Delft University of Technology

Delft Center for Systems and Control

Technical report 20-001

Analytic expressions in stochastic max-plus-linear algebra and their application in model predictive control*

T.J.J. van den Boom and B. De Schutter

If you want to cite this report, please use the following reference instead:
T.J.J. van den Boom and B. De Schutter, "Analytic expressions in stochastic max-plus-linear algebra and their application in model predictive control," *IEEE Transactions on Automatic Control*, vol. 66, no. 4, pp. 1872–1878, Apr. 2021. doi:10.1109/TAC.2020.2997851

Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

Analytic expressions in stochastic max-plus-linear algebra and their application in model predictive control

Ton J.J. van den Boom and Bart De Schutter *Fellow, IEEE*

***Abstract*—The class of max-plus-linear systems can model discrete event systems with synchronization but no choice. Model mismatch and/or disturbances can be characterized as stochastic uncertainties. In stochastic max-plus-linear systems one often needs to compute the expectation of a max-plus-scaling function or the chance constraint of a max-plus-scaling function. The algorithms available in literature are either computationally too expensive or only give an approximation. In this paper we derive an analytic expression for both the expectation and the chance constraint of a max-plus-scaling function. Both can be written in the form of a piece-wise polynomial function in the components of the control variables. The analytic function can be derived offline and can be evaluated online in a quick and efficient way. We also show how the expressions can be used in a model predictive control setting and show the efficiency of the proposed approach with a worked example.**

***Index Terms*—max-plus-linear systems, stochastic systems, nonlinear predictive control, discrete-event systems**

I. Introduction

Discrete-event models such as queuing systems, (extended) state machines, formal language models, automata, temporal logic models, generalized semi-Markov processes, Petri nets, etc. are in general nonlinear in conventional algebra. However, there exists an important class of discrete-event systems, namely the max-plus-linear systems, for which the model is linear in the max-plus algebra. The class of max-plus-linear systems consists of discrete-event systems with synchronization but no choice [2]. In stochastic discrete-event systems, processing times and/or transportation times are stochastic quantities, since in practice stochastic fluctuations in their values can be caused, e.g. by machine failure or depreciation [16]. To model this stochasticity in discrete-event systems we will often use stochastic max-plus-linear expressions or stochastic max-plus-scaling functions [2], [3], [11], [14], [16], [17], [18].

To control stochastic max-plus-linear systems, one efficient control approach is model predictive control (MPC) [13], [15], [10], [22]. Note that, due to the fact that the uncertainty enters the equations of the stochastic max-plus-linear expressions in a multiplicative way [20], it is not straightforward to use control techniques (e.g. residuation) other than MPC, and to the authors' best knowledge there is no literature on this specific topic.

MPC is an online model-based approach, in which at each event step an optimal control sequence is computed. This optimization is done over a finite sequence of events, and for each event step, only the first sample of the optimal control sequence will be applied to the system. For the next step, the horizon will be shifted forward and a new optimal control sequence will be computed.

In stochastic MPC an optimization problem has to be solved at each event step. In stochastic systems, the objective function defined in the MPC optimization problem usually consists of an expected value of a stochastic max-plus-scaling function [20], and is minimized subject to chance constraints. In general, the expected value is computed using either numerical integration or some available analytic approaches, which are all very time-consuming. Hence, solving this optimization problem creates a considerable computational complexity due to the presence of the expected value.

In literature two approaches have been proposed to reduce the computational burden of computing the expectation or the chance constraint of a max-plus-scaling function. The first approach [21] considers a method based on variability expansion. In particular, it has been shown that the computational load is reduced if one decreases the level of 'randomness' in the system. Three other methods use approximation of the expectation and/or the chance constraints. In [9] the approximation of the expectation is based on the raw moments of a random variable. This results in a much lower computational complexity and a much lower computation time while still guaranteeing a good performance. In [18] the chance constraints are approximated and substituted with a finite number of pointwise constraints at independently generated scenarios of the uncertainties. In [23] multivariate chance constraints are converted into univariate chance constraints using Boole's inequality or into linear constraints on the inputs using Chebyshev's inequality. The chance constraints can then be computed efficiently.

The main contribution of this paper is that we derive analytic expressions for the exact computation of the expectation and chance constraint of a stochastic max-plus-scaling function. We do this in a two-step algorithm: an offline computation of the analytic expressions and an online use of these expressions in e.g. MPC. The offline part only has to be done once. When the analytic expressions have been derived, they can easily be used in the computation of the MPC control law by means of a fast evaluation of a piecewise polynomial function for every event step. In this paper we have scaled up the algorithm from small size problems in [20] to medium size problems.

This paper is organized as follows: Section II presents some existing results on stochastic max-plus-scaling functions. In

Delft University of Technology, Delft Center for Systems and Control, Delft, The Netherlands (e-mail: {a.j.j.vandenboom, b.deschutter}@tudelft.nl).

Section III and IV we derive an analytic solutions to compute the expected value and the chance constraint of max-plus-scaling functions in the presence of a uniform distribution. Section VI shows how the derived analytic expressions can be used in model predictive control of max-plus-linear systems, finishing with a small case study.

II. STOCHASTIC MAX-PLUS-SCALING FUNCTIONS

In this section we will give some background, define the problem, and present a basic lemma on level sets.

A. Max-plus-linear systems and max-plus-scaling functions

First we give the basic definitions of the max-plus algebra [2], [7].

Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The max-plus-algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as $x \oplus y = \max(x, y)$ and $x \otimes y = x + y$ for any $x, y \in \mathbb{R}_\varepsilon$, and

$$[A \oplus B]_{i,j} = a_{i,j} \oplus b_{i,j} = \max(a_{i,j}, b_{i,j})$$

$$[A \otimes C]_{i,j} = \bigoplus_{k=1}^{n} a_{i,k} \otimes c_{k,j} = \max_{k=1,\ldots,n} (a_{i,k} + c_{k,j})$$

for matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$.

Let w be in a bounded polyhedral set $\mathcal{W} = \{w \in \mathbb{R}^p | C_w w \le d_w\}$. and let e be a uniformly distributed stochastic variable with probability density function

$$p(e) = \begin{cases} 2^{-n} & \text{for } |e_i| \le 1, \text{ for all } i = 1, \ldots, n \\ 0 & \text{elsewhere} \end{cases} \quad (1)$$

The domain $\mathcal{E}$ for e is n-dimensional unit hypercube. Let $\alpha \in \mathbb{R}^m$, $\Lambda \in \mathbb{R}^{m\times p}$, $\Gamma \in \mathbb{R}^{m\times n}$, $w \in \mathcal{W}$, $e \in \mathcal{E} \subset \mathbb{R}^n$. Define the Max-Plus-Scaling (MPS) function $f : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$f(w, e) = \max_j (\alpha_j + \Lambda_j\, w + \Gamma_j\, e) \quad (2)$$

where Λ_j and Γ_j stand for the jth row of Λ and Γ, respectively.

