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# Controller Design for Stabilizing Bilinear Systems With State-Based Switching

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**Abstract**

Some nonlinear systems can be approximated by switching bilinear systems. In this paper, we proposed a method to design state-based stabilizing controller for switching bilinear systems. Based on the similarity between switching bilinear systems and switching linear systems, corresponding switching linear systems are obtained for switching bilinear systems by applying state-based feedback control laws. Instead, we consider asymptotically stabilizing the corresponding switching linear system through solving a number of relaxed LMI conditions. Stabilizing controllers for switching bilinear systems can be derived based on the results of the corresponding switching linear systems. The stability of the controller is proved step by step through the decreasing of the multiple Lyapunov functions along the state trajectory. The effectiveness of the method is demonstrated by both a theoretical example and an example of urban traffic network with traffic signals.

*Keywords:*
Bilinear System with State-Based Switching, Switching Bilinear System Control, Lyapunov Stability, LMIs.

## 1. Introduction

In practice, many systems have nonlinear features and can be described by nonlinear models. For some of the systems, such as sociology systems, biology systems, water systems, power systems, and traffic systems, the nonlinearity can be described by multiple simple nonlinear models, which are combinations of linear terms and comparatively simpler nonlinear terms. The nonlinear terms usually are the coupling terms of system states and the system inputs, which means how the system inputs influence the system states not only depends on the value of the system inputs but also relays on the current system states. This term is considered as the bilinear term, and it will turn into a linear term once the system state becomes static. Some research works have been done to identify the simple models that build up the nonlinear systems [33, 34].

Bilinear systems are a kind of special nonlinear systems that have been investigated a lot since the 1960s [15, 20, 28]. They are very simple and close to linear systems, which makes it possible to extend some of the theory results for linear systems to bilinear systems. A bilinear system involves actually the addition of a linear term and a bilinear term. Due to the existence of the bilinear term, the structure of the system can be changed compared with the linear systems. Because of the variable structure of the bilinear systems, some uncontrollable linear systems may become controllable by simply adding the bilinear term. It has been proved that bilinear system had a better performance than linear system in optimal control, since the designed controller has more freedom to variate the structure of the system to achieve better performance due to the existence of the bilinear term [29].

For some nonlinear systems, they can be approximated by dividing them into multiple state-based linear or bilinear subsystems, and the nonlinearity of the systems can be approximated by switchings among these linear or bilinear subsystems [16]. In each state region, a subsystem, i.e. a linear or bilinear system, is activated, and the linear or bilinear subsystem switches to other subsystems when the corresponding state region is switched. This results in

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a state-based switching linear or bilinear systems [8]. Sociology systems, biology systems, water systems, power systems, and traffic systems [28] are all systems that naturally have the bilinear term, i.e. the states multiplying the control inputs, and can be approximated by state-based bilinear switching systems. For instance, take traffic system for example, the nonlinear traffic model can be divided into multiple simple traffic models for each specified traffic mode (or traffic state region), and every traffic dynamic model of its mode is a combined model with linear terms and bilinear terms. In other words, the dynamic of the traffic model is divided into several working zones, in each zone a bilinear model is established. The system state can switch among the different traffic modes, and thus to form the complex nonlinearity of the whole system. For more information, one can refer to [24]. The reason for the existence of the bilinear term in the traffic model is that the traffic signal settings (control inputs) and the traffic densities (the system states) are coupling with each other in the system dynamics, i.e. the impact of the traffic signals on the traffic system relays on the current traffic densities. This phenomenon can be easily found in other nonlinear systems, such as sociology systems, biology systems, water systems, and power systems. Consequently, in this paper, we designed a method to asymptotically stabilizing this kind of switching bilinear systems.

There were many research works investigating the problem in stability and design of switching systems. The existence of switching rules, and the design of switching sequences or switching laws, were studied to ensure stability of the switching systems [1, 5, 9, 10, 13, 14, 21, 30–32, 37–39]. Two families of switching laws were extensively studied, i.e. time-dependent and state-dependent switching laws. For both switching laws, the most celebrated design approaches are based on switching Lyapunov functions. If a switching quadratic Lyapunov function is shown to be strictly decreasing during the active intervals and at the switching instants, the switched linear systems are asymptotic stable for both switching laws, e.g. Yuan et al. [36] designed an interesting switching quadratic Lyapunov function with time-varying positive definite matrices to guarantee asymptotic stability for time-dependent switched system; Johansson et al. [18] introduced a piecewise quadratic Lyapunov function, incorporating S-procedure, to show asymptotic stability for state-dependent switched systems. To a more relaxed situation, multiple Lyapunov functions were designed to make sure the switching Lyapunov functions decreased during the whole dynamic process of the system, but not necessarily decreased for the switching of each time. Stabilizing controllers were designed for nonlinear systems by modeling them as piecewise affine (PWA) linear systems through splitting the state space into polyhedron regions [3, 11, 35, 36, 40]. Lyapunov-based controller design methods for switching systems were proposed by Ma et al. and Huo et al. in [17, 25]. Mhaskar et al. proposed Lyapunov-based methods to stabilize nonlinear switching systems by guaranteeing the reduction of the Lyapunov functions along the sequence of system switchings, and this method is carried out in several control synthesis including output feedback control, predictive control and robust predictive control [12, 26, 27]. For switching bilinear systems, studies had been done to design state feedback laws to guarantee the asymptotic stability of the systems [2, 4, 22]. In general, these works utilized the similarity of linear systems and bilinear systems, and investigated the stability of the bilinear systems by turning them into linear systems through adopting state feedback control laws. The control design was expressed in the form of a Linear Matrix Inequality (LMI) feasibility problem, which can be solved by LMI solvers efficiently. However, not much attention has been paid to study the state feedback control law design to asymptotically stabilize the state-based switching bilinear systems by multiple different Lyapunov functions.

