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A Directed Spanning Tree Adaptive Control Solution to Time-Varying Formations

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***Abstract*—In this paper, the time-varying formation and time-varying formation tracking problems are solved for linear multi-agent systems over digraphs without the knowledge of the eigenvalues of the Laplacian matrix associated to the digraph. The solution to these problems relies on an approach that generalizes the directed spanning tree adaptive method, which was originally limited to consensus problems. Necessary and sufficient conditions for the existence of solutions to the formation problems are derived. Asymptotic convergence of the formation errors is proved via graph theory and Lyapunov analysis.**

***Index Terms*—Adaptive control, directed graphs, multi-agent systems, formation control.**

I. Introduction

Formation control of multi-agent systems has captured increasing attention due to applications in spacecraft formation flying, search and rescue operations, intelligent transport system, to name a few [1], [2]. Up to now, many existing methods on formation control are based on a common assumption that each agent knows the formation information, e.g., the center and radius of the circular formation [3].

Meanwhile, the fertile framework of consensus [4]–[6] has motivated researchers to study consensus-based formation control in a *distributed* way, i.e. using local information from neighboring agents, so as to keep some formation offsets between each other. Along this line, recent results on time-varying formation (TVF) [7], [8], and time-varying formation tracking (TVFT) [7], [9]–[12] have provided a natural extension to the standard time-invariant formation case [13]. Several methods have been proposed to address the time-varying case: finite-time consensus techniques have been applied to the TVF/TVFT control for first-order multi-agent systems in [7] provided that the velocity information of the desired formation offsets are locally available for the agents. Adaptive neural networks have been used to achieve practical TVFT (with bounded tracking errors) for a class of second-order nonlinear multi-agent systems in [11]. For higher-order systems (linear time-invariant dynamics), necessary and sufficient conditions have been derived in [8] and [10] for TVF and TVFT with multiple leaders. The major benefit of these necessary and sufficient conditions is their interpretation in terms of formation feasibility conditions, which allow to remove the requirement of the velocity information of the formation offsets. Furthermore, these conditions provide extra design freedom for controlling the motion of the formation. More recently, some sufficient conditions for time-varying output formation tracking by output feedback control has been proposed in [12]. However, a requirement that still remains in [8]–[12] is the knowledge of the smallest nonzero eigenvalue of the communication Laplacian matrix, which might be unknown especially in large networks.

With respect to this requirement, it is known that the knowledge of the Laplacian eigenvalues can be overcome by suitably designing time-varying coupling weights in the network: this was shown for consensus [14]–[16], containment [17], or TVF [18]–[20] problems over undirected or detail-balanced/strongly-connected digraphs. It is well known in network science that nodes and edges are two interdependent elements of a network system. For undirected networks, distributed methods with adaptive coupling weights from both node and edge perspectives have been well understood. For example, the node-based method can be made fully distributed (without any global information), while the edge-based method can be applicable to switching connected graphs [15], [20]. However, the interdependence of nodes and edges is essentially more complex to understand in directed networks, especially in general digraphs where the only assumption is the presence of a directed spanning tree (DST). Along this direction, a fully distributed node-based method has been proposed to address tracking [27] and group TVFT [24]. On the other hand, a DST-based adaptive control method has recently been studied for synchronization/consensus problems in [21]–[23], which exploits the structure of a DST in the network. It should be mentioned that it is still not clear how to design fully distributed edge-based adaptive consensus algorithms even without the knowledge of a DST (Please refer to Conjecture 1 of [21]): in this sense, the DST-based adaptive method is currently the most relax edge-based method. As compared to the node-based method [24], [27], the DST adaptive method provides some interesting insights on how the structure of a complex directed network influences the network dynamics. However, to our best knowledge, a unifying

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DST-based adaptive control framework encompassing TVF and TVFT problems is not available. Most notably, it is unclear how to design appropriate feasibility conditions for time-varying formations in the DST framework. These observations motivate this study.

The main contribution of this paper is a unifying DST-based adaptive control solution addressing TVF and TVFT for linear MASs: not only does the proposed method avoid the knowledge of the Laplacian eigenvalues as compared with [8]–[12], but it also helps to establish necessary and sufficient conditions for such time-varying formations from a different perspective as compared with [18]–[20], [24]. For TVF without leaders, a novel class of feasibility conditions is proposed, which is more efficient to check than the feasibility conditions in the state of the art. The proposed conditions generalize in a natural unified way in the presence of one or more leaders.

The paper is organized as follows: Section II gives some preliminaries and formulates the problems. Sections III-IV present the main results for TVF and TVFT, respectively. Numerical examples are provided in Section V. Section VI concludes this paper.

II. Preliminaries and Problem Statement

A. Notations

Let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times p}$ represent the sets of real scalars, real positive scalars, n-dimensional column vectors, $n\times p$ matrices, respectively. Let $\mathbf{I}_n$ and $\mathbf{1}_n$ be the $n\times n$ identity matrix, and the column vector with n elements being one, respectively. Zero vectors and zero matrices are all denoted by 0. For a vector x, let $\|x\|$ denote the Euclidean norm. For a real symmetric matrix A, $\lambda_{\mathrm{M}}(A)$ (resp. $\lambda_{\mathrm{m}}(A)$) is its maximum (resp. minimum) eigenvalue, and $A>0$ (resp. $A\geq 0$) means that A is positive definite (resp. semi-definite). Denote $\mathcal{I}_N=\{1,2,\cdots,N\}$ as the set of natural numbers up to N. Denote $\mathrm{col}(x_1,\cdots,x_N)=({x_1}^T,\cdots,{x_N}^T)^T$ as the column vectorization. The abbreviation diag$(\cdot)$ is the diagonalization operator and 'N-S' is short for 'necessary and sufficient'. The cardinality of a set is denoted by $|\cdot|$ and the difference (resp. union) of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is denoted by $\mathcal{S}_1\setminus\mathcal{S}_2$ (resp. $\mathcal{S}_1\bigcup\mathcal{S}_2$). Moreover, $\otimes$ stands for the Kronecker product.

B. Graph Theory

A weighted digraph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{A})$ is specified by the node set $\mathcal{V}=\{1,\cdots,N\}$, the edge set $\mathcal{E}=\{e_{ij}|i\to j, i\neq j\}$ and the weighted adjacency matrix $\mathcal{A}=(a_{ij})\in\mathbb{R}^{N\times N}$. In the matrix $\mathcal{A}$, $a_{ij}>0$ if $e_{ji}\in\mathcal{E}$, indicating that j (resp. i) is an in-neighbor (resp. out-neighbor) of i (resp. j), which can be denoted by $j\in\mathcal{N}_1(i)$ (resp. $i\in\mathcal{N}_2(j)$). Let $\mathcal{D}_2(i)=|\mathcal{N}_2(i)|$ be the out-degree of i. Moreover, $\mathcal{L}=(\mathcal{L}_{ij})\in\mathbb{R}^{N\times N}$ is the Laplacian matrix of $\mathcal{G}$, which is defined as: $\mathcal{L}_{ij}=-a_{ij}$, if $i\neq j$, and $\mathcal{L}_{ii}=\sum_{k=1,k\neq i}^{N}a_{ik}$, $\forall i\in\mathcal{I}_N$. A path is a sequence of edges connecting a pair of nodes. A digraph $\mathcal{G}$ is *strongly connected* if any pair of nodes is connected by a directed path, and is *weakly-connected* if any pair of nodes is connected by a path disregarding the directions. A *directed spanning tree (DST)* of $\mathcal{G}$ is a subgraph where there is a node called the root, that has no in-neighbors, such that one can find a unique path from the root to every other node. In a DST, if j is an in-neighbor of i, one can also say that j is a parent node, and i is a child node. Moreover, a node is called a stem if it has at least one child, and a leaf otherwise.

C. Problem Statement

Let $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{A})$ denote the digraph that characterizes the communication topology among N agents, where the weights in $\mathcal{A}$ represent the communication strengths. The dynamics of the agents are given by

$$\dot{x}_i=Ax_i+Bu_i,\quad i\in\mathcal{I}_N \tag{1}$$

where $x_i\in\mathbb{R}^n$ is the state of agent i and $u_i\in\mathbb{R}^m$ is its control input to be designed. Let the pair (A,B) be stabilizable.

Definition 1 ([18],TVF): The multi-agent system (1) is said to achieve the time-varying formation (TVF) defined by the time-varying vector $h(t)=\mathrm{col}(h_1(t),h_2(t),\cdots,h_N(t))$ if, for any initial states, there holds

$$\lim_{t\to\infty}((x_i-h_i)-(x_j-h_j))=0,\ \forall i,j\in\mathcal{I}_N. \tag{2}$$

Now consider the case where there are M leader agents, $M\geq 1$, in the network $\mathcal{G}$. Without loss of generality, let the first M agents be the leaders, and the rest be the followers:

$$\begin{aligned}\dot{x}_l&=Ax_l, && l\in\mathcal{I}_M,\\ \dot{x}_i&=Ax_i+Bu_i, && i\in\mathcal{I}_N\setminus\mathcal{I}_M.\end{aligned} \tag{3}$$

As leaders have no in-neighbors, the Laplacian matrix of $\mathcal{G}$ can be partitioned as

$$\mathcal{L}=\begin{pmatrix}0 & 0\\ \mathcal{L}_1 & \mathcal{L}_2\end{pmatrix} \tag{4}$$

where $\mathcal{L}_1\in\mathbb{R}^{(N-M)\times M}$ and $\mathcal{L}_2\in\mathbb{R}^{(N-M)\times(N-M)}$.