In [20] we introduced the following stochastic max-plus-linear model:

$$\begin{aligned} x(k) &= A(k) \otimes x(k-1) \oplus B(k) \otimes u(k), \\ y(k) &= C(k) \otimes x(k) \end{aligned} \quad (3)$$

where the system matrices $A(k) \in \mathbb{R}_\varepsilon^{n_x\times n_x}$, $B(k) \in \mathbb{R}_\varepsilon^{n_x\times n_u}$, and $C(k) \in \mathbb{R}_\varepsilon^{n_y\times n_x}$ model the uncertainty (e.g. modeling errors or disturbances) in the system. All entries of the system matrices are assumed to be max-plus-scaling functions of an uncertainty vector $e(k)$, which gathers all uncertainties. The vector $e(k)$ is a stochastic variable.

Two stochastic quantities are important, namely the *expectation* of $f(w, e)$:

$$\mathbb{E}[f(w, e)] = \mathbb{E}[\max_j (\alpha_j + \Lambda_j\, w + \Gamma_j\, e)] \quad (4)$$

where $\mathbb{E}[\cdot]$ denotes the expectation, and the *chance constraint* that $f(w, e)$ is less than B, so

$$\mathbb{P}[f(w, e) \le B] = \mathbb{P}[\max_j (\alpha_j + \Lambda_j\, w + \Gamma_j\, e) \le B] \quad (5)$$

where $\mathbb{P}[\cdot]$ denotes the probability and $B \in \mathbb{R}$ is a constant.

In this paper we aim to

1) Find an analytic expression for the *expectation* $\mathbb{E}[f(w, e)]$.
2) Find an analytic expression for the *chance constraint* $\mathbb{P}[f(w, e) \le B]$.
3) Use the derived expressions for the expectation and the chance constraint in model predictive control (MPC) for stochastic max-plus-linear systems.

B. Level sets

In this subsection we discuss level sets. We introduce a variable z that can be either w or $\begin{bmatrix} w^T & B \end{bmatrix}^T$.

Lemma 1: Let $\mathcal{Z} = \{z \in \mathbb{R}^{p_z} | C_z z \le d_z\}$ be a bounded polyhedral set, and consider for a fixed value $z \in \mathcal{Z}$ the level set $\Phi(z) = \{\ e \mid A_z\, z + A_e\, e \le b\}$ where $A_e \in \mathbb{R}^{q\times n}$, $A_z \in \mathbb{R}^{q\times p_z}$, and $b \in \mathbb{R}^q$. Let $\mathcal{S} = \{S_1, \ldots, S_L\}$ be the set of all $n \times q$ matrices that consist of n rows of the $q \times q$ identity matrix such that the matrix $S_\ell\, A_e$ is invertible. For any matrix $S_\ell \in \mathcal{S}$ we can define a matrix $T_\ell \in \mathbb{R}^{(q-n)\times q}$ with the remaining rows of the $q \times q$ identity matrix. Define for $\ell = 1, \ldots, L$ the vectors $\sigma_\ell \in \mathbb{R}^n$, $g_\ell \in \mathbb{R}^{q-n}$ and matrices $\tau_\ell \in \mathbb{R}^{n\times p_z}$, $F_\ell \in \mathbb{R}^{(q-n)\times p_z}$ as follows:

$$\sigma_\ell = (S_\ell\, A_e)^{-1} S_\ell b \quad , \quad F_\ell = T_\ell\, (I - A_e (S_\ell\, A_e)^{-1} S_\ell) A_z$$

$$\tau_\ell = (S_\ell\, A_e)^{-1} S_\ell A_z \quad , \quad g_\ell = T_\ell\, (I - A_e (S_\ell\, A_e)^{-1} S_\ell) b$$

Now let ℓ be such that $z \in \mathcal{Z}_\ell = \{z | F_\ell z \le g_\ell\}$. Then

$$v_\ell(z) = \sigma_\ell + \tau_\ell\, z \quad (6)$$

is a vertex of $\Phi(z)$.

The proof is in Appendix A.

Finally we give the definition of a simplex and the geometric center of a polytope.

Definition 2: Consider $k + 1$ affinely independent[1] points $x_0, x_1, \ldots, x_k \in \mathbb{R}^n$. The simplex determined by them is given by $\mathbf{Co}(x_1, \ldots, x_n) = \{\lambda_1 x_1 + \ldots + \lambda_n x_n \mid \lambda_i \ge 0\ \ \lambda_1 + \ldots + \lambda_n = 1\ \}$.

Definition 3: Let $v_1, \ldots, v_N$ be the vertices of a polytope. The geometric center $v_{\rm gc}$ of the polytope is defined as $v_{\rm gc} = 1/N\, \sum_{i=1}^{N} v_i$.

III. AN ANALYTIC EXPRESSION FOR $\mathbb{E}[f(w, e)]$

Define $h(f, w) = \mathbb{E}[f(w, e)]$. In this section we will derive an analytic expression for $h(f, w)$.

Consider for a fixed $w \in \mathcal{W}$ the polyhedral sets

$$\Phi_j(w) = \{\ e \in \mathcal{E} \mid f(w, e) = \alpha_j + \Lambda_j w + \Gamma_j e\ \}$$

for $j = 1, \ldots, n$. This means that for a given w and $e \in \Phi_j(w)$ the jth inequality in $f(w, e)$ leads to the maximum; so

$$\alpha_j + \Lambda_j\, w + \Gamma_j\, e \ge \alpha_i + \Lambda_i\, w + \Gamma_i\, e\ , \ \ \forall i \ne j \quad (7)$$

Define for each j the set $\mathcal{I}_j = \{i_{j,1}, \ldots, i_{j,m-1}\} = \{1, 2, 3, \ldots, m\} \backslash \{j\}$, and define matrices $A'_{w,j} \in \mathbb{R}^{(m-1)\times p}$,

[1] Affinely independent means that $x_1 - x_0, \ldots, x_n - x_0$ are independent.

$A'_{e,j} \in \mathbb{R}^{(m-1)\times n}$, $b'_j \in \mathbb{R}^{m-1}$, such that $[b'_j]_s = \alpha_j - \alpha_{i_{j,s}}$, $[A'_{w,j}]_s = -\Lambda_j + \Lambda_{i_{j,s}}$, and $[A'_{e,j}]_s = -\Gamma_j + \Gamma_{i_{j,s}}$ for $i_{j,s} \in \mathcal{I}_j$, $s = 1, \dots, m-1$, where $[M]_s$ denotes the sth row of a matrix M. Define $A_{e,j} \in \mathbb{R}^{q\times n}$, $A_{w,j} \in \mathbb{R}^{q\times p}$, and $b_j \in \mathbb{R}^q$ as follows:

$$A_{e,j} = \begin{bmatrix} A'_{e,j} \\ C_e \\ 0 \end{bmatrix},\ A_{w,j} = \begin{bmatrix} A'_{w,j} \\ 0 \\ C_w \end{bmatrix},\ b_j = \begin{bmatrix} b'_j \\ d_e \\ d_w \end{bmatrix}$$

Now condition (7) for all $e \in \mathcal{E}$ and a fixed $w \in \mathcal{W}$ can be replaced by finding all $e \in \mathbb{R}^n$ and $w \in \mathbb{R}^p$ such that

$$A_{e,j}\, e \le b_j - A_{w,j}\, w \tag{8}$$

Note that $q > n$. Let $\mathcal{S}_j = \{S_{j,1}, \dots, S_{j,L_j}\}$ be the set of all $n \times q$ submatrices of the $q \times q$ identity matrix such that the matrix $S_{j,\ell}\, A_{e,j}$ is invertible. For any matrix $S_{j,\ell} \in \mathcal{S}_j$ the remaining part of the $q \times q$ identity matrix will be denoted by $T_{j,\ell} \in \mathbb{R}^{(q-n)\times q}$.

