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Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

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Optimizing the performance of the feedback controller for state-based switching bilinear systems

Shu Lin[a,*], Dewei Li[b], Bart De Schutter[c]

[a]School of Computer Science and Technology, University of Chinese Academy of Sciences, Beijing, China, and Key Laboratory of System Control and Information Processing, Ministry of Education, Shanghai.
[b]Department of Automation, Shanghai Jiao Tong University, Shanghai, China.
[c]B. De Schutter is with Delft Center for Systems and Control, Delft University of Technology, Delft, Netherlands.

Abstract

This paper is concerned with the design and performance optimization of feedback controllers for state-based switching bilinear systems, where subsystems take the form of bilinear systems in different state space polyhedra. First, by further dividing the subregions into smaller regions and designing region dependent feedback controllers in the resulting regions, the switching bilinear systems can be transformed into corresponding switching linear systems. Then, for these switching linear systems, by imposing contractility conditions on the Lyapunov functions, an upper bound on the infinite horizon quadratic cost can be obtained. Optimizing this upper bound yields the controller design. The optimization problem is formulated as an LMI optimization problem, which can be solved efficiently. Finally, the stability of the close-loop system under the proposed controller is established step by step through a decreasing overall Lyapunov function.

Keywords:
Bilinear System with State-Based Switching, Switching Bilinear System Control, Lyapunov Stability, LMIs.

1. INTRODUCTION

Most of the problems found in practice are normally nonlinear problems, which are usually complex. In order to optimize or control these kind of realistic problems, the nonlinear complex systems are usually described by multiple simple models, such as linear models, bilinear models, Markov models, statistic models, etc. Many research works have been done to identify the individual simple models and their connections that build up the nonlinear systems [16–18, 20–23, 26, 27].

A special kind of nonlinear system, i.e. the bilinear system, contains the sum of a linear term and a bilinear term. Bilinear systems have been investigated a lot since the 1960s [4, 6–8, 12, 14]. It has been proved bilinear systems have a better performance than linear systems in optimal control [15], since bilinear systems have a variable structure due to the existence of the bilinear term. In practice, there are systems that naturally have a bilinear term with the states multiplying the control inputs, such as in the field of sociology, biology, power systems, etc. [13, 14]. Usually, the reason for the existence of the term is that the influence of the control input on the system depends on the current system state.

In practice, some complex nonlinear systems can be approximated by dividing into multiple state-based bilinear subsystems [5, 19]. In each state region, a bilinear subsystem is activated, and the bilinear subsystems switch between each other according to the switching of the state regions. This results in a state-based switching bilinear system [3]. Developing the theory on stabilizing controllers for the state-based switching bilinear systems provides a methodology to design controllers for systems with complex nonlinear features in practice. Inspired by this, a stabilizing controller

*Corresponding author
Email addresses: slin@ucas.ac.cn (Shu Lin), dwli@sjtu.edu.cn (Dewei Li), b.deschutter@tudelft.nl (Bart De Schutter)

design based on linear matrix inequalities (LMI) has been addressed in [11]. It should be pointed out that generally many controllers can be designed to achieve the stabilizability of bilinear systems [24, 25], but maybe more work needs to be done to improve the close-loop performance by utilizing the remaining degrees of freedom. For bilinear systems, optimal control problems have attracted much attention [1, 2, 9, 14]. However, to the best knowledge of the authors, for state-based switching bilinear systems few results exist focusing on the performance optimization of the controller. Motivated by this, this paper is devoted to optimizing the performance of stabilizing controllers for state-based switching bilinear systems.

To deal with the state-based switching bilinear systems, the subregions where subsystems are activated are further divided into some multiple regions, then region-dependent controllers are designed for the resulting subregions, which transforms the bilinear systems into linear ones. For the resulting state-based switching linear systems, the infinite-horizon quadratic cost is difficult to calculate explicitly. To solve this problem, contractility conditions on the Lyapunov function are used to derive an upper bound on the quadratic cost. Then instead of directly optimizing the infinite-horizon quadratic cost, an LMI optimization problem is formulated to optimize this upper bound.

The remainder of the paper is organized as follows. In Section 2, the problem statement is given. The main results including the transformation of the bilinear systems into linear ones and the derivation of an upper bound on the quadratic cost are given in Section 3. In the end, a numerical example is given in Section 4 to illustrate the proposed approach. Finally, some conclusions are drawn in Section 5.