Definition 2 ([10]): A follower is called *well-informed* if all leaders are its in-neighbors, and is *uninformed* if no leader is its in-neighbor.

Definition 3 (TVFT): The multi-agent system (3) is said to achieve the time-varying formation tracking (TVFT) defined by the time-varying vector $h^F(t)=\mathrm{col}(h_{M+1}(t),h_{M+2}(t),\cdots,h_N(t))$ and by positive constants β_l, $l\in\mathcal{I}_M$, satisfying $\sum_{l=1}^{M}\beta_l=1$ if, for any initial states, there holds

$$\lim_{t\to\infty}\big(x_i-h_i-\sum_{l=1}^{M}\beta_l x_l\big)=0,\ \forall i\in\mathcal{I}_N\setminus\mathcal{I}_M. \tag{5}$$

For the special case $M=1$, (5) becomes

$$\lim_{t\to\infty}\big(x_i-h_i-x_1\big)=0,\quad i=2,\cdots,N. \tag{6}$$

Remark 1: TVFT with multiple leaders was firstly formulated in [10] as: given a predefined $h^F(\cdot)$, find a group of hyperparameters β_l, such that (5) holds for any initial x_i. This formulation was adapted to group TVFT in [24] and to time varying output formation tracking in [12]. Definition 3 has a slightly different formulation: given a predefined $h^F(\cdot)$ and any predefined combination of convex coefficients β_l, determine whether (5) can hold for any initial x_i. This formulation of TVFT is motivated by practical consideration. In fact, the methods in [10], [12], [24] lead to a specific group of β_l determined by the communication topology, and cannot predefine the convex combination.

The goal of this paper is to solve the problems outlined by (2), (5) and (6) by consistently generalizing the DST idea. Along this paper, $h_i(\cdot)$ (β_l) is assumed only known by follower i (leader l).

III. DST-Based Distributed Adaptive TVF

This section appropriately extends the DST-based adaptive control method to solve the TVF problem of Definition 1. The following is a standard connectivity assumption ([8]).

Assumption 1: The digraph $\mathcal{G}$ has at least one DST.

Under Assumption 1, one can select a DST $\bar{\mathcal{G}}(\mathcal{V}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ of $\mathcal{G}$. As in [22], we assume that $\bar{\mathcal{G}}$ is known. Without loss of generality, let node 1 be the root of $\bar{\mathcal{G}}$. Correspondingly, let $\bar{\mathcal{L}}$ be the Laplacian matrix of $\bar{\mathcal{G}}$ and $\bar{\mathcal{N}}_2(i)$ be the set of out-neighbors of i in $\bar{\mathcal{G}}$.

Let i_k denote the unique parent of node $k+1$ in $\bar{\mathcal{G}}$ for $k \in \mathcal{I}_{N-1}$, then $\bar{\mathcal{E}} = \{e_{i_k,k+1} | k \in \mathcal{I}_{N-1}\} \subset \mathcal{E}$. For compactness, define $d_i(t) = x_i(t) - h_i(t)$ as the formation state, i.e., the distance between the current state and the desired formation offset of agent i. Denote $x = \mathrm{col}(x_1, \cdots, x_N)$, $d = \mathrm{col}(d_1, \cdots, d_N)$.

We propose the DST-based adaptive TVF controller as:

$$u_i = K_0 x_i + K_1 d_i + K_2 \sum_{j \in \mathcal{N}_1(i)} \alpha_{ij}(t)(d_i - d_j) \tag{7}$$

with the time-varying coupling weights

$$\alpha_{ij}(t) = \begin{cases} a_{ij}, & \text{if} \quad e_{ji} \in \mathcal{E} \setminus \bar{\mathcal{E}}, \\ \bar{a}_{k+1,i_k}(t), & \text{if} \quad e_{ji} \in \bar{\mathcal{E}}. \end{cases} \tag{8}$$

$$\dot{\bar{a}}_{k+1,i_k} = \rho_{k+1,i_k}\Big((d_{i_k} - d_{k+1}) - \sum_{j \in \bar{\mathcal{N}}_2(k+1)} (d_{k+1} - d_j)\Big)^T \Gamma (d_{i_k} - d_{k+1}). \tag{9}$$

In (7)-(9), K_0, K_1, K_2, and Γ are gains to be designed, and $\rho_{k+1,i_k} \in \mathbb{R}^+$. In (7), $\alpha_{ij}(t)$ is the coupling weight between agent i and its in-neighbor j, which is time-varying only if the corresponding edge appears in $\bar{\mathcal{G}}$, i.e., $j = i_k$ and $i = k+1$ for some $k \in \mathcal{I}_{N-1}$, and constant otherwise.

Remark 2: The structure of controller (7) is as follows. The gain K_0 is to be designed to make the time-varying formation $h(\cdot)$ feasible; the gain K_1 is needed to control the average formation signal $d_{\text{ave}} = \frac{1}{N}\sum_{j \in \mathcal{I}_N} d_j$ [8]; the gain K_2 is a consensus gain. Different from the related literature [8], the DST structure is explicitly used in the control law (7)-(9).

A. Technical lemmas

Lemma 1 (N-S condition for TVF): Under Assumption 1, and for any DST $\bar{\mathcal{G}}$, define $\Xi \in \mathbb{R}^{(N-1)\times N}$ as

$$\Xi_{kj} = \begin{cases} -1, & \text{if} \quad j = k+1, \\ 1, & \text{if} \quad j = i_k, \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

Then, the TVF for multi-agent system (1) can be achieved if and only if

$$\lim_{t \to \infty} \|(\Xi \otimes \mathbf{I}_n) d(t)\| = 0. \tag{11}$$

Proof: From Lemma 3.2 in [22], (11) holds if and only if $\lim_{t\to\infty} \|d_i(t) - d_j(t)\| = 0, \forall i, j \in \mathcal{I}_N$. Then, Lemma 1 holds following Definition 1 and the definition of $d_i(t)$. In fact, Ξ^T is the incidence matrix associated to $\bar{\mathcal{G}}$. ■

Lemma 2 (Auxiliary matrix Q): Under Assumption 1, and for any DST $\bar{\mathcal{G}}$, define $Q \in \mathbb{R}^{(N-1)\times(N-1)}$ as $Q = \tilde{Q} + \bar{Q}$ with $\tilde{Q}_{kj} = \sum_{c \in \bar{\mathcal{V}}_{j+1}} (\tilde{\mathcal{L}}_{k+1,c} - \tilde{\mathcal{L}}_{i_k,c})$ and $\bar{Q}_{kj} = \sum_{c \in \bar{\mathcal{V}}_{j+1}} (\bar{\mathcal{L}}_{k+1,c} - \bar{\mathcal{L}}_{i_k,c})$. Here, $\bar{\mathcal{V}}_{j+1}$ represents the vertex set of the subtree rooting at node $j+1$ and $\tilde{\mathcal{L}} = \mathcal{L} - \bar{\mathcal{L}}$. Then, there holds

$$\Xi \mathcal{L} = Q \Xi \tag{12}$$

where Ξ is defined in (10). Moreover, $\bar{Q}$ can be explicitly written as

$$\bar{Q}_{kj} = \begin{cases} \bar{a}_{j+1,i_j}, & \text{if} \quad j = k, \\ -\bar{a}_{j+1,i_j}, & \text{if} \quad j = i_k - 1, \\ 0, & \text{otherwise}. \end{cases} \tag{13}$$

Proof: See the appendix. The proof revises and completes the results in [22], [23], since step 1) of the proof ($\mathcal{L} = \mathcal{L}J\Xi$) is missing there. ■

Remark 3: Lemma 2 states that the information of the Laplacian $\mathcal{L}$ can be transferred into a reduced-order matrix Q through a commutative-like multiplication law (12). For the off-diagonal elements of $\bar{Q}$, $\bar{Q}_{kj} = -\bar{Q}_{jj}$ if and only if $j+1$ is the parent of $k+1$ in $\bar{\mathcal{G}}$. Note that the existence of a solution X to the more general matrix equation $YZ = XY$ was discussed in Lemma 9 of [25]. As compared to [25], the merit of our Lemma 2 is to give the explicit solution Q to (12), and to explicitly reveal the relation between this solution with the weights on the DST by (13). This relation is used in the proof of Theorem 1 to design the adaptation law.