Now define for $\ell = 1, \dots, L_j$ the vectors $\sigma_{j,\ell} \in \mathbb{R}^n$, $g_{j,\ell} \in \mathbb{R}^{q-n}$ and matrices $\tau_{j,\ell} \in \mathbb{R}^{n\times p}$, $F_{j,\ell} \in \mathbb{R}^{(q-n)\times p}$ according to Lemma 1. Let ℓ be such that $w \in \mathcal{W}_{j,\ell} = \{w | F_{j,\ell} w \le g_{j,\ell}\}$. Then we find that

$$v_{j,\ell}(w) = \sigma_{j,\ell} + \tau_{j,\ell}\, w \tag{9}$$

is a vertex of $\Phi_j(w)$.

The next step in our algorithm is to divide the polytope $\Phi_j(w)$ into $K_j(w)$ simplices $\Omega_{j,k}(w)$, $k = 1, \dots, K_j(w)$. We now use the following recursive procedure: We start by considering each 2-dimensional face of the polytope. We select the geometric center of the face and connect that to each of the vertices of the given faces. In this way each 2-dimensional face can be partitioned into simplices with 3 vertices. We consider all 3-dimensional faces and construct 3-dimensional simplices by connecting the geometric center of each of the 3-dimensional faces with all the vertices of the simplices of the 2-dimensional subfaces of the given 3-dimensional face. We continue in this way until the full n-dimensional polytope $\Phi_j(w)$ has been divided into n-dimensional simplices. Note that the geometric center of a polytope is a convex combination of the vertices of that polytope. This means that the vertices of the n-dimensional simplices are convex combinations of the vertices of the polytope $\Phi_j(w)$.

The partitioning of a 3-dimensional polytope into simplices is illustrated in Fig. 1.

Consider one of the simplices $\Omega_{j,k}(w)$ and denote the vertices of this simplex by $\bar{v}_{j,k,0}, \bar{v}_{j,k,1}, \bar{v}_{j,k,2}, \cdots, \bar{v}_{j,k,n}$. The simplex $\Omega_{j,k}(w)$ is now given by $\Omega_{j,k}(w) = \mathbf{Co}(\bar{v}_{j,k,0}, \bar{v}_{j,k,1}, \bar{v}_{j,k,2}, \cdots, \bar{v}_{j,k,n})$. Define

$$h_{j,k}(f, w) = \int \cdots \int_{\Omega_{j,k}(w)} \Big(\alpha_j + \Lambda_j w + \Gamma_j e\Big) p(e)\, de_1 \cdots de_n$$

Then $h(w)$ can be computed by

$$h(f, w) = \sum_{j=1}^{m} \sum_{k=1}^{K_j(w)} h_{j,k}(f, w) \tag{10}$$

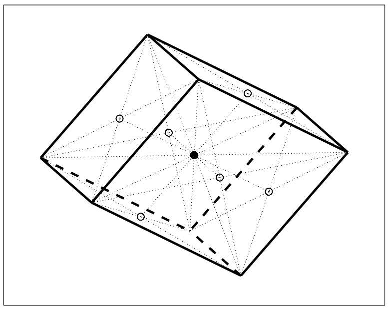

Fig. 1. Partitioning of polytope $\Phi_j(w)$ into simplices. The circles (○) denote the geometric centers of the 2-dimensional faces, the bullet (●) denotes the geometric centers of the 2-dimensional vertices.

The next step is to derive an expression for the terms $h_{j,k}(f, w)$ for all j, k, and w. For a fixed j and $w \in \mathcal{W}$, let $v_{j,\ell}(w)$, $\ell = 1, \dots, L_j$ be the vertices of the polytope $\Phi_j(w)$. The vertices $\bar{v}_{j,k,i}(w)$, $i = 0, \dots, n$ of the simplex $\Omega_{j,k}(w)$, $k \in \{1, \dots, K_j\}$ will be convex combinations of the vertices $v_{j,\ell}(w)$, $\ell = 1, \dots, L_j$. In other words, there exist parameters $\lambda_{j,i,k,\ell}$ such that $\bar{v}_{j,k,i}(w) = \sum_{\ell=1}^{L_j} \lambda_{j,i,k,\ell} v_{j,\ell}(w)$, where $\lambda_{j,i,k,\ell}$ does not depend on w (because we used geometric centers to construct the simplices). Now define $\bar{\sigma}_{j,k,i} = \sum_{\ell=1}^{L_j} \lambda_{j,i,k,\ell}\sigma_{j,\ell}$ and $\bar{\tau}_{j,k,i} = \sum_{\ell=1}^{L_j} \lambda_{j,i,k,\ell}\tau_{j,\ell}$. Then using (9) we find

$$\bar{v}_{j,k,i}(w) = \bar{\sigma}_{j,k,i} + \bar{\tau}_{j,k,i}\, w\,. \tag{11}$$

The following lemma gives an analytic expression for the value $h_{j,k}(f, w)$.

Lemma 4: [19] Consider the simplex

$$\Omega_{j,k}(w) = \mathbf{Co}(\bar{v}_{j,k,0}(w), \bar{v}_{j,k,1}(w), \cdots, \bar{v}_{j,k,n}(w)) \tag{12}$$

with vertices that are affine in w according to (11), and

$$\bigcup_{j=1}^{m} \bigcup_{k=1}^{K_j(w)} \Phi_j(w) = \mathcal{W}\,.$$

Define

$$V_{j,k}(w) = \begin{bmatrix} \bar{v}^T_{j,k,1}(w) - \bar{v}^T_{j,k,0}(w) \\ \bar{v}^T_{j,k,2}(w) - \bar{v}^T_{j,k,0}(w) \\ \vdots \\ \bar{v}^T_{j,k,n}(w) - \bar{v}^T_{j,k,0}(w) \end{bmatrix}^T = V_{j,k,0} + \sum_{\ell=1}^{p} V_{j,k,\ell}\, w_\ell$$

then for a constant $p(e) = 2^{-n}$ we have

$$\begin{aligned} h_{j,k}(f, w) &= \int \cdots \int_{\Omega_{j,k}(w)} \Big(\alpha_j + \Lambda_j w + \Gamma_j e\Big) p(e)\, de_1 \cdots de_n \\ &= \frac{\det V_{j,k}(w)}{(n+1)!}\, 2^{-n} \Big((n+1)(\alpha_j + \Lambda_j w) \\ &\qquad + \sum_{i=0}^{n} \Gamma_j \bar{\sigma}_{j,k,i} + \Gamma_j \bar{\tau}_{j,k,i}\, w \end{aligned}$$

Hence, $h_{j,k}(f, w)$ is an $(n+1)$st order polynomial function in w.