2. PROBLEM STATEMENT

Consider a Switching Bilinear System (SBLS)

$$\dot{x} = A_i x + \sum_{j=1}^{m_i} (G_{i,j}x + b_{i,j})u_{i,j}, \text{ if } x \in \Omega_i, i \in \Lambda, \tag{1}$$

where A_i and $G_{i,j}$ are $[n \times n]$ matrices, $b_{i,j}$ is an $[n \times 1]$ vector, Ω_i is the corresponding state space polyhedron with $i \in \Lambda$ the state space partition of $\Omega \subset \mathbb{R}^n$ ($\cup_{i\in\Lambda}\Omega_i = \Omega, \Omega_i \neq \emptyset, \forall i \in \Lambda, \Omega_i \cup \Omega_j = \emptyset, \forall i,j \in \Lambda, i \neq j$), $j \in M_i = \{1,\ldots,m_i\}$, and $U_i = [u_{i,1}\ u_{i,2} \cdots u_{i,m_i}]^{\mathrm{T}} \in \mathbb{R}^{m_i}$ is an m_i-dimensional control input.

In order to find the relationship between the bilinear term and the linear term, the bilinear system can be further adapted. Since each control input $u_{i,j}$ is a scalar, then $\mathrm{rank}(G_{i,j}) = 1$, so it can be expressed as the inner product of two vectors. Then, we can write (1) as

$$\dot{x} = A_i x + \sum_{j=1}^{m_i} b_{i,j}(c_{i,j}^{\mathrm{T}}x + 1)u_{i,j}, \text{ if } x \in \Omega_i, i \in \Lambda. \tag{2}$$

Due to the similarity between switching bilinear systems and switching linear systems, we could define the control inputs as

$$u_{i,j} = \frac{k_{i,j}x}{c_{i,j}^{\mathrm{T}}x + 1}, \quad \text{if } x \in \Omega_i, i \in \Lambda. \tag{3}$$

so as to obtain a corresponding switching linear system (SLS) for the original switching bilinear system (SBLS), as

$$\dot{x} = A_i x + \sum_{j=1}^{m_i} b_{i,j}k_{i,j}x, \text{ if } x \in \Omega_i, i \in \Lambda. \tag{4}$$

Herein, the controller can be designed for the derived corresponding SLS.

3. OPTIMIZED STATE-FEEDBACK CONTROL DESIGN FOR SLSs

Instead of designing a controller for the SBLS directly, we consider designing a state-feedback controller for the corresponding SLS of the original system. Based on the similarity between the two systems, the derived controller can be extended to be used for the SBLS easily.

3.1. *Corresponding Switching Linear System*

For switching bilinear systems, in order to design stabilizing switching division controllers for each bilinear subsystem $i \in \Lambda$, we need to partition the state space polyhedron Ω_i into more subregions. If for sub-bilinear system $i \in \Lambda$, the control input is $u_{i,j}(i \in \Lambda, j \in M_i)$, then for each control input $u_{i,j}$ two state-feedback controllers should be designed. The polyhedral partition of Ω_i $(i \in \Lambda)$ for bilinear subsystem i can be defined as $\{\Omega_{i,l}\}_{i\in\Lambda, l\in\Gamma_i}$, where $\cup_{l\in\Gamma_i}\Omega_{i,l} = \Omega_i, \Omega_{i,l} \neq \emptyset, \forall l \in \Gamma_i, \Omega_{i,l_1} \cap \Omega_{i,l_2} \neq \emptyset, \forall l_1 \neq l_2, l_1, l_2 \in \Gamma_i$.

Based on the polyhedral partition of the state space and defining the equilibrium as the origin, the controller is designed for each polyhedron $\Omega_{i,l}$ as

$$U_{i,l} = [u_{i,l,1}\ u_{i,l,2} \cdots u_{i,l,m_i}]^{\mathrm{T}}, \quad i \in \Lambda, \tag{5}$$

where each control element is designed according to (3). If we substitute (3) into (2), then the bilinear terms are eliminated, and the bilinear system in (2) becomes a switching linear system, which is the corresponding SLS of the SBLS. In order to control the SBLS, we can first consider to design a stabilizing state-feedback controller for the following corresponding SLS:

$$\dot{x} = (A_i + B_i K_{i,l})x, \quad \text{if } x \in \Omega_{i,l}, l \in \Gamma_i, i \in \Lambda, \tag{6}$$

where

$$\begin{aligned} B_i &= [b_{i,1}\ b_{i,2} \cdots b_{i,m_i}], \\ K_{i,l} &= [k_{i,l,1}\ k_{i,l,2} \cdots k_{i,l,m_i}]^{\mathrm{T}}. \end{aligned} \tag{7}$$

Therefore, by dividing the state space into more subregions, the SBLS can be adapted into the corresponding SLS. The corresponding SLS and the SBLS are actually the same model working on different divisions of state space.