Lemma 3 (Feasibility conditions): Under Assumption 1, let us consider controller (7) with time-varying coupling weights (8) for any DST $\bar{\mathcal{G}}$. Suppose that the origin of the linear time-varying system

$$\dot{d}_L = (\mathbf{I}_{N-1} \otimes (A + BK_0 + BK_1) + Q(t) \otimes BK_2) d_L \tag{14}$$

is globally asymptotically stable, where $Q(t) = \tilde{Q} + \bar{Q}(t)$ with fixed $\tilde{Q}$ defined as in Lemma 2, and

$$\bar{Q}_{kj}(t) = \begin{cases} \bar{a}_{j+1,i_j}(t), & \text{if} \quad j = k, \\ -\bar{a}_{j+1,i_j}(t), & \text{if} \quad j = i_k - 1, \\ 0, & \text{otherwise}. \end{cases} \tag{15}$$

Then, the TVF problem can be solved by controller (7) if and only if

$$\lim_{t\to\infty} (A + BK_0)(h_{i_k}(t) - h_{k+1}(t)) - (\dot{h}_{i_k}(t) - \dot{h}_{k+1}(t)) = 0 \tag{16}$$

holds $\forall k \in \mathcal{I}_{N-1}$.

Proof: Let $\bar{d}_k(t) = d_{i_k}(t) - d_{k+1}(t)$ be the error vector between the parent and the child nodes of the directed edge

$e_{i_k,k+1}$, $k \in \mathcal{I}_{N-1}$, and denote $\bar{d} = \mathrm{col}(\bar{d}_1, \cdots, \bar{d}_N)$. Then, $\bar{d} = (\Xi \otimes \mathbf{I}_n)d$. From Lemma 1, it remains to prove that $\lim_{t\to\infty} \|\bar{d}(t)\| = 0$ under the given conditions.

Based on (1) and (7), the dynamics of $x(t)$ is given by

$$\dot{x} = (\mathbf{I}_N \otimes (A + BK_0 + BK_1))x + (\mathcal{L}(t) \otimes BK_2)d - (\mathbf{I}_N \otimes BK_1)h \tag{17}$$

where $\mathcal{L}(t)$ is the Laplacian matrix of $\mathcal{G}$ at time t due to the adaptive mechanisms. Then, it follows from (17) and the definitions of d and $\bar{d}$ that

$$\begin{aligned}\dot{\bar{d}} =&(\mathbf{I}_{N-1} \otimes (A + BK_0 + BK_1))\bar{d} + (\Xi\mathcal{L}(t) \otimes BK_2)d \\ &+ (\Xi \otimes (A + BK_0))h - (\Xi \otimes \mathbf{I}_n)\dot{h} \\ =&(\mathbf{I}_{N-1} \otimes (A + BK_0 + BK_1) + Q(t) \otimes BK_2)\bar{d} \\ &+ (\Xi \otimes (A + BK_0))h - (\Xi \otimes \mathbf{I}_n)\dot{h}\end{aligned} \tag{18}$$

where Lemma 2 is used to get the second equality. Given that the linear system (14) asymptotically converges to zero, one knows that $\lim_{t\to\infty} \|\bar{d}(t)\| = 0$ if and only if

$$\lim_{t\to\infty} (\Xi \otimes (A + BK_0))h(t) - (\Xi \otimes \mathbf{I}_n)\dot{h}(t) = 0. \tag{19}$$

From the definition of Ξ, condition (16) is equivalent to (19). This completes the proof. ∎

B. Main result

The design process of the TVF controller is summarized in Algorithm 1, and analyzed in the following theorem.

Algorithm 1 TVF Controller Design

1) Find a constant K_0 such that the formation feasibility condition

$$(A + BK_0)(h_{i_k}(t) - h_{k+1}(t)) - (\dot{h}_{i_k}(t) - \dot{h}_{k+1}(t)) = 0 \tag{20}$$

holds $\forall k \in \mathcal{I}_{N-1}$ for any DST $\bar{\mathcal{G}}$. If such K_0 exists, continue; else, the algorithm terminates without solutions;

2) Choose K_1 such that $(A+BK_0+BK_1, B)$ is stabilizable (using, e.g., pole placement). For some η, $\theta \in \mathbb{R}^+$, solve the following LMI:

$$(A + BK_0 + BK_1)P+P(A + BK_0 + BK_1)^T - \eta BB^T + \theta P \le 0 \tag{21}$$

to get a $P > 0$;

3) Set $K_2 = -B^T P^{-1}$, $\Gamma = P^{-1}BB^T P^{-1}$ and choose scalars $\rho_{k+1,i_k} \in \mathbb{R}^+$.

Theorem 1 (Main result for TVF): Under Assumption 1, and feasibility condition (20), the TVF problem in Definition 1 is solved by controller (7) with adaptive coupling weights (8)-(9), along the designs in Algorithm 1.

Proof: The feasibility condition (20) guarantees that (16) holds $\forall k \in \mathcal{I}_{N-1}$. Moreover,

$$\dot{\bar{d}} =(\mathbf{I}_{N-1} \otimes (A + BK_0 + BK_1) + Q(t) \otimes BK_2)\bar{d}, \tag{22}$$

where $Q(t)$ is defined as in Lemma 3 based on $\bar{\mathcal{G}}$. In the following, it will be proved that the designed controller guarantees $\lim_{t\to\infty} \bar{d}(t) = 0$. As such, the proof of the theorem will be complete according to Lemma 3.

Consider the Lyapunov candidate

$$V_1(t) = \frac{1}{2}\bar{d}^T(\mathbf{I}_{N-1} \otimes P^{-1})\bar{d} + \sum_{k=1}^{N-1} \frac{1}{2\rho_{k+1,i_k}}(\bar{a}_{k+1,i_k}(t) - \phi_{k+1,i_k})^2 \tag{23}$$

where P is a solution to (21) and $\phi_{k+1,i_k} \in \mathbb{R}^+$, $k \in \mathcal{I}_{N-1}$ are to be decided later.

By (22) and (9), the derivative of V_1 is

$$\begin{aligned}\dot{V}_1 =&\bar{d}^T(\mathbf{I}_{N-1} \otimes P^{-1}(A + BK_0 + BK_1) \\ &+ Q(t) \otimes P^{-1}BK_2)\bar{d} \\ &+ \sum_{k=1}^{N-1} (\bar{a}_{k+1,i_k} - \phi_{k+1,i_k})(\bar{d}_k - \sum_{j+1\in\bar{\mathcal{N}}_2(k+1)} \bar{d}_j)^T\Gamma\bar{d}_k.\end{aligned} \tag{24}$$

Based on Lemma 2, one has

$$\begin{aligned}&\sum_{k=1}^{N-1} \bar{a}_{k+1,i_k}(\bar{d}_k - \sum_{j+1\in\bar{\mathcal{N}}_2(k+1)} \bar{d}_j)^T\Gamma\bar{d}_k \\ =&\sum_{k=1}^{N-1} (\bar{Q}_{kk}(t)\bar{d}_k + \sum_{j=1,j\neq k}^{N-1} \bar{Q}_{jk}(t)\bar{d}_j)^T\Gamma\bar{d}_k \\ =&\sum_{k=1}^{N-1}\sum_{j=1}^{N-1} \bar{Q}_{jk}(t)\bar{d}_j^T\Gamma\bar{d}_k\end{aligned} \tag{25}$$

Let us define $\Phi \in \mathbb{R}^{(N-1)\times(N-1)}$ as

$$\Phi_{kj} = \begin{cases} \phi_{j+1,i_j}, & \text{if} \quad j = k, \\ -\phi_{j+1,i_j}, & \text{if} \quad j = i_k - 1, \\ 0, & \text{otherwise}. \end{cases} \tag{26}$$

Then, it follows from (24)-(26) that

$$\begin{aligned}\dot{V}_1 =&\bar{d}^T(\mathbf{I}_{N-1} \otimes P^{-1}(A + BK_0 + BK_1) \\ &+ Q(t) \otimes P^{-1}BK_2)\bar{d} \\ &+ \sum_{k=1}^{N-1}\sum_{j=1}^{N-1} (\bar{Q}_{jk}(t) - \Phi_{jk})\bar{d}_j^T\Gamma\bar{d}_k \\ =&\bar{d}^T(\mathbf{I}_{N-1} \otimes P^{-1}(A + BK_0 + BK_1) \\ &+ Q(t) \otimes P^{-1}BK_2)\bar{d} \\ &+ \bar{d}^T((\bar{Q}(t) - \Phi) \otimes \Gamma)\bar{d}.\end{aligned} \tag{27}$$

Define $\tilde{d} = (\mathbf{I}_{N-1} \otimes P^{-1})\bar{d}$, and substitute K_2, Γ designed in Algorithm 1 into (27). Then, one has

$$\begin{aligned}\dot{V}_1 =&\tilde{d}^T(\mathbf{I}_{N-1}\otimes(A+BK_0+BK_1)P \\ &\qquad\qquad - Q(t)\otimes BB^T)\tilde{d} \\ &+\tilde{d}^T((\bar{Q}(t)-\Phi)\otimes BB^T)\tilde{d} \\ =&\tilde{d}^T(\mathbf{I}_{N-1}\otimes(A+BK_0+BK_1)P)\tilde{d} \\ &-\tilde{d}^T((\tilde{Q}+\Phi)\otimes BB^T)\tilde{d} \\ =&\frac{1}{2}\tilde{d}^T\Big(\mathbf{I}_{N-1}\otimes((A+BK_0+BK_1)P \\ &\qquad\qquad + P(A+BK_0+BK_1)^T) \\ &-(\tilde{Q}+\tilde{Q}^T+\Phi+\Phi^T)\otimes BB^T\Big)\tilde{d}. \end{aligned} \tag{28}$$