$w \in [-\infty, -8)$:	$h(w) = 2w + 6$
$w \in [-8, 0)$:	$h(w) = 0.00208w^3 + 0.05w^2 + 2.4w + 7.067$
$w \in [0, 0.556)$:	$h(w) = 0.05w^2 + 2.4w + 7.067$
$w \in [0.556, 1)$:	$h(w) = 0.169w^3 - 0.231w^2 + 2.56w + 7.03$
$w \in [1, 1.5)$:	$h(w) = 0.167w^3 - 0.225w^2 + 2.55w + 7.4$
$w \in [1.5, 1.78)$:	$h(w) = -0.167w^3 + 1.275w^2 + 0.3w + 8.1648$
$w \in [1.78, 2)$:	$h(w) = 1.52w^3 - 7.72w^2 + 16.3w - 1.32$
$w \in [2, 2.67)$:	$h(w) = 0.0917w^3 - 0.275w^2 + 3.65w + 5.62$
$w \in [2.67, 3)$:	$h(w) = -0.0208w^3 + 0.625w^2 + 1.25w + 7.75$
$w \in [3, 6)$:	$h(w) = -0.0208w^3 + 0.375w^2 + 2.75w + 5.5$
$w \in [6, \infty)$:	$h(w) = 5w + 1$

TABLE I
THE FUNCTION $h(w)$ FOR DIFFERENT RANGES OF w IN THE EXAMPLE OF SECTION III. (FOR EASE OF NOTATION WE LIST ROUNDED VALUES FOR THE PARAMETERS)

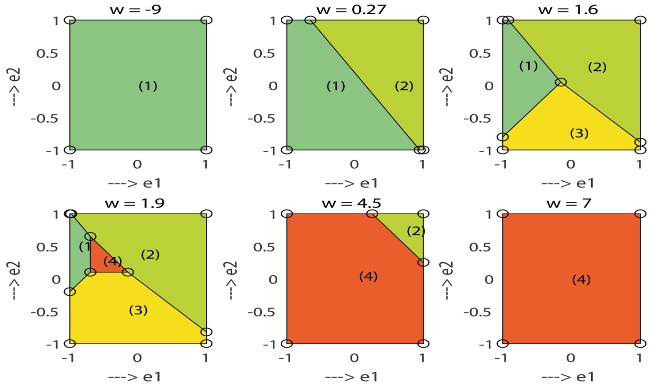

Fig. 2. The regions $\Phi_j(w)$, $j = 1, \ldots, 4$ for different values of w in the example of Section III. The vertices are denoted by small circles (○) .

In the computation of the determinant of a matrix $V_{j,k}(w)$ in Lemma 4 (and Lemma 6 in Section IV) we take sums and products of entries of the matrix $V_{j,k}(w)$. Every entry is affine in the control variable w and the determinant will thus be a nth-order polynomial in the control variable w. Note that this computation is in the offline part of the algorithm, so we only have to do this once for every region.

Theorem 5: Given the function f as defined in (2). For a fixed $w \in \mathcal{W}$, let $s_1, s_2, \ldots, s_n$ be such that

$$w \in \mathcal{W}_{f,j,s_j} \text{ for } j = 1, \ldots, n$$

where $\mathcal{W}_{f,j,s_j}$ has been defined in Lemma 1. Define for $w \in \mathcal{W}_{f,j,s_j}$, $j = 1, \ldots, n$:

$$h_j(f, w) = h_{j,s_j}(f, w) \tag{13}$$

Then for $w \in \mathcal{W}_{f,j,s_j}$ we find that $h(f, w) = \sum_{j=1}^{m} h_j(f, w)$ is a piecewise $(n + 1)$st-order polynomial function in w.

Proof: This immediately follows from (10) combined with Lemmas 1 and 4. ⋄

Example

In this example we compute a piecewise polynomial expression for the following expression:

$$h(w) = \mathbb{E}\Big[\max(6+2w+e_2, 5+3w+5e_1+5e_2, 3+4w+e_1, 1+5w+e_1+e_2)\Big]$$

so for

$$\alpha = \begin{bmatrix} 6 \\ 5 \\ 3 \\ 1 \end{bmatrix} \quad , \quad \Lambda = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad , \quad \Gamma = \begin{bmatrix} 0 & 1 \\ 5 & 5 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{14}$$

if the MPS function is written in the form (4).

We compute the vertices of the regions $\Phi_j(w)$ for $j = 1, \ldots, 4$. Fig. 2 shows these regions $\Phi_j(w)$ for different values of w. The functions $F_{j,k}$, $g_{j,k}$, $\tau_{j,k}$, and $\sigma_{j,k}$ can be computed using Lemma 1, and with these values we can compute $h_{j,k}$ and using Theorem 5 we then compute h_j and h. The resulting function h is given in Table I. We clearly see that the function h is a piecewise polynomial function in the variable w. In [19] we have already proven that the function h is convex.

IV. An Analytic Expression for $\mathbb{P}[f(w, e) \leq B]$

Define $\chi(f, w, B) = \mathbb{P}[f(w, e) \leq B]$. In this section we will derive an analytic expression for $\chi(f, w, B)$. Introduce the set $\mathcal{Z} = \mathcal{W} \times \mathcal{B}$, where $\mathcal{B} = [B_{\min}, B_{\max}]$ and the variable $z = \begin{bmatrix} w^T & B^T \end{bmatrix}^T$. The set $\mathcal{Z}$ is now given by $\mathcal{Z} = \{z \in \mathbb{R}^{p+1} | C_z z \leq d_z\}$, where

$$C_z = \begin{bmatrix} C_w & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \quad , \quad d_z = \begin{bmatrix} d_w \\ -B_{\min} \\ B_{\max} \end{bmatrix}$$

To ease the notation in this section we will use $\chi(f, z)$ instead of $\chi(f, w, B)$.

Let $z \in \mathcal{Z}$ be fixed. The first step is to compute the set $\Phi(z) = \{\, e \,|\, f(w, e) \leq B \,\}$. Now $\chi(f, z)$ is given by

$$\chi(f, z) = \int\cdots\int_{\Phi(z)} p(e)\, de_1 \cdots de_n \tag{15}$$

To compute the set $\Phi(z) = \{\, e \,|\, f(w, e) \leq B \,\}$, let us consider all $e \in \mathcal{E}$ such that

$$\alpha + \Lambda\, w + \Gamma\, e \leq B\bar{1}\ , \tag{16}$$

where $\bar{1} = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T \in \mathbb{R}^q$. Define $A_e \in \mathbb{R}^{q \times n}$, $A_z \in \mathbb{R}^{q \times (p+1)}$, and $b \in \mathbb{R}^q$ as follows:

$$A_e = \begin{bmatrix} \Gamma \\ C_e \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; A_z = \begin{bmatrix} \Lambda & -\bar{1} \\ 0 & 0 \\ C_w & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}, \; b = \begin{bmatrix} -\alpha \\ d_e \\ d_w \\ -B_{\min} \\ B_{\max} \end{bmatrix}$$

Now condition (16) for all $e \in \mathcal{E}$ can be replaced by finding all $e \in \mathbb{R}^n$ and $w \in \mathbb{R}^p$ such that

$$A_e\, e \leq b - A_z\, z \tag{17}$$

Note that $q > n$. Let $\mathcal{S} = \{S_1, \ldots, S_L\}$ be the set of all $n \times q$ submatrices of the $q \times q$ identity matrix such that the matrix

$S_\ell A_e$ is invertible. For any matrix $S_\ell \in \mathcal{S}$ the remaining part of the $q \times q$ identity matrix will be denoted by $T_\ell \in \mathbb{R}^{(q-n)\times q}$.