3.2. *Lyapunov Functions and Boundary Constraints*

Each polyhedral region $\Omega_{i,l}$ can be described as a system of linear inequalities:

$$\underbrace{[F_{i,l} \quad f_{i,l}]}_{\bar{F}_{i,l}} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0, \quad \text{if } x \in \Omega_{i,l}, \tag{8}$$

and the boundary hyperplane for two neighboring regions $\Omega_{i,l}$ and $\Omega_{i',l'}$ is characterized by an equality and inequality as

$$\underbrace{[\Psi_{ii',ll'} \quad \psi_{ii',ll'}]}_{\bar{\Psi}_{ii',ll'}} \begin{bmatrix} x \\ 1 \end{bmatrix} = 0, \text{ and } \underbrace{[\Phi_{ii',ll'} \quad \phi_{i',ll'}]}_{\bar{\Phi}_{ii',ll'}} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0, \qquad \forall\, x \in \Omega_{i,l} \cap \Omega_{i',l'}. \tag{9}$$

Lyapunov functions are defined for each polyhedral region $\Omega_{i,l}$ $(l \in \Gamma_i, i \in \Lambda)$ with the following format

$$V_{i,l}(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix} P_{i,l} & \star \\ s_{i,l}^{\mathrm{T}} & r_{i,l} \end{bmatrix}}_{\bar{P}_{i,l}} \underbrace{\begin{bmatrix} x \\ 1 \end{bmatrix}}_{\bar{x}}, \quad \forall\, l \in \Gamma_i, i \in \Lambda, x \in \Omega_{i,l}, \tag{10}$$

with $\bar{x} = [x\ 1]^{\mathrm{T}}$, $P_{i,l} \in \mathbb{R}^{n\times n}$ a symmetric matrix, $s_{i,l}$ an $[n \times 1]$ dimensional vector, and $r_{i,l} \in \mathbb{R}$. $\star$ stands for the transpose of its symmetrical element.

3.3. *Optimized State-feedback Control for SLS*

In this part, optimal switching state-feedback control laws will be designed for the SLS in (6), to asymptotically steer any state in the feasible region to the origin, and to guarantee the minimization of a given objective function along the system state trajectory at the same time. More related reference work could be found in [5, 10].

The following theorem gives a sufficient condition to design optimal switched state-feedback control laws for the SLS in (6) that, to asymptotically bring the state to the origin (the equilibrium for at least one of the subsystems), and to optimize the objective function along the system state trajectory. Since the switchings are unknown among the subregions, and the objective function along the system state trajectory is not certain, it is not possible to explicitly optimize the objective function along the state trajectory as

$$J(\infty) = \int_0^\infty [x^{\mathrm{T}} Q_J x + u^{\mathrm{T}} R_J u] dt. \tag{11}$$

Therefore, instead of optimizing the infinite objective function, we optimize the upper bound of the infinite objective function in a min max format, and prove the realization with LMIs in the following theorems.

In the description below, the augmented system matrices are used to describe the linear affine systems as follows:

$$\bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \; \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}. \tag{12}$$

Theorem 1. *For the optimization problem*

$$\begin{aligned} \min_u \max_x \quad & J(\infty) = \int_0^\infty [x^{\mathrm{T}} Q_J x + u^{\mathrm{T}} R_J u] dt \\ s.t. \quad & (16) - (19), \end{aligned} \tag{13}$$

if there exists a solution satisfying all the constraints, with positive definite matrices $\bar{Q}_{i,l}$, $Q_{i,l}$, $R_{i,l}$, and $M_{i,l}$, then taking the state-feedback control laws with gains as

$$\bar{K}_{i,l} = \bar{N}_{i,l} \bar{Q}_{i,l}^{-1} \quad \forall\, (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{14}$$

and

$$K_{i,l} = N_{i,l} Q_{i,l}^{-1} \quad \forall\, (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \in \Omega_{i,l}\}, \tag{15}$$

asymptotically stabilizes the SLS system in (6)*, and guarantees the minimization of the objective function along the state trajectory.*

Proof: First, using the Schur complement for (16), and multiplying the result from the right and left side by $\bar{Q}_{i,l}^{-1} = \bar{P}_{i,l}$, yields

$$\bar{P}_{i,l} - \bar{F}_{i,l}^{\mathrm{T}} R_{i,l}^{-1} \bar{F}_{i,l} > 0, \quad \forall\, (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{20}$$

which guarantees that the Lyapunov function on each state polyhedron is positive because of the positiveness of the matrix $R_{i,l}$, i.e.

$$V_{i,l} > 0, \quad \text{if } \bar{F}_{i,l}\bar{x}_{i,l} \geq 0 \text{ and } \bar{x}_{i,l} \neq 0, \quad \forall\, (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}. \tag{21}$$

For the case for the subsystem containing the origin, i.e. for the polyhedron with $0 \in \Omega_{i,l}$, the LMIs in (17) and (18) are applied to make sure obtain a positive Lyapunov function and a negative derivative of Lyapunov function on the region. The row and column corresponding to the augmented variable are removed here, to guarantee that the derivative of the Lyapunov function $\dot{V}_{i,l}$ would be zero only when the state x is zero.