Now we show that by appropriately selecting ϕ_{k+1,i_k}, $k\in\mathcal{I}_{N-1}$, it can be fulfilled that

$$\Phi+\Phi^T = \begin{pmatrix} 2\phi_{2,i_1} & \phi_{21} & \cdots & \phi_{N-2,1} & \phi_{N-1,1} \\ \phi_{21} & 2\phi_{3,i_2} & \cdots & \cdots & \phi_{N-1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{N-2,1} & \vdots & \cdots & 2\phi_{N-1,i_{N-2}} & \phi_{N-1,N-2} \\ \phi_{N-1,1} & \phi_{N-1,2} & \cdots & \phi_{N-1,N-2} & 2\phi_{N,i_{N-1}} \end{pmatrix} \tag{29}$$

is positive definite. To see this, let us denote $\Psi_1 = \begin{pmatrix} 2\phi_{2,i_1}\end{pmatrix}$ and $\Psi_k = \begin{pmatrix}\Psi_{k-1} & \varphi_k \\ \varphi_k^T & 2\phi_{k+1,i_k}\end{pmatrix}$, where $\varphi_k = (\phi_{k1},\phi_{k2},\cdots,\phi_{k,k-1})^T$, $k=2,\cdots,N-1$. Clearly, $\Psi_1>0$ by choosing $\phi_{2,i_1}>0$. Now suppose $\Psi_{k-1}>0$, $k\geq 2$. Note that $|\phi_{kj}|\leq|\phi_{j+1,i_j}|$, $\forall j\in\mathcal{I}_{k-1}$. Then, one has $\varphi_k^T\Psi_{k-1}^{-1}\varphi_k \leq \lambda_{\mathrm{M}}(\Psi_{k-1}^{-1})\sum_{j=2}^{k}\phi_{j,i_{j-1}}^2$. By choosing $\phi_{k+1,i_k} > \frac{\sum_{j=2}^{k}\phi_{j,i_{j-1}}^2}{2\lambda_{\mathrm{m}}(\Psi_{k-1})}$, one has $\Psi_k>0$ according to the Schur complement [26, Chapter 2.1]. By mathematical induction, $\Phi+\Phi^T=\Psi_{N-1}$ is positive definite.

Moreover, since $\tilde{Q}$ is fixed, one can always choose sufficiently large ϕ_{k+1,i_k}, $k\in\mathcal{I}_{N-1}$, such that $\lambda_{\mathrm{m}}(\tilde{Q}+\tilde{Q}^T+\Phi+\Phi^T)\geq\eta$ where η is defined in (21). Then, it follows from (28) and (21) that

$$\begin{aligned}\dot{V}_1 \leq&\frac{1}{2}\tilde{d}^T\Big(\mathbf{I}_{N-1}\otimes((A+BK_0+BK_1)P \\ &+P(A+BK_0+BK_1)^T-\eta BB^T)\Big)\tilde{d} \\ \leq&-\frac{\theta}{2}\tilde{d}^T(\mathbf{I}_{N-1}\otimes P)\tilde{d} = -\frac{\theta}{2}\bar{d}^T(\mathbf{I}_{N-1}\otimes P^{-1})\bar{d}\leq 0 \end{aligned} \tag{30}$$

which implies that the signals $\bar{d}(t)$ and $\bar{a}_{k+1,i_k}(t)$ in $V_1(t)$ are bounded. Note that $\dot{V}_1(t)=0$ implies that $\bar{d}=0$, thus by LaSalle's invariance principle, one has $\lim_{t\to\infty}\bar{d}(t)=0$. This completes the proof. ∎

Remark 4: The LMI (21) is feasible for some $P>0$ if and only if $(A+BK_0+BK_1,B)$ is stabilizable, which can be realized since (A,B) is stabilizable. Note that different formation vectors $h(\cdot)$ might lead to different solutions P,K_0,K_1.

Remark 5: In state-of-the-art TVF, the number of feasibility conditions is of the order $\frac{N(N-1)}{2}$ (i.e., one condition for each pair of connected agents) [8], [18]. The proposed number of feasibility conditions in (20) is $N-1$, i.e., exploiting the DST structure leads to the minimum number of conditions: note that $N-1$ is the minimum number of edges such that $\mathcal{G}$ is weakly-connected. Consider the example of three agents communicating via a directed ring with dynamics $A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and $B=\begin{pmatrix}0\\1\end{pmatrix}$. Assume that $h_1(t)=(\sin(t)+\cos(t),\cos(t))^T$, $h_2(t)=(\cos(t),0)^T$, $h_3(t)=(\cos(t)-\sin(t),-\cos(t))^T$. In this case, Condition (20) can be satisfied with $K_0=(0,0)$ for any of the DSTs, i.e., two feasibility conditions are sufficient instead of three.

IV. DST-Based Distributed Adaptive TVFT

In this section, we propose a novel generalized DST-based adaptive controller to solve the TVFT problem of Definition 3. We address the general case with multiple leaders, and give a corollary for the special case with a single leader.

Definition 4: The digraph $\mathcal{G}$ is said to have a generalized DST rooting at the leadership, if the followers are either well-informed or uninformed, and for each uninformed follower, there exists at least one well-informed follower that has a directed path to it.

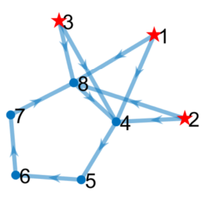
Assumption 2: The digraph $\mathcal{G}$ has at least one generalized DST rooting at the leadership.

Remark 6: TVFT with multiple leaders is also considered in [10], [24], where it is required that the coupling weights from any leader to different well-informed followers are identical and known a priori. Assumption 2 relaxes that requirement.

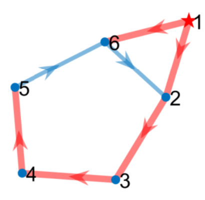
A. Auxiliary system, technical lemma and control law

Let us introduce an auxiliary multi-agent system with an induced communication graph $\mathcal{G}'(\mathcal{V}',\mathcal{E}',\mathcal{A}')$. Define $\mathcal{V}'=\mathcal{I}_{N-M+1}$ where the agent with index 1 is the leader and $\mathcal{E}'=\{e'_{1j},j>1|j+M-1 \text{ is well-informed in } \mathcal{G}\}\bigcup\{e'_{jp},j,p>1|e_{j+M-1,p+M-1}\in\mathcal{E}\}$. The adjacency matrix $\mathcal{A}'=(a'_{jp})$ where $a'_{jp}>0$ if $e'_{pj}\in\mathcal{E}'$, and $a'_{jp}=0$ otherwise.

To clarify Assumption 2 and the induced graph $\mathcal{G}'$, see Fig. 1. It is clear that the multiple leaders are merged as a single joint leader in $\mathcal{G}'$.



(a) $\mathcal{G}$



(b) $\mathcal{G}'$

Fig. 1. A communication graph $\mathcal{G}$ with three leaders (with indexes 1, 2, 3) which satisfies Assumption 2, and the induced graph $\mathcal{G}'$ with a single leader (with index 1).

In the auxiliary multi-agent system, let y_j and v_j be the state and control input of agent j. For the leader, define $y_1=\sum_{l=1}^{M}\beta_l x_l$ and $h'_1\equiv 0$. For the followers, define $y_j=x_{j+M-1}$, $h'_j=h_{j+M-1}$, for $j=2,\cdots,N-M+1$.

Let $d'_j = y_j - h'_j$, $j \in \mathcal{I}_{N-M+1}$. Then, the dynamics of y_j satisfies

$$\begin{aligned} \dot{y}_1 &= Ay_1, \\ \dot{y}_j &= Ay_j + Bv_j \quad j = 2, \cdots, N-M+1, \end{aligned} \tag{31}$$

where $v_j = u_{j+M-1}$, and the initial state values are determined by those of multi-agent system (3).

Lemma 4 (N-S condition for TVFT): Under Assumption 2, the multi-agent system (3) achieves the TVFT with multiple leaders defined by $h^F(t) = \mathrm{col}(h_{M+1}(t), h_{M+2}(t), \cdots, h_N(t))$ and by β_l, $l \in \mathcal{I}_M$, if and only if the auxiliary system (31) achieves the TVFT defined by $h'_F(t) = \mathrm{col}(h'_2(t), h'_3(t), \cdots, h'_{N-M+1}(t))$ with a single leader.