Now define for $\ell = 1, \ldots, L$ the vectors $\sigma_\ell \in \mathbb{R}^n$, $g_\ell \in \mathbb{R}^{q-n}$ and matrices $\tau_\ell \in \mathbb{R}^{n\times(p+1)}$, $F_\ell \in \mathbb{R}^{(q-n)\times(p+1)}$ according to Lemma 1. Let ℓ be such that $z \in \mathcal{Z}_\ell = \{z | F_\ell z \leq g_\ell\}$. Then we find that

$$v_\ell(z) = \sigma_\ell + \tau_\ell z \tag{18}$$

is a vertex of $\Phi(z)$.

We now use the same recursive procedure as in the previous section to divide the polytope $\Phi(z)$ into $K(z)$ simplices $\Omega_k(z)$, $k = 1, \ldots, K(z)$.

Consider one of the simplices $\Omega_k(z)$ and denote the vertices of this simplex by $\bar{v}_{k,0}, \bar{v}_{k,1}, \bar{v}_{k,2}, \cdots, \bar{v}_{k,n}$. The simplex $\Omega_k(z)$ is now given by:

$$\Omega_k(z) = \mathbf{Co}(\bar{v}_{k,0}, \bar{v}_{k,1}, \bar{v}_{k,2}, \cdots, \bar{v}_{k,n})$$

Define $\chi_k(f,z) = \int\cdots\int_{\Omega_k(z)} p(e)\, de_1 \cdots de_n$. Then $\chi(f,z)$ can be computed by

$$\chi(f,z) = \sum_{k=1}^{K(z)} \chi_k(f,z) \tag{19}$$

The following lemma gives an analytic expression for the value $\chi_k(f,z)$.

Lemma 6: Consider the simplex

$$\Omega_k(z) = \mathbf{Co}(\bar{v}_{k,0}(z), \bar{v}_{k,1}(z), \cdots, \bar{v}_{k,n}(z)) \tag{20}$$

with vertices affine in z according to (11). Define

$$V_k(z) = \begin{bmatrix} \bar{v}_{k,1}^T(z) - \bar{v}_{k,0}^T(z) \\ \bar{v}_{k,2}^T(z) - \bar{v}_{k,0}^T(z) \\ \vdots \\ \bar{v}_{k,n}^T(z) - \bar{v}_{k,0}^T(z) \end{bmatrix}^T = V_{k,0} + \sum_{\ell=1}^{p} V_{k,\ell}\, z_\ell$$

Then $\chi_k(f,z) = \int\cdots\int_{\Omega_k(z)} p(e)\, de_1 \cdots de_n = \frac{\det V_k(z)}{n!} 2^{-n}$. Hence, $\chi_k(f,z)$ is an nth-order polynomial function in z.

The proof immediately follows from Lemma 4 for $\alpha = 1$, $\Lambda = 0$, and $\Gamma = 0$.

Theorem 7: Let $\mathcal{Z}_s$ be defined as in Lemma 1, and for a fixed $z \in \mathcal{Z}$, let s be such that $z \in \mathcal{Z}_s$. Then we find that

$$\chi(f,z) = \chi_s(f,z)$$

is a piecewise nth-order polynomial function in z.

Proof: This immediately follows from (19) combined with Lemmas 1 and 6. ◇

Properties of $\chi(f,w,B)$: Recall that $\mathcal{Z}$ is a convex set. For a quasi-concave function $g : \mathcal{Z} \to \mathbb{R}$ it holds that

$$g(\lambda z_1 + (1-\lambda) z_2) \geq \min(g(z_1), g(z_2))$$

for any $z_1, z_2 \in \mathcal{Z}$ and $\lambda \in [0,1]$ [5].

Lemma 8:
Define the function $\phi : \mathbb{R} \to \mathbb{R}$:

$$\phi(a) = \begin{cases} 0 & \text{for } a < 0 \\ 1 & \text{for } a \geq 0 \end{cases} \tag{21}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a concave function. Then $\phi(g(x))$ is quasi-concave in x.

Proof: For a concave function g we have

$$g(\lambda z_1 + (1-\lambda) z_2) \geq \lambda g(z_1) + (1-\lambda) g(z_2)$$

for any $z_1, z_2 \in \mathcal{Z}$ and $\lambda \in [0,1]$.
Since ϕ is nondecreasing, this means that

$$\begin{aligned} \phi(g(\lambda z_1 + (1-\lambda) z_2)) &\geq \phi(\lambda g(z_1) + (1-\lambda) g(z_2)) \\ &\geq \min\Big(\phi(g(z_1)), \phi(g(z_2))\Big) \end{aligned}$$

So $\phi(g(\cdot))$ is a quasi-concave function. ◇

Lemma 9: Given the function f as defined in (2), let χ_o be a fixed scalar with $0 < \chi_o < 1$. Then the constraint $\chi(f,z) \geq \chi_o$ is a convex set in z.
The proof is in Appendix B.

Example

In this example we compute a piecewise polynomial expression for the following expression:

$$\chi(f,w,B) = \mathbb{P}\Big[\max(6+2w+e_2, 5+3w+5e_1+5e_2, 3+4w+e_1, 1+5w+e_1+e_2) < B\Big]$$

so for α, Λ, and Γ as defined in (14) if the MPS function is written in the form (5).

We compute the vertices of the region $\Phi(w)$. Fig. 3 shows these regions $\Phi(w)$ for different values of w. The functions F_k, g_k, τ_k, and σ_k can be computed using Lemma 1, and with these values we can compute χ_k and using Theorem 5 we then compute χ. The resulting function χ is given in Table II. We clearly see that the function χ is a piecewise polynomial function in the variable w.