Second, the Schur complement is applied on (18), and the obtained result is multiplied from left and the right side by $Q_{i,l}^{-1} = P_{i,l}$. With the feedback laws (14), we obtain

$$\begin{aligned} P_{i,l}(A_i + B_i K_{i,l}) + (A_i + B_i K_{i,l})^{\mathrm{T}} P_{i,l} < -F_{i,l}^{\mathrm{T}} M_{i,l}^{-1} F_{i,l} - Q_J - K_{i,l}^{\mathrm{T}} R_J K_{i,l}, \\ \forall\, (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \end{aligned} \tag{22}$$

$$\begin{bmatrix} \bar{Q}_{i,l} & \star \\ \bar{F}_{i,l}\bar{Q}_{i,l} & R_{i,l} \end{bmatrix} > 0, \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{16}$$

$$\begin{bmatrix} Q_{i,l} & \star \\ F_{i,l}Q_{i,l} & R_{i,l} \end{bmatrix} > 0,\ \bar{Q}_{i,l} = \begin{bmatrix} Q_{i,l} & \star \\ 0 & q_{i,l} \end{bmatrix} > 0, \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \in \Omega_{i,l}\}, \tag{17}$$

$$\begin{bmatrix} A_iQ_{i,l} + Q_{i,l}A_i^{\mathrm{T}} + B_iN_{i,l} + N_{i,l}^{\mathrm{T}}B_i^{\mathrm{T}} & \star & \star & \star \\ F_{i,l}Q_{i,l} & -M_{i,l} & 0 & 0 \\ Q_{i,l} & 0 & -Q_J^{-1} & 0 \\ K_{i,l}Q_{i,l} & 0 & 0 & -R_J^{-1} \end{bmatrix} < 0, \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i\}, \tag{18}$$

$$\begin{bmatrix} \bar{Q}_{i,l} & \star & \star & \star \\ \bar{Q}_{i,l} & \bar{Q}_{i',l'} & \star & \star \\ \bar{\Psi}_{ii',ll'}\bar{Q}_{i,l} & 0 & -\lambda_{ii',ll'} & \star \\ \bar{\Phi}_{ii',ll'}\bar{Q}_{i,l} & 0 & 0 & -\Theta_{ii',ll'} \end{bmatrix} > 0, \quad \text{if } d_{i,j} > d_{i',j'}, \text{ and } \{\Omega_{i,l} \cap \Omega_{i',l'}\} \neq \emptyset,\ \forall i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} \tag{19}$$

since the parameter matrices in the objective functions (Q_J and R_J) are positive definite, and $M_{i,l}$ is also positive definite; therefore it guarantees that the derivative of the Lyapunov function on each state polyhedron is negative, as

$$\dot{V}_{i,l} < 0, \quad \text{if } F_{i,l}x_{i,l} \geq 0 \text{ and } x_{i,l} \neq 0, \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}. \tag{23}$$

The LMI that makes sure the derivative of the Lyapunov function is negative is written in the format of (18), because for the linear affine switching subsystems, the derivative of the affine offset is 0.

Then, we perform the Schur complement on (19) 3 times, each time with respect to the last row and column. Similarly, we multiply the result from the right and left side by $\bar{Q}_{i,l}^{-1} = \bar{P}_{i,l}$; and use (14), then we obtain the following inequalities to guarantee the boundary condition:

$$\begin{aligned} &\bar{P}_{i,l} - \bar{P}_{i',l'} + \lambda_{ii',ll'}^{-1}\bar{\Psi}_{ii',ll'}^{\mathrm{T}}\bar{\Psi}_{ii',ll'} + \bar{\Phi}_{ii',ll'}^{\mathrm{T}}\Theta_{ii',ll'}^{-1}\bar{\Phi}_{ii',ll'} > 0, \\ &\qquad \text{if } d_{i,j} > d_{i',j'}, \text{ and } \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset, \quad \forall i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'}, \end{aligned} \tag{24}$$

which ensures that $V_{i,l} \geq V_{i',l'}$ for all the states $\bar{x} \in \mathscr{S}_{ii',ll'}$ on the boundary of $\Omega_{i,l}$ and $\Omega_{i',l'}$. $d_{i,j}$ is the shortest distance between the origin and the polyhedron $\Omega_{i,j}$.The augmented $\bar{Q}_{i,l}$ is defined for the polyhedron $\Omega_{i,l}$ containing the origin in (17), to make it comparable on the boundary conditions with other polyhedron without the origin.