Proof: According to the definitions of y_j and h'_j, it is obvious that $\lim_{t\to\infty}(y_j(t) - h'_j(t) - y_1(t)) = 0$, $j = 2, \cdots, N-M+1$, is equivalent to $\lim_{t\to\infty}(x_i(t) - h_i(t) - \sum_{l=1}^{M} \beta_l x_l(t)) = 0$, $\forall i \in \mathcal{I}_N \setminus \mathcal{I}_M$. ■

Under Assumption 2, there is at least one DST in $\mathcal{G}'$ rooting at the leader. Then, one can choose such a DST $\hat{\mathcal{G}}'(\mathcal{V}', \hat{\mathcal{E}}', \hat{\mathcal{A}}')$. Let j_k denote the unique parent of node $k+1$ in $\hat{\mathcal{G}}'$ for $k \in \mathcal{I}_{N-M}$. Let $\mathcal{N}'_1(j)$ be the set of in-neighbors of j in $\mathcal{G}'$ and $\hat{\mathcal{N}}'_2(j)$ be the set of out-neighbors of j in $\hat{\mathcal{G}}'$.

The generalized DST-based distributed adaptive TVFT controller for follower i of (3), $i \in \mathcal{I}_N \setminus \mathcal{I}_M$, is proposed as:

$$u_i = v_{i-M+1}, \tag{32}$$

$$v_j = K_0 h'_j + K_2 \sum_{p \in \mathcal{N}'_1(j)} \alpha'_{jp}(t)(d'_j - d'_p), \tag{33}$$

$$\alpha'_{jp}(t) = \begin{cases} a'_{jp}, & \text{if} \quad e_{pj} \in \mathcal{E}' \setminus \hat{\mathcal{E}}', \\ \hat{a}'_{k+1,j_k}(t), & \text{if} \quad e_{pj} \in \hat{\mathcal{E}}' \end{cases} \tag{34}$$

$$\begin{aligned} \dot{\hat{a}}'_{k+1,j_k} = \rho_{k+1,j_k} \Big(& (d'_{j_k} - d'_{k+1}) - \\ & \sum_{p \in \hat{\mathcal{N}}'_2(k+1)} (d'_{k+1} - d'_p) \Big)^T \Gamma (d'_{j_k} - d'_{k+1}). \end{aligned} \tag{35}$$

In order to illustrate the idea of the auxiliary multi-agent system, the information flow of the closed-loop system x_i, $i \in \mathcal{I}_N \setminus \mathcal{I}_M$, is sketched in Fig. 2. Instead of directly designing the controllers for multi-agent system (3), an auxiliary multi-agent system is defined as in (31), and some interaction between them is constructed: at stage (33), each leader x_l of (3) broadcast its β_l-scaled state to the single leader of (31), and each follower broadcast its state to the corresponding follower, respectively; at stage (32), each follower of (31) responds to the corresponding follower of (3) with its control input. Then, the original TVFT problem in (3) is successfully transformed into the TVFT with a single leader in (31). It should be pointed out that only the local information, i.e., the states of x_s, $s \in \mathcal{N}_1(i)$, are included in the loop of x_i from Fig. 2.

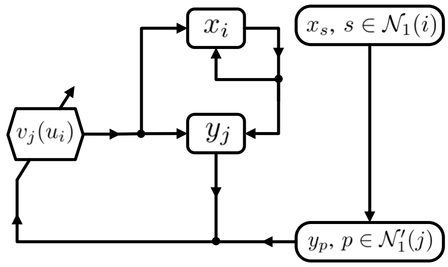


Fig. 2. The information flow of the closed-loop system x_i, $i \in \mathcal{I}_N \setminus \mathcal{I}_M$.

Algorithm 2 TVFT Controller Design

1) Find a constant K_0 such that the formation tracking feasibility condition

$$(A + BK_0)h_i(t) - \dot{h}_i(t) = 0 \tag{36}$$

holds $\forall i \in \mathcal{I}_N \setminus \mathcal{I}_M$. If such K_0 exists, continue; else, the algorithm terminates without solutions;

2) Choose η, $\theta \in \mathbb{R}^+$, and solve the following LMI:

$$AP + PA^T - \eta BB^T + \theta P \leq 0 \tag{37}$$

to get a $P > 0$;

3) Set $K_2 = -B^T P^{-1}$, $\Gamma = P^{-1} BB^T P^{-1}$ and choose scalars $\rho_{k+1,i_k} \in \mathbb{R}^+$.

B. Main result

The design process of the TVFT controller is summarized in Algorithm 2, and analyzed in the following theorem.

Theorem 2 (Main result for TVFT): Under Assumption 2, and feasibility condition (36). The TVFT problem in Definition 3 can be solved by controller (32)-(35) with $\rho_{k+1,j_k} \in \mathbb{R}^+$, and K_2, Γ designed as in Algorithm 2.

Proof: The condition that (36) holds $\forall i \in \mathcal{I}_N \setminus \mathcal{I}_M$ is equivalent to $(A + BK_0)h'_j(t) - \dot{h}'_j(t) = 0$, $\forall j \in \{2, \cdots, N-M+1\}$, which means that the TVFT defined by $h'_F = \mathrm{col}(h'_2, h'_3, \cdots, h'_{N-M+1})$ is feasible for the auxiliary multi-agent system (31). According to Lemma 4, it remains to show that (33)-(35) solves the TVFT for multi-agent system (31) defined by h'_F with a single leader.

Extensions of Lemma 1 and Lemma 2 apply to $\mathcal{G}'$ and $\hat{\mathcal{G}}'$, and are not repeated for compactness. Let $h' = \mathrm{col}(h'_1, h'_F)$ $\hat{d}'_k(t) = d'_{i_k}(t) - d'_{k+1}(t)$ be the error vector between the parent and the child nodes of the directed edge $\hat{e}'_{i_k,k+1}$, $k \in \mathcal{I}_{N-M}$, and denote $\hat{d}' = \mathrm{col}(\hat{d}'_1, \cdots, \hat{d}'_{N-M+1})$. Then $\hat{d}' = (\Xi' \otimes \mathbf{I}_n)d'$. Let $Q'(t) = \tilde{Q}' + \hat{Q}'(t)$ where Ξ' and $\tilde{Q}'$ is defined as in Lemma 1 and 2, respectively, based on $\hat{\mathcal{G}}'$ and

$$\hat{Q}'_{kj}(t) = \begin{cases} \hat{a}'_{j+1,i_j}(t), & \text{if} \quad j = k, \\ -\hat{a}'_{j+1,i_j}(t), & \text{if} \quad j = i_k - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

where the time-varying weights are defined in (33).

With (32), the closed-loop state dynamics of the leader-following multi-agent system (31) can be obtained as

$$\begin{aligned} \dot{y} =& (\mathbf{I}_{N-M+1} \otimes A)y + (\mathcal{L}'(t) \otimes BK_2)d' \\ &+ (\mathbf{I}_{N-M+1} \otimes BK_0)h'. \end{aligned} \tag{39}$$

Then, it follows from (39) and the definitions of d and $\hat{d}$ that

$$\begin{aligned}\dot{\hat{d}}' =&(\mathbf{I}_{N-M} \otimes A)\hat{d}' + (\Xi'\mathcal{L}'(t) \otimes BK_2)d' \\ &+ (\Xi' \otimes (A + BK_0))h' - (\Xi' \otimes \mathbf{I}_n)\dot{h}' \\ =&(\mathbf{I}_{N-M} \otimes A + Q'(t) \otimes BK_2)\hat{d}' \\ &+ (\Xi' \otimes (A + BK_0))h' - (\Xi \otimes \mathbf{I}_n)\dot{h}' \qquad (40)\end{aligned}$$

where $\mathcal{L}'(t)$ is the time-varying Laplacian matrix of $\mathcal{G}'(t)$. Under the feasibility condition (36), one has

$$\dot{\hat{d}}' = (\mathbf{I}_{N-M} \otimes A + Q'(t) \otimes BK_2)\hat{d}'. \qquad (41)$$

Consider the Lyapunov candidate as

$$\begin{aligned}V_2(t) =& \frac{1}{2}\hat{d}'^T(\mathbf{I}_{N-M} \otimes P^{-1})\hat{d}' \\ &+ \sum_{k=1}^{N-M} \frac{1}{2\rho_{k+1,i_k}}(\hat{a}'_{k+1,i_k}(t) - \delta_{k+1,i_k})^2\end{aligned} \qquad (42)$$

where P is a solution of (37) and $\delta_{k+1,i_k} \in \mathbb{R}^+$, $k \in \mathcal{I}_{N-M}$. Following similar steps as in the proof of Theorem 1, one has $\lim_{t\to\infty} \hat{d}'(t) = 0$. In this case, the TVFT with a single leader is realized in (31), meanwhile, the TVFT with multiple leaders is realized in (3). This completes the proof. ■

In the special case when $M = 1$, the auxiliary multi-agent system (31) coincides with the original one, thus, it can be removed. The DST-based adaptive TVFT controller can be directly designed for follower i, $i = 2, \cdots, N$, as:

$$u_i = K_0 h_i + K_2 \sum_{j\in\mathcal{N}_1(i)} \alpha_{ij}(t)(d_i - d_j) \qquad (43)$$

$$\alpha_{ij}(t) = \begin{cases} a_{ij}, & \text{if} \quad e_{ji} \in \mathcal{E} \setminus \hat{\mathcal{E}}, \\ \hat{a}_{k+1,i_k}(t), & \text{if} \quad e_{ji} \in \hat{\mathcal{E}} \end{cases} \qquad (44)$$

and adaption law $\dot{\hat{a}}_{k+1,i_k}$ as in (9). Here, $d_1(t) = x_1(t)$. Immediately, we have the following corollary.

