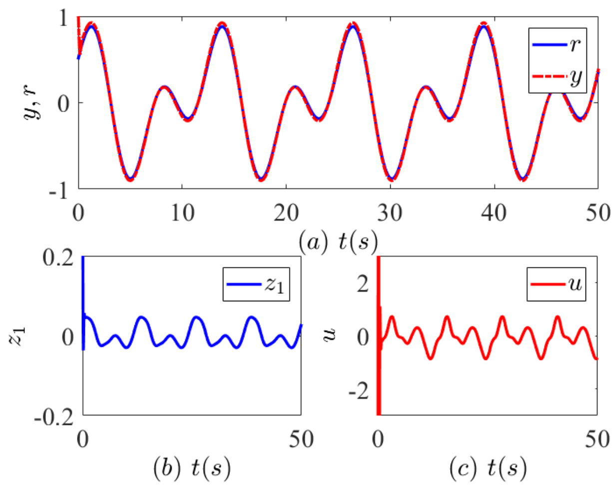

Fig. 5: (a): System output y and desired trajectory r; (b): System output tracking error z_1; (c): Control input u.

From Fig. 5 (a) and (b), we can see that the system output y tracks the desired trajectory r with a small tracking error. Fig. 5 (c) shows bounded control input. Fig. 6 illustrates the boundedness of adaptation parameters $\hat{\psi}_1$, $\hat{\psi}_2$, $\hat{\Xi}_1$, and $\hat{\Xi}_2$, respectively. From Fig. 7, we can see that, compared to the method of [33], the tracking errors have a comparable order of magnitude; however, the proposed approach provides smaller peaks and a smoother response.

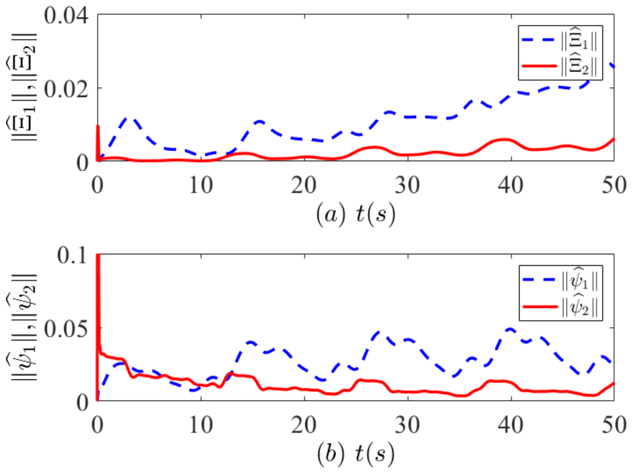

Fig. 6: Evolutions of $\left\|\hat{\Xi}_1\right\|$, $\left\|\hat{\Xi}_2\right\|$, $\left\|\hat{\psi}_1\right\|$, and $\left\|\hat{\psi}_2\right\|$.

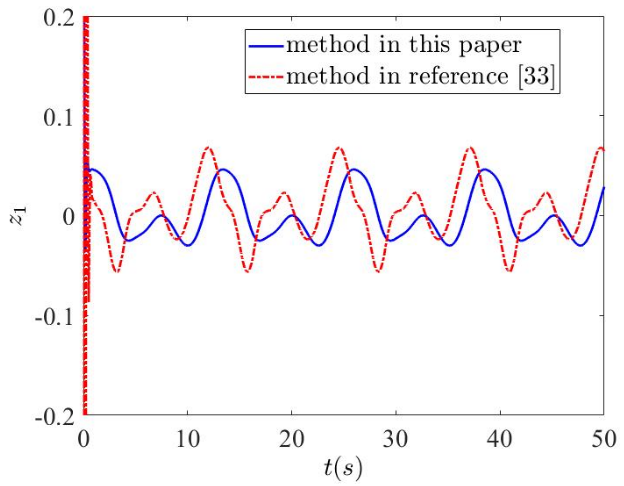

Fig. 7: Tracking errors of our approach and that of reference [33].

VI. Conclusions

In this paper, we have proposed a novel adaptive fuzzy control scheme based on DSC for a more general class of strict-feedback nonlinear systems with periodic perturbations of the control gain functions. In comparison with the existing results, the restrictive assumption that upper and lower bounds of control gain functions are assumed to be known a priori has been removed. This has been achieved by introducing appropriate compact invariant sets where maximum and minimum values of the continuous control gain functions can be defined a posteriori and used for stability analysis. This significantly relaxes a severe limitation while still guaranteeing controllability and thus well-posedness of the adaptive control problem. Interesting future work might involve to investigate if the invariant set mechanism holds also in the framework of finite time stability as [3] and [5], since many engineering applications often require to achieve control objective in finite time.

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