Because the Lyapunov functions reduce during the switchings of the regions in the state space, there is a sequence of polyhedra in Ω, whose distances to the origin are reducing, which satisfy

$$d_p \geq d_{p-1} \geq \cdots \geq d_1 \geq 0, \tag{25}$$

with p as the total number of polyhedron $\Omega_{i,l}$, $\forall\, i \in \Lambda, l \in \Gamma_i$ in Ω, which is corresponding to a sequence of decreasing Lyapunov functions for all the polyhedra as

$$V_p(x_p^*) \geq V_{p-1}(x_{p-1}^*) \geq \cdots \geq V_1(x_1^*) \geq 0, \tag{26}$$

that can make the system state asymptotically converge to the origin, from an initial state x_0 within any of the polyhedra in Ω. At the same time, the upper bound of the cost function is minimized along the trajectory, to make sure the objective function of the worst case is minimized under the uncertain switchings. Consequently, by solving the optimization in Theorem 1, it is possible to design the optimal switched state-feedback control laws for the SLS in (6) that, asymptotically bring the state to the origin (the equilibrium for at least one of the subsystems), and optimize the objective function along the system state trajectory. □

In order to solve the min max optimization problem, the upper bound of the objective function is derived to further solve the optimization with LMIs.

Since the objective function is

$$J(\infty)=\int_0^{\infty}[x^{\mathrm{T}}Q_Jx+u^{\mathrm{T}}R_Ju]dt, \tag{27}$$

and for the derived sequence of polyhedra in Ω, according to (22), we have

$$\dot{V}_p(x)<-x^{\mathrm{T}}F_p^{\mathrm{T}}M_p^{-1}F_px-x^{\mathrm{T}}Q_Jx-x^{\mathrm{T}}K_p^{\mathrm{T}}R_JK_px, \tag{28}$$

for polyhedron Ω_p. Integrating along the trajectory for Ω_p on both sides of (28), yields

$$\int_{x_{p,\mathrm{s}}}^{x_{p,\mathrm{e}}}x^{\mathrm{T}}F_p^{\mathrm{T}}M_p^{-1}F_px\,dx+\int_{x_{p,\mathrm{s}}}^{x_{p,\mathrm{e}}}[x^{\mathrm{T}}Q_Jx+x^{\mathrm{T}}K_p^{\mathrm{T}}R_JK_px]dx<\int_{\Omega_p}\dot{V}_p(x)dx, \tag{29}$$

that is

$$C_p+J_p<V_p(x_{p,\mathrm{s}})-V_p(x_{p,\mathrm{e}}), \tag{30}$$

where $x_{p,\mathrm{s}}$ and $x_{p,\mathrm{e}}$ are the starting and ending states on polyhedron Ω_p. For the sequence of polyhedra in Ω, we have

$$\begin{aligned}
&C_1+J_1<V_1(x_{1,\mathrm{s}})-V_1(x_{1,\mathrm{e}}),\\
&C_2+J_2<V_1(x_{2,\mathrm{s}})-V_2(x_{2,\mathrm{e}}),\\
&\vdots\\
&C_p+J_p<V_p(x_{p,\mathrm{s}})-V_p(x_{p,\mathrm{e}}),
\end{aligned} \tag{31}$$

where $x_{1,\mathrm{s}}=x_0$. In addition, according to (19), we have

$$\begin{aligned}
&V_1(x_{1,\mathrm{e}})>V_0(x_{0,\mathrm{s}}),\\
&V_0(x_{0,\mathrm{e}})>V_3(x_{3,\mathrm{s}}),\\
&\vdots\\
&V_{p-1}(x_{p-1,\mathrm{e}})>V_p(x_{p,\mathrm{s}}).
\end{aligned} \tag{32}$$

If we sum up (31) along the switching sequence for all the polyhedra to the equilibrium, then we have

$$C+J<V_1(x_0)-V_p(x_{p,\mathrm{e}}), \tag{33}$$

where C is the integrating of $x^{\mathrm{T}}F_p^{\mathrm{T}}M_p^{-1}F_px$ along the state trajectory, which is larger than 0 because the matrices M_p are positive definite. Since $\lim_{t\to\infty}V_p(x_{p,\mathrm{e}})=0$, and C is positive, thus $J<V_1(x_0)$, an upper bound of the objective function is $V_1(x_0)=x_0^{\mathrm{T}}P_1x_0$. Therefore the min max problem can be solved by minimizing the upper bound of the objective function with the following theorem satisfying the following constraint

$$\begin{bmatrix}\gamma & \star\\ x_0 & Q_1\end{bmatrix}\geq 0, \tag{34}$$

which guarantees $x_0^{\mathrm{T}}P_1x_0\leq\gamma$.