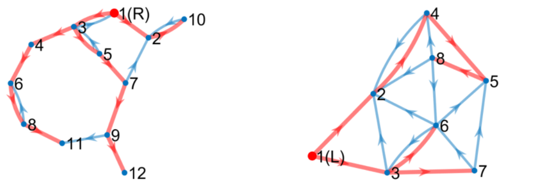
Corollary 1 (Single leader case): Suppose there exists a DST $\hat{\mathcal{G}}$ rooting at the leader. Under feasibility condition (36), the TVFT with a single leader is solved by (43)-(44) and $\dot{\hat{a}}_{k+1,i_k}$ as in (9), along the designs in Algorithm 2.

Remark 7: With a single leader, Assumption 2 degenerates to the standard assumption of existence of a DST rooting at the leader ([7], [9], etc). The benefit of Theorem 2 is thus to provide a natural unifying solution for the DST adaptive method in the presence of one or more leaders.

Remark 8: The TVFT problem with a single leader can be seen as a special type of the TVF problem where $h_1(\cdot) \equiv 0$ for the leader. In this sense, the feasibility condition (36) is a direct consequence of condition (20). By comparing (7) with (43), it can be seen that $K_1 = -K_0$ in (43). This means that there is no separate term for the average formation signal, since the formation reference is known a prior as the of leader's trajectory.

V. Numerical examples

In this section, three numerical examples for TVF, TVFT with three leaders and with a single leader are implemented to validate the theoretical results. In all three examples, the initial positions of the agents (followers) are chosen from a Gaussian distribution with standard deviation 5, and the initial coupling weights of the edges are chosen from a uniform distribution in the interval $(0, 0.1)$.



(a) $\mathcal{G}_1$ for Example 1 (b) $\mathcal{G}_2$ for Example 3

Fig. 3. Communication graphs. The DSTs are highlighted with red color, and (R), (L) are the root and leader nodes.

Example 1 (TVF): Consider a second-order system modeled by (1) with $N = 12$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The agents interact on the digraph $\mathcal{G}_1$ in Fig. 3. The required TVF is a pair of nested hexagons with $h_i(t) = (8\sin(t + \frac{(i-1)\pi}{3}), 8\cos(t + \frac{(i-1)\pi}{3}))^T$ for $i \in \mathcal{I}_6$, and $h_i(t) = (4\sin(t + \frac{(i-1)\pi}{3}), 4\cos(t + \frac{(i-1)\pi}{3}))^T$ for $i \in \mathcal{I}_{12} \setminus \mathcal{I}_6$.

Let $K_0 = (0, -2)$. It can be verified via condition (20) that the desired formation is feasible for the selected DST. Let $K_1 = (0, 0.1)$, $\eta = 2$, $\theta = 1$, and solve LMI (21) to give a solution $P = \begin{pmatrix} 0.2934 & -0.3074 \\ -0.3074 & 0.6175 \end{pmatrix}$. Following Algorithm 1, one has $K_2 = (-3.5470, -3.3852)$, and $\Gamma = \begin{pmatrix} 12.5813 & 12.0074 \\ 12.0074 & 11.4598 \end{pmatrix}$. Let $\rho_{k+1,i_k} = 0.1$.

The trajectories of the agents are in Fig. 4, showing how the nested hexagons are formed and rotate. Let $e_i(t) = d_i(t) - d_{\text{ave}}$ (see Remark 2), $i \in \mathcal{I}_N$. The global formation error $E(t) = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \|e_i(t)\|^2}$ converges to zero, as shown in Fig. 5. Fig. 5 also shows that the weights α_{ij} are time-varying on the DST (solid lines) and kept constant otherwise (dashed lines). For comparison, Fig. 6 shows that if all weights are kept constant $(\alpha_{ij} = \alpha_{ij}(0))$, no TVF may be achieved (the global formation error diverges).

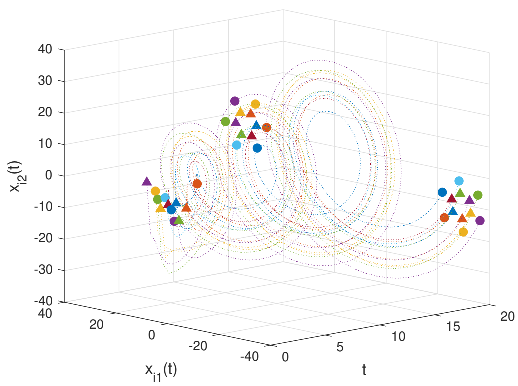


Fig. 4. Example 1 (TVF): Trajectories of the agents $x_i(t)$, where the circles and triangles are used to mark the agents $i \in \mathcal{I}_6$ and the agents $i \in \mathcal{I}_{12} \setminus \mathcal{I}_6$, respectively, at $t = 0$, 10 and 20.

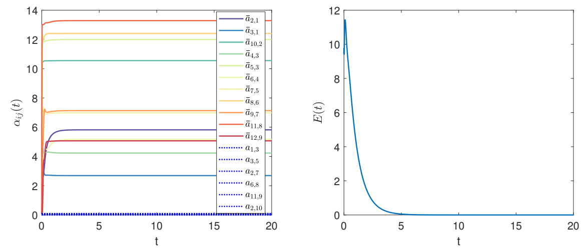


Fig. 5. Example 1 (TVF): Coupling weights $\alpha_{ij}(t)$ and global formation error $E(t)$ with proposed adaptive method.

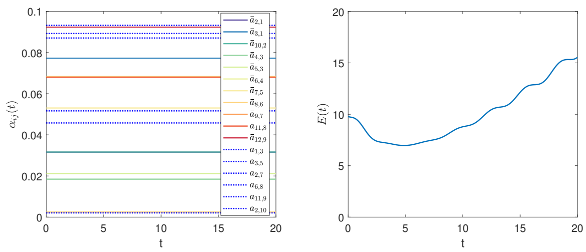


Fig. 6. Example 1 (TVF): Coupling weights α_{ij} and global formation error $E(t)$ with nonadaptive adaptive method (same initial α_{ij} as in Fig. 5).

Example 2 (TVFT with Three Leaders): Consider a third-order multi-agent system modeled by (3) with $N = 8$, $M = 3$, and

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -10 & -3 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The communication graph is the digraph $\mathcal{G}$ in Fig. 1. The followers are required to form a time-varying pentagram described by

$$h_i(t) = \begin{pmatrix} 3\sin(t + \frac{2(i-4)\pi}{5}) \\ -3\cos(t + \frac{2(i-4)\pi}{5}) \\ 6\cos(t + \frac{2(i-4)\pi}{5}) \end{pmatrix}, \quad i = 4, 5 \cdots, 8,$$

while tracking the average of the leaders, i.e., $\beta_1 = \beta_2 = \beta_3 = 1/3$.

Let $K_0 = (0, 4, 0)$. It can be verified that the defined $h_i(\cdot)$ is feasible. Let $\eta = 2$, $\theta = 1$, and $\rho_{k+1,j_k} = 0.1$. Following Algorithm 2, one has $K_2 = (-2.3066, -6.8257, -2.4970)$, and $\Gamma = \begin{pmatrix} 5.3206 & 15.7444 & 5.7596 \\ 15.7444 & 46.5895 & 17.0434 \\ 5.7596 & 17.0434 & 6.2349 \end{pmatrix}$.

The initial value of the leaders are chosen as $x_1(0) = (5, 5, 10)^T$, $x_2(0) = (-10, -5, -5)^T$, $x_3(0) = (5, -10, 5)^T$. Several snapshots of the agents are in Fig. 7, showing that the pentagram emerges and rotates around the average of the three leaders. Similarly, we define the global formation tracking error $E(t) = \sqrt{\frac{1}{N-3}\sum_{i=4}^{N}\|d_i(t) - \sum_{l=1}^{3}\beta_l x_l(t)\|^2}$. The trajectories of α'_{ij} in $\mathcal{G}'$ (see Fig. 1) and $E(t)$ are provided in Fig. 8. Once more, a constant coupling strategy fails to accomplish the TVFT task, as shown in Fig. 9.

Example 3 (TVFT with a Single Leader): Consider a network of second-order agents with $N = 8$, $M = 1$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and digraph $\mathcal{G}_2$ in Fig. 3.