$w \in [-\infty, -2.33)$:	$\chi(f,w,B) = 1$
$w \in [-2.33, 0.5)$:	$\chi(f,w,B) = 0.755 - 0.21w - 0.045w^2$
$w \in [0.5, 1)$:	$\chi(f,w,B) = 0.78 - 0.16w - 0.245w^2$
$w \in [1, 1.1852)$:	$\chi(f,w,B) = -0.62 + 2.64w - 1.645w^2$
$w \in [1.1825, 1.5)$:	$\chi(f,w,B) = 4.5 - 6w + 2w^2$
$w \in [1.5, \infty)$:	$\chi(f,w,B) = 0$

TABLE II
THE FUNCTION $\chi(f,w,B)$ WITH FIXED $B = 8$ FOR DIFFERENT RANGES OF w IN THE EXAMPLE OF SECTION IV (FOR EASE OF NOTATION WE LIST ROUNDED VALUES FOR THE PARAMETERS)

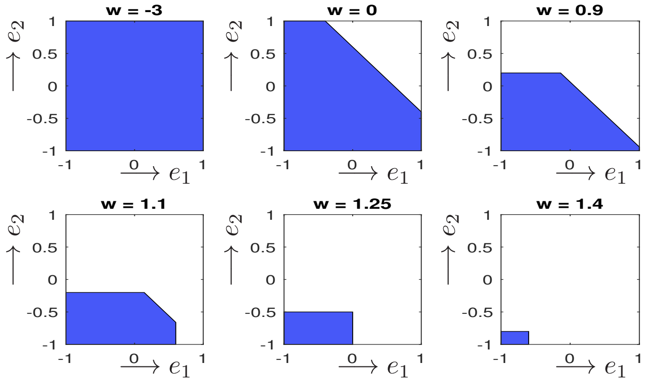

Fig. 3. The regions $\Phi(w)$ with fixed $B = 8$ for different values of w in the example of Section IV.

V. Model predictive control

Consider the stochastic max-plus-linear system (3). Define the vectors

$$\tilde{u}(k) = \begin{bmatrix} u^T(k) & \cdots & u^T(k+N_\mathrm{p}-1) \end{bmatrix}^T \quad ,$$
$$\tilde{e}(k) = \begin{bmatrix} e^T(k) & \cdots & e^T(k+N_\mathrm{p}-1) \end{bmatrix}^T \quad \in \mathbb{R}^{n_{\tilde{e}}} \quad .$$

In [20] we derived that $y(k+j)$ is a max-plus-scaling function in the variables $x(k-1)$, $\tilde{u}(k)$, and $\tilde{e}(k)$, and so we can compute the expectation and chance constraint of $y(k+j)$ using the techniques of the previous sections.

Let $\mathcal{I} = \{1, \dots, n_y\}$ and $\mathcal{J} = \{0, \dots, N_\mathrm{p}–1\}$. We introduce the cost criterion J in the event period $\{k, \dots, k+N_\mathrm{p}-1\}$ as follows:

$$J(k) = \mathbb{E}\Big[\max_{i\in\mathcal{I}, j\in\mathcal{J}} f_{i,j}(\tilde{u},\tilde{e})\Big] - \lambda \sum_{\ell=0}^{(N_\mathrm{p}-1)n_y} \tilde{u}_\ell(k) \qquad (22)$$

where $f_{i,j}(\tilde{u},\tilde{e}) = \max(y_i(k+j) - r_i(k+j), 0)$ is the tracking error between the the reference signal r_i and the ouput y_i, and the scalar variable $\lambda > 0$ is fixed. Since the entries of the vector $y(k+j)$ are max-plus-scaling functions in $\tilde{u}$ and $\tilde{e}$, also the functions $f_{i,j}$ will be max-plus-scaling functions in $\tilde{u}$ and $\tilde{e}$.

The optimization of the cost function will usually be subject to inequality chance constraints:

$$\mathbb{P}\Big[\max_i\Big(F_i(k)\tilde{u}(k) + G_i(k)\tilde{y}(k) - g_i(k)\Big) \leq 0\Big] \geq \chi_c \qquad (23)$$

where $F_i(k)$, $G_i(k)$, and $g_i(k)$ refer to the ith row of $F(k)$, $G(k)$, and $g(k)$, respectively, and χ_c is some fixed probability level with $0 < \chi_c < 1$. Note that constraint (23) means that the probability that the constraints are satisfied is greater than or equal to χ_c. Furthermore the function $\max_i \Big(F_i(k)\tilde{u}(k) + G_i(k)\tilde{y}(k) - g_i(k)\Big)$ is a max-plus-scaling function in $\tilde{u}$ and $\tilde{e}$. Another important set of constraints are given by

$$\Delta u(k+j) \geq 0 \quad \text{for } j = 0, \dots, N_\mathrm{p}-1 \qquad (24)$$
$$\Delta^2 u(k+j) = 0 \quad \text{for } j = N_\mathrm{c}, \dots, N_\mathrm{p}-1 \qquad (25)$$

where (24) reflects the fact that $u(k)$ must be an increasing sequence and (25) is the control horizon which gives $u(k)$ a constant rate beyond the control horizon.
The problem given by minimizing (22) subject to constraints (24), (25) and (23) will be called the stochastic max-plus linear MPC problem for event step k.

MPC uses a receding horizon principle. This means that after computation of the optimal control sequence $u(k), \dots, u(k+N_\mathrm{c}-1)$, only the first control sample $u(k)$ will be implemented, subsequently the horizon is shifted one sample, and the optimization is restarted with new information of the measurements.

Note that the stochastic max-plus linear MPC problem results in an optimization problem with a convex cost function subject to constraints. If we assume that the mapping $\tilde{y} \to G(k)\tilde{y}$ is a monotonically non-decreasing function of $\tilde{y}$ (this happens often in practice), constraint (23) will be convex, which is taken from [8]. This means that the problem can then be solved using fast and reliable convex optimization algorithms.

Example

In this example we consider the production system of [20] with two machines M_1 and M_2 operating in batches. The raw material in the kth cycle is fed to machine M_1 at time $x_1(k)$ where preprocessing is done. Afterwards the intermediate product is fed to machine M_2 at time $x_2(k)$ and finally the finished product leaves the system at time $y(k)$. More details of the system are given in [20].

Consider the cost function (22) and inequality constraint

$$\mathbb{P}[\max(y(k) - r(k), y(k+1) - r(k+1)) \leq 0] \geq \chi_c \qquad (26)$$

We first solve the unconstrained MPC problem with cost criterion (22) for $N_\mathrm{p} = N_\mathrm{c} = 2$, $\lambda = 0.2$, $\chi_c = 0.98$. Next we solve the constrained MPC problem with cost criterion (22) subject to constraint (26) for $N_\mathrm{p} = N_\mathrm{c} = 2$ and $\lambda = 0.2$, and $\chi = 0.98$. For comparison purposes we also consider the residuation controller [4] where we assume the nominal undisturbed max-plus-linear model to compute the optimal input signal $u(k)$.

The optimal input sequence is computed for $k = 1, \dots, 50$ for the initial state $x(0) = \begin{bmatrix} 18 & 4 \end{bmatrix}^T$. In the experiments, the true system is simulated for a uniformly distributed stochastic variable $e(k)$ for the given range. Figure 4 gives the difference between the output date signal y and the due date signal r. We see that, as expected, the residuation controller cannot handle the parameter variation. The unconstrained MPC controller performs better but still the tracking error $y(k) - r(k)$ is often larger than zero. The constrained MPC controller with the chance constraint leads to only one violation of the due-date constraint (but that violation is due to the initial state, so it is unavoidable anyway).