Theorem 2. *For the optimization problem*

$$\begin{aligned}
\min\quad & \gamma\\
\textit{s.t.}\quad & (16)-(19)\ \textit{and}\ (34),
\end{aligned} \tag{35}$$

if there exists a solution satisfying all the constraints (16)-(19) *and* (34)*, with positive definite matrices* $\bar{Q}_{i,l}$, $Q_{i,l}$, $R_{i,l}$, *and* $M_{i,l}$*, then taking the state-feedback control laws with gains as*

$$\bar{K}_{i,l}=\bar{N}_{i,l}\bar{Q}_{i,l}^{-1}\quad\forall\,(i,l)\in\{(i,l)\mid i\in\Lambda,l\in\Gamma_i,0\notin\Omega_{i,l}\}, \tag{36}$$

and

$$K_{i,l}=N_{i,l}Q_{i,l}^{-1}\quad\forall\,(i,l)\in\{(i,l)\mid i\in\Lambda,l\in\Gamma_i,0\in\Omega_{i,l}\}, \tag{37}$$

asymptotically stabilizes the SLS system in (6)*, and guarantees the minimization of the upper bound of the objective function along the state trajectory with uncertain switchings.*

4. EXAMPLE

In this section, an example is presented to evaluate the performance of the optimal controller designed for a SBLS based on Theorem 2.

In the example, we use the conditions presented in Theorem 2 to design state-feedback control laws optimizing the upper bound of the infinite objective function. We directly use the SBLS model in (2) with the following vectors and matrices:

$$\begin{aligned}
A_1 &= \begin{bmatrix} -3 & 1 \\ -5 & -8 \end{bmatrix},\ b_{1,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ c_{1,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
A_2 &= \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix},\ b_{2,1} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ c_{2,1} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
Q_J &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ R_J = 0.1.
\end{aligned}$$

There are 2 bilinear subsystems separated by $x_1 - x_2 = 0$. According to Sec. 3.1, the state space is partitioned into 4 regions with $\Lambda = \{1,2\}$ and $\Gamma_1 = \{1,2\}, \Gamma_2 = \{1,2\}$. Then, the parameters for the obtained corresponding SLS with the format (6) are:

$$\begin{aligned}
A_1 &= \begin{bmatrix} -3 & 1 \\ -5 & -8 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
F_{1,1} &= \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix},\ \bar{F}_{1,2} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}, \\
\bar{\Psi}_{11,12} &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},\ \bar{\Phi}_{11,12} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\
F_{2,1} &= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},\ \bar{F}_{2,2} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \\
\bar{\Psi}_{22,12} &= \begin{bmatrix} 0 & -1 & 1 \end{bmatrix},\ \bar{\Phi}_{22,12} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Using the Yalmip toolbox (with the SeDuMi solver) to solve the optimization problem (i.e. the LMIs) (16)-(19) and (34), an decreasing overall Lyapunov function is obtained as given in Fig. 1. As the overall Lyapunov function shows, the Lyapunov function is smooth with each subregion, positive, and decreasing gradually to the origin of the state space. In addition, the Lyapunov function is able to jump and decrease on the boundaries of the state space switchings of the sequence of the switching state polyhedra. As a result, applying the derived Lyapunov function, the controllers are obtained according to (3), (36) and (37), as

$$\begin{aligned}
U_{1,1} &= \frac{K_{1,1}x}{x_1+1},\ U_{1,2} = \frac{\bar{K}_{1,2}\bar{x}}{x_1+1},\ U_{1,12} = 0, \\
U_{2,1} &= \frac{K_{2,1}x}{-x_2+1},\ U_{1,2} = \frac{\bar{K}_{2,2}\bar{x}}{-x_2+1},\ U_{2,12} = 0,
\end{aligned}$$

where

$$\begin{aligned}
K_{1,1} &= [-0.2048\ 0.0080],\ \bar{K}_{1,2} = [-0.2980\ 0.0082\ 0], \\
K_{2,1} &= [0.0967\ -0.1556],\ \bar{K}_{2,2} = [0.0967\ -0.1556\ 0].
\end{aligned}$$

The simulation show that the designed controllers are able to steer state to the origin for different initial conditions, as in Fig. 2 and 3.