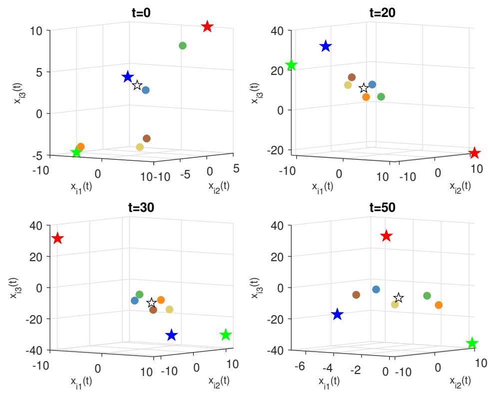


Fig. 7. Example 2 (TVFT with Three Leaders): Snapshots at $t = 0$, 20, 30, and 50. Three filled pentagrams, five circles and an unfilled pentagram are used to mark leaders, followers, and the average of the leaders, respectively.

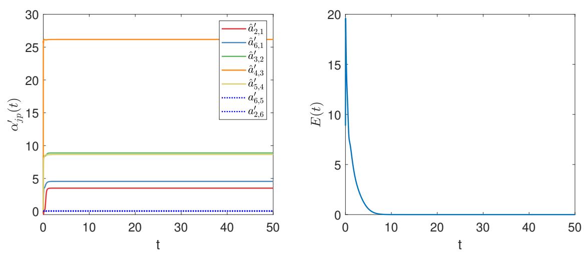


Fig. 8. Example 2 (TVFT with Three Leaders): Coupling weights $\alpha'_{jp}(t)$ in $\mathcal{G}'$, and global formation tracking error $E(t)$ with proposed adaptive method.

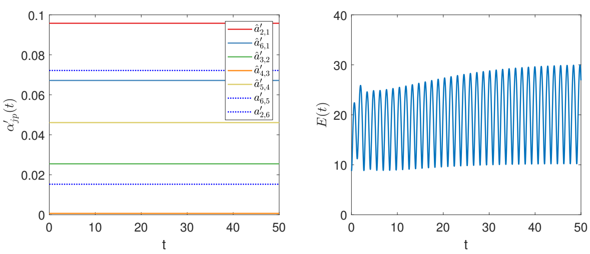


Fig. 9. Example 2 (TVFT with Three Leaders): Coupling weights α'_{jp} in $\mathcal{G}'$, and global formation tracking error $E(t)$ with nonadaptive control (same initial α'_{jp} as in Fig. 8).

The desired formation is an equilateral triangle-like formation around the leader, which is specified by $h_i(t) = (4\sin(t + \frac{2(i-2)\pi}{3} + \pi), 4\cos(t + \frac{2(i-2)\pi}{3} + \pi))^T$ for $i \in \{2, 3, 4\}$, and $h_i(t) = (2\sin(t + \frac{(i-5)\pi}{2}), 2\cos(t + \frac{(i-5)\pi}{2}))^T$ for $i \in \{5, 6, 7, 8\}$.

Let $K_0 = (-1, 0)$. It can be verified via condition (36) that the desired formation is feasible. Let $\eta = 2$, $\theta = 1$, and solve the LMI (37) to give a solution $P = \begin{pmatrix} 0.6513 & -0.6513 \\ -0.6513 & 0.8256 \end{pmatrix}$. Following Algorithm 2, one has $K_2 = (-5.7356, -5.7356)$, and $\Gamma = \begin{pmatrix} 32.8969 & 32.8969 \\ 32.8969 & 32.8969 \end{pmatrix}$. We choose $\rho_{k+1,i_k} = 0.1$.

The initial value of the leader is chosen as $x_1(0) = (0.5, 0.5)^T$. The trajectories of the agents are in Fig. 10, showing how the triangle emerges and rotates around the leader. If we define the global formation tracking error as $E(t) = \sqrt{\frac{1}{N-1}\sum_{i=2}^{N}\|d_i(t) - x_1(t)\|^2}$, we can see from Fig. 11 that it converges to zero (see also the time-varying weights α_{ij} on the DST). Fig. 12 (left) shows that also in this case the TVFT may not be achieved with nonadaptive control.

As is indicated in Section I, the DST-based adaptive method is not the only possible solution to remove the knowledge of the Laplacian eigenvalues: a node-based method have been proposed for consensus [27] and group TVFT [24]. It should be noted that even though the node-based method does not explicitly rely on a DST, the existence of a DST is required as a basic assumption. Nevertheless, in order to verify this assumption, the designer has to find at least one DST (some classic algorithms for finding a DST are well known in network science [28]). From this perspective, the knowledge of a DST in the DST-based method is perfectly reasonable. Besides, let us include a comparison with the adaptive method used in [24], [27], which can be written as:

$$\begin{aligned} u_i &= K_0 h_i + K_2(c_i(t) + \xi_i P^{-1}\xi_i)\xi_i \\ \dot{c}_i &= \rho_i \xi_i^T \Gamma \xi_i \qquad \xi_i = \sum_{j\in\mathcal{N}_1(i)} a_{ij}(d_i - d_j). \end{aligned} \tag{45}$$

Note that (45) makes all coupling weights in the network adaptive. We select the same initial states of the agents, same parameters P, Γ, K_0, K_2 and $\rho_i = 0.1$. Since it is not straightforward to select the same initial coupling weights due to the different nature of (45), we select three different initial conditions $c_i(0) = 10, 30, 100$, respectively. The global formation tracking error is shown in Fig. 12 (right). As compared to Fig. 11 (right), it is interesting to note that high gains are required in (45) to attain fast convergence and reduced oscillations of the global formation error.

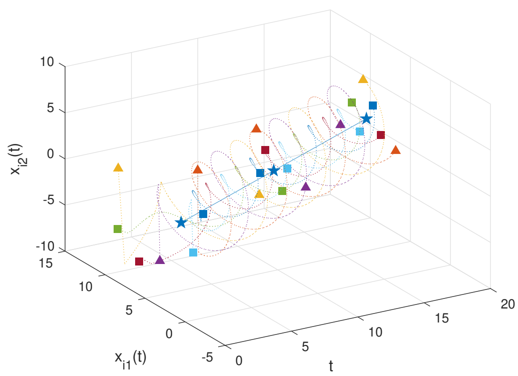


Fig. 10. Example 3 (TVFT with a Single Leader): Trajectories of the agents $x_i(t)$, where three triangles, four squares and a pentagram are used to mark the agents $i \in \{2,3,4\}$, $i \in \{5,6,7,8\}$, and the leader $i = 1$, respectively, at $t = 0$, 10 and 20.

Remark 9: Sinusoidal $h(\cdot)$ are the most widely used class of functions to solve the feasibility conditions (see the simulations in [7]–[12], [18]–[20], [24]). One important reason for this is that sinusoidal functions allow to solve the feasibility conditions analytically instead of numerically: in addition, this class of function can be used to describe a wide variety of

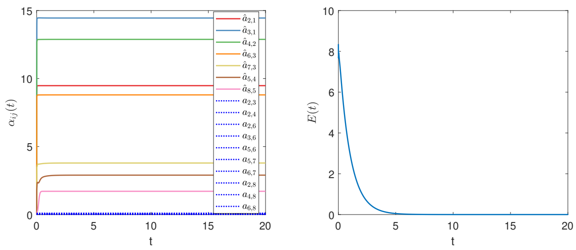


Fig. 11. Example 3 (TVFT with a Single Leader): Coupling weights $\alpha_{ij}(t)$ and global formation tracking error $E(t)$ with proposed adaptive method.

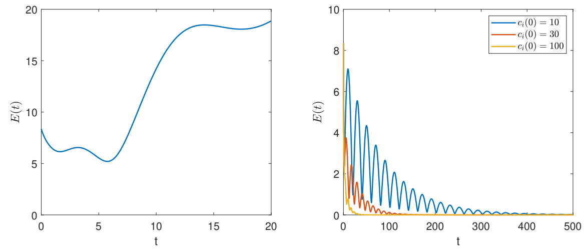


Fig. 12. Example 3 (TVFT with a Single Leader): Global formation tracking error $E(t)$ with nonadaptive control (left) and with adaptive controller (45) (right).

periodic time-varying formations, e.g., circle [3], square [19], triangle [20], and so on.

VI. Conclusions

A directed spanning tree (DST) adaptive method has been developed for time-varying formation and formation tracking of linear multi-agent systems. The proposed method provides a natural generalization of the DST based adaptive method. Necessary and sufficient conditions for solving TVF/TVFT with DST adaptive method have been derived. Future topics may include generalizing the proposed method in the sense of cluster formation, collision avoidance, partial state information, nonlinear agents and nonzero inputs of the leaders. Some solutions proposed in literature for these settings require undirected or strongly-connected digraphs [19], [20]. For more general digraphs, the problem seems open and not trivial.

Appendix
Proof of Lemma 2

Inspired by [21] and [22], an auxiliary matrix J is introduced to analyze Lemma 2. Define $J \in \mathbb{R}^{N\times(N-1)}$ as

$$J_{ik} = \begin{cases} 0, & \text{if} \quad i \in \bar{\mathcal{V}}_{k+1}, \\ 1, & \text{otherwise} \end{cases}$$

where $\bar{\mathcal{V}}_{k+1}$ represents the vertex set of the subtree of $\bar{\mathcal{G}}$ rooting at node $k+1$. The proof will proceed along three steps:

1) Proving that $\mathcal{L} = \mathcal{L}J\Xi$;
2) Proving that $Q = \Xi\mathcal{L}J$;
3) Proving (12) and (13), i.e., the statements of the lemma.