The simulation were done on a DELL Latitude 7480 computer (Intel(R) Core(TM) i7-6600 CPU @2.6-2.8MHz) using MATLAB R2017b and the MPC optimization in MATLAB is done with a quasi-Newton algorithm (fminunc) for the unconstrained case and an interior-point method (fmincon) for the constrained case. If we compare the computation time of the unconstrained MPC controller using the analytic

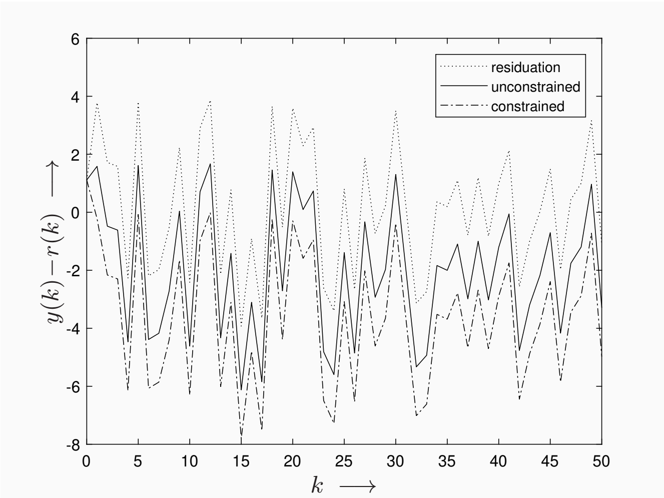

Fig. 4. The difference $y(k) - r(k)$ between the output date signal y and the due date signal r for residuation, unconstrained MPC, and constrained MPC.

expression with the MPC controller derived in [20] we see that the average computation time for the controller in [20] was 51.9 seconds and for the new analytic controller it was 5.3 seconds, which is a factor 9.8 lower.

VI. Computational Complexity

MPC of stochastic max-plus linear systems is often time consuming because a straightforward evaluation of the expectation of the cost function and the chance constraints are computationally expensive. The analytic expressions for the expectation and the chance constraint, derived in this paper, can be computed offline as piecewise polynomial functions. In the online operation of the model predictive controller the computation can now be done very fast, efficiently and precisely (as an approximation is not necessary).

The offline computation is certainly computationally expensive and both computation time and necessary memory will increase for high-order max-plus linear functions and longer predictions horizons in MPC. Consider the computation of the vertices of set $\Phi_j(w)$ (see Section III). Let $\mathcal{W}$ be a hypercube, then we find $q = m - 1 + 2n + 2p$. Following Lemma 1 we find $L_{\max} = \begin{pmatrix} q \\ n \end{pmatrix}$ possibilities to choose n out of q equalities and so for the number of vertices of $\Phi_j(w)$ we find $L_j \leq L_{\max}$, where for all $j = 1, \ldots, m$. From [12] we learn that the number of simplices $K_j \leq K_{\max}$ where $K_{\max} = L_{\max}^{\lceil q \rceil}$, where $\lceil q \rceil$ is the the ceiling function that maps q to the least integer greater than or equal to q. For the expectation we have $n_{\exp} \leq mK_{\max}$ simplices.

For the chance constraint we can use the same expression to compute $L_{\max}$ and $K_{\max}$ using $q = m + 2n + 2p + 2$ and $j = 1$. For each simplex we easily derive the $(n+1)$st order (for the chance constraint an nth order) polynomial using a parameterized expression for determinant of an $n \times n$ matrix.

Summarizing we find that the offline algorithm is in the order of $\mathcal{O}\left(\mu \cdot \begin{pmatrix} q \\ n \end{pmatrix}^{\lceil q \rceil}\right)$, where for the expectation we have $q = m - 1 + 2n + 2p$ and $\mu = m$ and for the chance constraint we have $q = m + 2n + 2p + 2$ and $\mu = 1$.

In the online part the expectation is represented by

$$h_j(f, w) = \begin{cases} h_{j,\text{pol},1} & \text{if } M_{j,1} w \leq m_{j,1} \\ \vdots & \vdots \\ h_{j,\text{pol},L} & \text{if } M_{j,L} w \leq m_{j,L} \end{cases} \tag{27}$$

for $j = 1 \ldots, m$ and $h(f, w) = \sum_{j=1}^{m} h_j(f, w)$. (For the chance constraints we have a similar representation.) This means that the extensive computation of level sets and analytic integrals is reduced to a simple evaluation of (27). The complexity of the online part of the algorithm is characterized by the number mK_o of regions that form the analytic solution. For MPC problems with a large prediction horizon, many states and many disturbance inputs, this number L may be so high that in the online part of the algorithm the search for the right function in (27) becomes too computationally expensive. This problem is similar to the problem of explicit MPC for conventional time-driven systems. Techniques to obtain a drastic reduction of complexity are discussed in [1]. These techniques can also be used in our setting.

VII. Conclusions

We have presented an analytic piecewise polynomial function expression for both the expectation and the chance constraint of a max-plus-linear expression, where we assume that the max-plus expression is affine in a control variable and affine in a stochastic variable. We have shown how to apply the resulting analytic expressions in stochastic max-plus linear model predictive control. Using a case study we have shown that the computation time in the online part has been reduced with nearly a factor 10 with respect to the computation time using the controller of [20].

In future research we will further improve the proposed two-step algorithm. Hereby we can follow up on the results achieved in explicit MPC (see [1]). Further we will compare the method derived in this paper with approximate methods [9], [18], [21], [23]. We will also consider simulation techniques in combination with gradient estimation techniques like infinitesimal perturbation techniques (IPA) [6]. Moreover, we will measure the loss of precision by the approximation in relation to the gain in computation speed.

References

[1] A. Alessio and A. Bemporad. A survey on explicit model predictive control. In D.M. Raimondo L. Magni and F. Allgöwer (Eds.), editors, *Nonlinear Model Predictive Control*, LNCIS 384, pages 345–369. Springer-Verlag Berlin Heidelberg, 2009.

[2] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity*. John Wiley & Sons, (accessible via www.maxplus.org), New York, 1992.

[3] A. Bemporad, F. Borrelli, and M.Morari. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control*, 48(9):1600–1606, 2003.

[4] J.L. Boimond and J.L. Ferrier. Internal model control and max-algebra: Controller design. *IEEE Transactions on Automatic Control*, 41(3):457–461, March 1996.

[5] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, New York, NY, USA, 2004.

[6] X.R. Cao. Perturbation analysis of discrete event systems: concepts, algorithms, and applications. *European Journal of Operational Research*, 91:1–13, 1995.

[7] R.A. Cuninghame-Green. *Minimax Algebra*, volume 166 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 1979.

[8] B. De Schutter and T. van den Boom. Model predictive control for max-plus-linear discrete event systems. *Automatica*, 37(7):1049–1056, July 2001.

[9] S.S. Farahani, T. van den Boom, H. van der Weide, and B. De Schutter. An approximation approach for model predictive control of stochastic max-plus linear systems. In *Proceedings of the 10th International Workshop on Discrete Event Systems (WODES 2010)*, pages 386–391, Berlin, Germany, August–September 2010.

[10] M. Farina and R. Scattolini. Model predictive control of linear systems with multiplicative unbounded uncertainty and chance constraints. *Automatica*, 70:258–265, 2016.