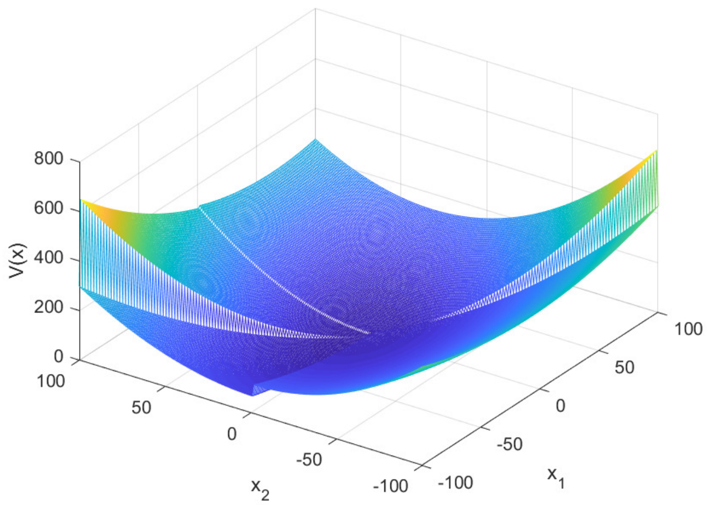

Figure 1: Illustration for the overall Lyapunov function

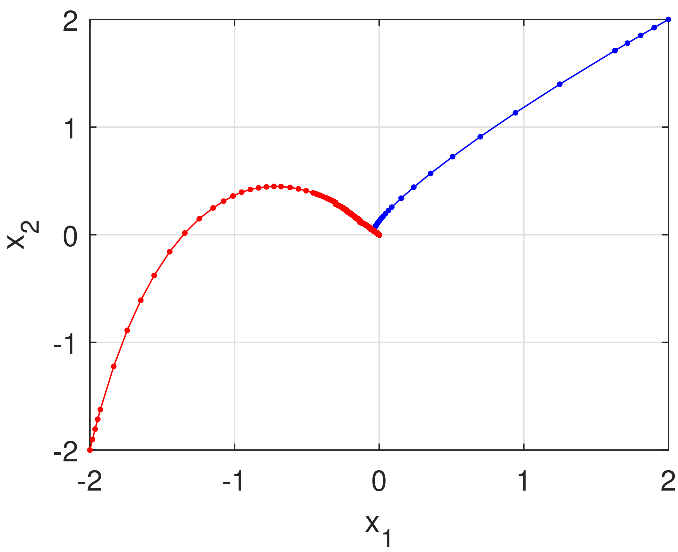

Figure 2: The closed-loop trajectories with initial states $[2\ 2]^{\mathrm{T}}$ and $[-2\ -2]^{\mathrm{T}}$

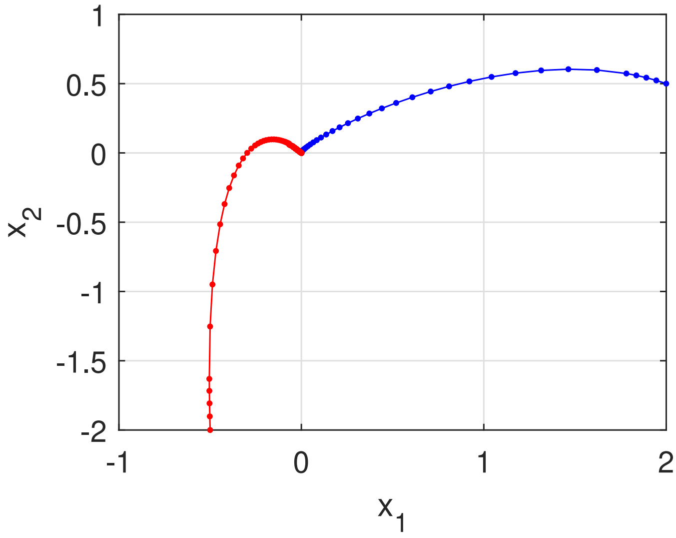

Figure 3: The closed-loop trajectories with initial states $[2\ 0.5]^{\mathrm{T}}$ and $[-0.5\ \ -2]^{\mathrm{T}}$

5. CONCLUSIONS

In practice, there are some complex nonlinear systems that can be approximated by switching bilinear systems. Designing stabilizing controller for switching bilinear systems makes it possible to better control these kind of nonlinear systems. To deal with the state-based switching bilinear systems, the subregions where subsystems are activated are further divided into some regions utilizing the special features of bilinear systems. And then, region dependent controllers are designed in resulting subregions, which transform the bilinear systems into linear ones. Based on the linear property of the system, a state-feedback controller design method is proposed considering the infinite horizon quadratic cost function to minimize the total cost along the state trajectory. By solving the series of derived LMIs, optimized switching state-feedback control laws will be obtained for the SBLS, to asymptotically steer any state in the feasible region to the equilibrium, and can guarantee the minimization of the upper bound of the objective function along the system state trajectory at the same time. The numerical result shows that the designed controller is able to stabilize the system. In the future, we will apply the proposed method in traffic flow control, and try to use it solving real traffic problems.

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7. reference

[1] Bichiou, S., Bouafoura, M. K., Benhadj Braiek, N., 2018. Time optimal control laws for bilinear systems. Mathematical Problems in Engineering 2018.
[2] Breiten, T., Kunisch, K., Pfeiffer, L., 2018. Infinite-horizon bilinear optimal control problems: Sensitivity analysis and polynomial feedback laws. SIAM Journal on Control and Optimization 56 (5), 3184–3214.
[3] Cheng, D., 2005. Controllability of switched bilinear systems. IEEE Transactions on Automatic Control 50 (4), 511–515.
[4] Gutman, P.-O., 1981. Stabilizing controllers for bilinear systems. IEEE Transactions on Automatic Control 26 (4), 917–922.