Step 1) Let us denote $X = J\Xi$. Then, $X_{ij} = \sum_{k=1}^{N-1} J_{ik}\Xi_{kj}$, $i, j \in \mathcal{I}_N$. We classify the discussions according to the value of j in order to clarify the matrix X.

Case 1: $j = 1$. Then, $X_{i1} = \sum_{k=1,i_k=1}^{N-1} J_{ik}$.

Since $J_{1k} = 1$, $\forall k$, then $X_{11} = \bar{\mathcal{D}}_2(1)$, which is the out-degree of the root in $\bar{\mathcal{G}}$; When $i > 1$, there exists a unique $\bar{k} \in \mathcal{I}_{N-1}$ satisfying $i_{\bar{k}} = 1$, such that $i \in \bar{\mathcal{V}}_{\bar{k}+1}$, implying that $J_{i\bar{k}} = 0$. Thus, $X_{i1} = \bar{\mathcal{D}}_2(1) - 1$.

To sum up, $X_{i1} = \begin{cases} \bar{\mathcal{D}}_2(1), & i = 1, \\ \bar{\mathcal{D}}_2(1) - 1, & i > 1. \end{cases}$

Case 2: j is a stem. Then, $X_{ij} = \sum_{k=1,i_k=j}^{N-1} J_{ik} - J_{i,j-1}$.

i. When $i \notin \bar{\mathcal{V}}_j$, $X_{ij} = \sum_{k=1,i_k=j}^{N-1} J_{ik} - 1$. Then, $\forall k$ satisfying $i_k = j$, $i \notin \bar{\mathcal{V}}_{k+1}$. Thus, $X_{ij} = \bar{\mathcal{D}}_2(j) - 1$.

ii. When $i \in \bar{\mathcal{V}}_j$, $X_{ij} = \sum_{k=1,i_k=j}^{N-1} J_{ik}$. If $i = j$, then $\forall k$ satisfying $i_k = j$, $J_{ik} = 1$. Thus $X_{jj} = \bar{\mathcal{D}}_2(j)$. If $i \neq j$, there exists a unique $\bar{k}$ satisfying $i_{\bar{k}} = j$, such that $i \in \bar{\mathcal{V}}_{\bar{k}+1}$, implying that $J_{i\bar{k}} = 0$. Then, $X_{ij} = \bar{\mathcal{D}}_2(j) - 1$.

To sum up, $X_{ij} = \begin{cases} \bar{\mathcal{D}}_2(j), & i = j, \\ \bar{\mathcal{D}}_2(j) - 1, & i \neq j \end{cases}$ when j is a stem.

Case 3: j is a leaf. Then, $X_{ij} = -J_{i,j-1}$.

In this case, $\bar{\mathcal{V}}_j = \{j\}$, meaning that $J_{i,j-1} = 0$ if and only if $i = j$. Then $X_{ij} = \begin{cases} 0, & i = j, \\ -1, & i \neq j. \end{cases}$

Summarizing all three cases, the matrix X can be written in a unified way as $X_{ij} = \begin{cases} \bar{\mathcal{D}}_2(j), & i = j, \\ \bar{\mathcal{D}}_2(j) - 1, & i \neq j. \end{cases}$ Then,

$$\begin{aligned}(\mathcal{L}X)_{ij} &= \sum_{k=1}^{N} \mathcal{L}_{ik} X_{kj} \\ &= \sum_{k \neq j} \mathcal{L}_{ik}(\bar{\mathcal{D}}_2(j) - 1) + \mathcal{L}_{ij}\bar{\mathcal{D}}_2(j) \\ &= (\bar{\mathcal{D}}_2(j) - 1)\sum_{k=1}^{N} \mathcal{L}_{ik} + \mathcal{L}_{ij} = \mathcal{L}_{ij}.\end{aligned}$$

So, $\mathcal{L} = \mathcal{L}J\Xi$ is proved.

Step 2) Let us denote $Y = \Xi\mathcal{L}J$. Then,

$$\begin{aligned}Y_{kj} &= \sum_{i=1}^{N} (\Xi\mathcal{L})_{ki} J_{ij} = \sum_{i=1}^{N} (\sum_{s=1}^{N} \Xi_{ks}\mathcal{L}_{si}) J_{ij} \\ &= \sum_{s=1}^{N} \Xi_{ks} \sum_{i=1}^{N} \mathcal{L}_{si} J_{ij} = \sum_{i=1}^{N} \mathcal{L}_{i_k,i} J_{ij} - \sum_{i=1}^{N} \mathcal{L}_{k+1,i} J_{ij} \\ &= \sum_{i=1, i \notin \bar{\mathcal{V}}_{k+1}}^{N} (\mathcal{L}_{i_k,i} - \mathcal{L}_{k+1,i})\end{aligned}$$

where the definitions of Ξ and J are used to get the last two equalities, respectively. Since $\mathcal{L}$ has zero row sums, we have

$$\begin{aligned}Y_{kj} &= \sum_{c \in \bar{\mathcal{V}}_{k+1}} (\mathcal{L}_{k+1,c} - \mathcal{L}_{i_k,c}) \\ &= \sum_{c \in \bar{\mathcal{V}}_{j+1}} (\tilde{\mathcal{L}}_{k+1,c} - \tilde{\mathcal{L}}_{i_k,c}) + \sum_{c \in \bar{\mathcal{V}}_{j+1}} (\bar{\mathcal{L}}_{k+1,c} - \bar{\mathcal{L}}_{i_k,c}) \\ &= \tilde{Q}_{kj} + \bar{Q}_{kj} = Q_{kj}.\end{aligned}$$

Then, $Q = \Xi\mathcal{L}J$ is proved.

Step 3) Let both sides $Q = \Xi\mathcal{L}J$ multiply Ξ, one has $Q\Xi = \Xi\mathcal{L}J\Xi = \Xi\mathcal{L}$, then (12) holds. To prove the explicit form of $\bar{Q}$ in (13), one can distinguish three cases based on the relationships between the edge $\bar{e}_{i_k,k+1}$ and the subtree $\bar{\mathcal{V}}_{j+1}$:

Case 1: $k + 1 \notin \bar{\mathcal{V}}_{j+1}$. Then, it is obvious that $\bar{Q}_{kj} = 0$.

Case 2: $k + 1 \in \bar{\mathcal{V}}_{j+1}$ and $i_k \notin \bar{\mathcal{V}}_{j+1}$. In this case, the only possible value of k is $k = j$. Then,

$$\begin{aligned}\bar{Q}_{kj} &= \sum_{c \in \bar{\mathcal{V}}_{j+1}} (\bar{\mathcal{L}}_{k+1,c} - \bar{\mathcal{L}}_{i_k,c}) \\ &= \bar{\mathcal{L}}_{k+1,k+1} = \bar{\mathcal{L}}_{j+1,j+1} = \bar{a}_{j+1,i_j}.\end{aligned}$$

Case 3: $i_k \in \bar{\mathcal{V}}_{j+1}$. Then,

i. When $i_k = j + 1$,

$$\begin{aligned}\bar{Q}_{kj} &= \sum_{c \in \bar{\mathcal{V}}_{j+1}} (\bar{\mathcal{L}}_{k+1,c} - \bar{\mathcal{L}}_{i_k,c}) \\ &= \bar{\mathcal{L}}_{k+1,i_k} - \bar{\mathcal{L}}_{i_k,i_k} + \bar{\mathcal{L}}_{k+1,k+1} - \bar{\mathcal{L}}_{i_k,k+1} \\ &= -\bar{\mathcal{L}}_{i_k,i_k} = -\bar{a}_{j+1,i_j}.\end{aligned}$$

ii. When $i_k > j + 1$,

$$\begin{aligned}\bar{Q}_{kj} &= \sum_{c \in \bar{\mathcal{V}}_{j+1}} (\bar{\mathcal{L}}_{k+1,c} - \bar{\mathcal{L}}_{i_k,c}) \\ &= \bar{\mathcal{L}}_{k+1,i_{i_k-1}} - \bar{\mathcal{L}}_{i_k,i_{i_k-1}} + \bar{\mathcal{L}}_{k+1,i_k} - \bar{\mathcal{L}}_{i_k,i_k} \\ &\quad + \bar{\mathcal{L}}_{k+1,k+1} - \bar{\mathcal{L}}_{i_k,k+1} \\ &= -\bar{\mathcal{L}}_{i_k,i_{i_k-1}} + \bar{\mathcal{L}}_{k+1,i_k} - \bar{\mathcal{L}}_{i_k,i_k} + \bar{\mathcal{L}}_{k+1,k+1} = 0.\end{aligned}$$

Summarizing all three cases, the matrix $\bar{Q}$ can also be given in a unified way as $\bar{Q}_{kj} = \begin{cases} \bar{a}_{j+1,i_j}, & \text{if } j = k, \\ -\bar{a}_{j+1,i_j}, & \text{if } j = i_k - 1, \\ 0, & \text{otherwise.} \end{cases}$

Then (13) is proved, which completes the proof.

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