[11] B. Heidergott, G.J. Olsder, and J. van der Woude. *Max Plus at Work: Modeling and Analysis of Synchronized Systems*. Princeton University Press, Princeton, 2006.

[12] Carl W. Lee and Francisco Santos. Subdivisions and triangulations of polytopes. In J.E. Goodman, J. O'Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry, 3rd edition*, pages 416–447. CRC Press, Boca Raton, FL, 2017.

[13] J.M. Maciejowski. *Predictive Control with Constraints*. Prentice Hall, Pearson Education Limited., Harlow, UK, 2002.

[14] J. Mairesse. Stochastic linear systems in the (max,+) algebra. In *11th International Conference on Analysis and Optimization of Systems Discrete Event Systems*, pages 257–265, vol. 199 of Lecture Notes in Control and Information Sciences, Springer, London, UK, 1994.

[15] A. Mesbah. Stochastic model predictive control: An overview and perspectives for future research. *IEEE Control Systems Magazine*, 61(6):30–44, December 2016.

[16] G.J. Olsder, J.A.C. Resing, R.E. de Vries, M.S. Keane, and G. Hooghiemstra. Discrete event systems with stochastic processing times. *IEEE Transactions on Automatic Control*, 35(3):299–302, March 1990.

[17] J.A.C. Resing, R.E. de Vries, G. Hooghiemstra, M.S. Keane, and G.J. Olsder. Asymptotic behavior of random discrete event systems. *Stochastic Processes and their Applications*, 36:195–216, 1990.

[18] V. Rostampour, D. Adzkiya, S. Soudjani, B. De Schutter, and T. Keviczky. Chance constrained model predictive controller synthesis for stochastic max-plus linear systems. In *IEEE Int. Conf. Syst. Man Cybern.*, Budapest, Hungary, Oct. 2016.

[19] T. van den Boom and B. De Schutter. Analytic expressions in stochastic max-plus-linear algebra. In *53rd IEEE Conference on Decision and Control (CDC 2014)*, pages 1608–1613, Los Angeles, California, USA, December 2014.

[20] T.J.J. van den Boom and B. De Schutter. Model predictive control for perturbed max-plus-linear systems: A stochastic approach. *International Journal of Control*, 77(3):302–309, February 2004.

[21] T.J.J. van den Boom, B. Heidergott, and B. De Schutter. Complexity reduction in MPC for stochastic max-plus-linear systems by variability expansion. *Automatica*, 43(6):1058–1063, June 2007.

[22] M. Vitus, Z. Zhou, and C. Tomlin. Stochastic control with uncertain parameters via chance constrained control. *IEEE Transactions on Automatic Control*, 61(10):2892—-2905, October 2016.

[23] J. Xu, T.J.J. van den Boom, and B. De Schutter. Model predictive control for stochastic max-plus linear systems with chance constraints. *IEEE Transactions on Automatic Control*, 64(1):337–342, 2019.

Appendix A: Proof of Lemma 1

Proof: Let us consider the equation

$$A_e\, e + A_z\, z \leq b \tag{28}$$

with $e \in \mathcal{E}$ and a fixed $z \in \mathcal{Z}$.

Now consider a vertex v_ℓ of the polyhedral set $\Phi(z)$. The matrix S_ℓ will select constraints from (28) that are active for v_ℓ, and T_ℓ will select constraints from (28) that may be inactive for v_ℓ. For v_ℓ to be a vertex of the polyhedral set $\Phi(z)$, we therefore have the following properties:

$$S_\ell\, A_e\, v_\ell = S_\ell(b - A_z\, z) \tag{29}$$

$$T_\ell\, A_e\, v_\ell \leq T_\ell(b - A_z\, z) \tag{30}$$

From (29) and from the fact that $\det(S_\ell\, A_e) \neq 0$ we derive

$$v_\ell = (S_\ell\, A_e)^{-1}\, S_\ell(b - A_z\, z) \tag{31}$$

and substituting this in (30) we find

$$T_\ell\, A_e(S_\ell\, A_e)^{-1}\, S_\ell(b - A_z\, z) \leq T_\ell(b - A_z\, z) \tag{32}$$

This means that (31) is a vertex if $\det(S_\ell\, A_e) \neq 0$ and if (32) is satisfied. ◇

Appendix B: Proof of Lemma lem:chiconvex

Proof: Combining (15) and (21) we derive

$$\chi(f,w,B) = \int_{\mathcal{E}}\!\cdots\!\int \phi\Big(B - \max_j(\alpha_j + \Lambda_j w + \Gamma_j e)\Big)\, p(e)\, de_1 \cdots de_n$$

Now we have with $\lambda_1 + \lambda_2 = 1$:

$$\begin{aligned}
&\chi\Big(f, \lambda_1 w_1 + \lambda_2 w_2, \lambda_1 B_1 + \lambda_2 B_2\Big) = \\
&= \int_{\mathcal{E}}\!\cdots\!\int \phi\Big((\lambda_1 B_1 + \lambda_2 B_2) - \max_j(\alpha_j + \Lambda_j(\lambda_1 w_1 + \lambda_2 w_2) + \Gamma_j e)\Big)\, p(e)\, de_1 \cdots de_n\,, \\
&\geq \int_{\mathcal{E}}\!\cdots\!\int \phi\Big(\lambda_1(B_1 - \max_j(\alpha_j + \Lambda_j w_1 + \Gamma_j e)) + \lambda_2(B_2 - \max_j(\alpha_j + \Lambda_j w_2 + \Gamma_j e))\Big)\, p(e)\, de_1 \cdots de_n\,, \\
&\geq \int_{\mathcal{E}}\!\cdots\!\int \min\Big\{\phi\Big(B_1 - \max_j(\alpha_j + \Lambda_j w_1 + \Gamma_j e)\Big)\,, \phi\Big(B_2 - \max_i(\alpha_i + \Lambda_i w_2 + \Gamma_i e)\Big)\Big\}\, p(e)\, de_1 \cdots de_n \\
&\geq \min\Big\{\int_{\mathcal{E}}\!\cdots\!\int \phi\Big(B_1 - \max_j(\alpha_j + \Lambda_j w_1 + \Gamma_j e)\Big) \cdot p(e)\, de_1 \cdots de_n\,, \int_{\mathcal{E}}\!\cdots\!\int \phi\Big(B_2 - \max_i(\alpha_i + \Lambda_i w_2 + \Gamma_i e)\Big) \cdot p(e)\, de_1 \cdots de_n\Big\} \\
&\geq \min\Big\{\chi\Big(f, \lambda_1 w_1 + \lambda_2 w_2, \lambda_1 B_1 + \lambda_2 B_2\Big)\,, \chi\Big(f, \lambda_1 w_1 + \lambda_2 w_2, \lambda_1 B_1 + \lambda_2 B_2\Big)\Big\}
\end{aligned}$$

where we used $\max_j(a_j + b_j) \leq \max_j a_j + \max_j b_j$ and $\int \min \leq \min \int$. From the derivation we can conclude that $\chi(f,z)$ is a quasi-concave function in variable z.

From the fact that $\chi(f,z)$ is quasi-concave we can conclude that the constraint $\chi(f,z) \geq \chi_o$ is a convex set.

◇