[5] Hajiahmadi, M., De Schutter, B., Hellendoorn, H., 2016. Design of stabilizing switching laws for mixed switched affine systems. IEEE Transactions on Automatic Control 61 (6), 1676–1681.

[6] Khapalov, A., Mohler, R., 1998. Asymptotic stabilization of the bilinear time-invariant system via piecewise-constant feedback. Systems & Control Letters 33 (1), 47–54.

[7] Khapalov, A. Y., Mohler, R., 1996. Reachable sets and controllability of bilinear time-invariant systems: A qualitative approach. IEEE Transactions on Automatic Control 41 (9), 1342–1346.

[8] Koditschek, D., Narendra, K., 1983. Stabilizability of second-order bilinear systems. IEEE Transactions on Automatic Control 28 (10), 987–989.

[9] Langson, W., Alleyne, A., 1997. Infinite horizon optimal control of a class of nonlinear systems. In: Proceedings of the 1997 American Control Conference. Vol. 5. IEEE, pp. 3017–3022.

[10] Lin, H., Antsaklis, P. J., 2009. Stability and stabilizability of switched linear systems: a survey of recent results. IEEE Transactions on Automatic control 54 (2), 308–322.

[11] Lin, S., De Schutter, B., Li, D., 2018. Stabilizing controller design for state-based switching bilinear systems. In: 2018 IEEE Conference on Decision and Control (CDC). IEEE, pp. 6452–6457.

[12] Longchamp, R., 1980. Stable feedback control of bilinear systems. IEEE Transactions on Automatic Control 25 (2), 302–306.

[13] Mohler, R., Khapalov, A., 2000. Bilinear control and application to flexible ac transmission systems. Journal of Optimization Theory and Applications 105 (3), 621–637.

[14] Mohler, R. R., 1973. Bilinear Control Processes: With Applications to Engineering, Ecology and Medicine. Academic Press, Inc., London.

[15] Mohler, R. R., Kolodziej, W., 1980. An overview of bilinear system theory and applications. IEEE Transactions on Systems, Man and Cybernetics 10 (10), 683–688.

[16] Wan, Y., Keviczky, T., 2019. Real-time fault-tolerant moving horizon air data estimation for the reconfigure benchmark. IEEE Transactions on Control Systems Technology 27 (3), 997–1011.

[17] Wan, Y., Keviczky, T., Verhaegen, M., 2018. Fault estimation filter design with guaranteed stability using markov parameters. IEEE Transactions on Automatic Control 63 (4), 1132–1139.

[18] Xin, J., Negenborn, R. R., Lin, X., 2018. Piecewise affine approximations for quality modeling and control of perishable foods. Optimal Control Applications and Methods 39 (2), 860–872.

[19] Xin, J., Negenborn, R. R., van Vianen, T., 2018. A hybrid dynamical approach for allocating materials in a dry bulk terminal. IEEE Transactions on Automation Science and Engineering 15 (3), 1326–1336.

[20] Xu, J., van den Boom, T., De Schutter, B., Dec. 2016. Optimistic optimization for model predictive control of max-plus linear systems. Automatica 74, 16–22.

[21] Xu, J., van den Boom, T., De Schutter, B., 2019. Model predictive control for stochastic max-plus linear systems with chance constraints. IEEE Transactions on Automatic Control 64 (1), 337–342.

[22] Yu, C., Chen, J., Verhaegen, M., 2019. Subspace identification of individual systems in a large-scale heterogeneous network. Automatica 109, 108517.

[23] Yu, C., Ljung, L., Wills, A., Verhaegen, M., 2019. Constrained subspace method for the identification of structured state-space models. IEEE Transactions on Automatic Control, In press.

[24] Zhang, Z., Shi, Y., Zhang, Z., Yan, W., 2018. New results on sliding-mode control for takagi–sugeno fuzzy multiagent systems. IEEE Transactions on Cybernetics 49 (5), 1592–1604.

[25] Zhang, Z., Yan, W., Li, H., Li, L., 2019. Consensus control of linear systems with optimal performance on directed topologies. Journal of the Franklin Institute.

[26] Zheng, H., Negenborn, R. R., Lodewijks, G., 2017. Fast ADMM for distributed model predictive control of cooperative waterborne AGVs. IEEE Transactions on Control Systems Technology 25 (4), 1406 – 1413.

[27] Zheng, H., Wu, J., Wu, W., Wang, Y., 2019. Integrated motion and powertrain predictive control of intelligent fuel cell/battery hybrid vehicles. IEEE Transactions on Industrial Informatics 16, 3397–3406.