In this paper, a method is proposed for designing controllers to stabilize state-based switching bilinear systems. By considering the similarity between linear systems and bilinear systems, feedback control laws are designed for the switching bilinear systems to turn the close-loop systems into corresponding state-based switching linear systems. Instead of studying the method to stabilize the switching bilinear system directly, we propose a theorem to asymptotically stabilize the close-loop switching linear system based on multiple Lyapunov functions. In addition, we further relax the conditions by allowing the Lyapunov functions to jump on the boundary of the neighboring state regions. In other words, the value of the Lyapunov functions does not necessarily continuously decrease along the state trajectory. A stabilizing controller for switching bilinear systems can be obtained by checking the feasibilities of LMI conditions. In order to test the effectiveness of the results, we apply the method to a second-order switching bilinear system, and to a two-link urban traffic system with traffic signals.

The paper is organized as follows: Section 2 presents the problem statement; Section 3 gives some preliminary knowledge on state-based controller design for switching bilinear systems; the controller design and proof are presented in Section 4; examples are given in Section 5; and Section 6 concludes the paper.

## 2. Problem Statement

Consider a Switching Bilinear System (SBLS)

$$\dot{x} = A_i x + \sum_{j=1}^{m_i} (G_{i,j} x + b_{i,j}) u_{i,j}, \text{ if } x \in \Omega_i, i \in \Lambda, \tag{1}$$

where $A_i$ and $G_{i,j}$ are $n \times n$ matrices, $b_{i,j}$ is an $n \times 1$ vector, $\Lambda$ is the set of the state space partitions, $\Omega_i$ is the corresponding polyhedral region with $i \in \Lambda$ in the set of the state space partition $\Omega \subset \mathbb{R}^n$ $(\cup_{i\in\Lambda}\Omega_i = \Omega, \Omega_i \neq \emptyset, \forall i \in \Lambda)$, $j \in M_i = \{1, \ldots, m_i\}$, and $U_i = [u_{i,1}\ u_{i,2} \cdots u_{i,m_i}] \in \mathbb{R}^{m_i}$ is an $m_i$-dimensional control input with $u_{i,1}$, $u_{i,2}$ ... $u_{i,m_i}$ are scalars.

Eq. (1) can be written as

$$\dot{x} = A_i x + \sum_{j=1}^{m_i} b_{i,j} (c_{i,j}^{\mathrm{T}} x + 1) u_{i,j}, \text{ if } x \in \Omega_i, i \in \Lambda, \tag{2}$$

if matrix $G_{i,j}$ can be written as the inner product of two vectors. Particularly, when the rank of $G_{i,j}$ is

$$\mathrm{rank}(G_{i,j}) = 1, \tag{3}$$

then matrix $G_{i,j}$ can be expressed as

$$G_{i,j} = b_{i,j} c_{i,j}^{\mathrm{T}}, \text{ if } i \in \Lambda, j \in M_i. \tag{4}$$

This means that each of the control inputs $u_{i,j}(j \in M_i, i \in \Lambda)$ only corresponds to one system state, i.e. element $x_i$ $(i \in \{1, \ldots, n\})$, in other words, the bilinear terms in (1) only exist for the local control inputs and states having direct correlations with each other (indirect correlations are considered by the dynamic model). In practice, it is easy to find systems with this property, like flow dynamic systems, such as water networks, traffic networks, power networks, etc.

In general, smooth nonlinear functions can be approximated with multiple simpler functions, such as piecewise affine (PWA) linear functions and bilinear functions. The switchings among multiple simple linear or nonlinear functions can construct a complex nonlinear feature. In other words, a complex nonlinear function is actually the addition of a number of simple linear and nonlinear functions. In [16], switched nonlinear systems with smooth nonlinear subsystem functions are approximated with piecewise affine (PWA) functions under a novel framework, and a controllable switching signal that orchestrates the switching between PWA subsystems. In practice, a nonlinear system has multiple working zones, and in each working zone the system dynamic can be described by a simple model. If the models are linear affine, then the nonlinear system can be approximated by switched PWA linear functions; If the models are bilinear, then the nonlinear system can be approximated by switched bilinear functions; More complex case could be the nonlinear system is approximated with different kinds of simple functions. When the model can be described by bilinear functions, usually it is because the system states are coupled with the control input variables, for example in traffic flow system. By separating the state space according to the working zones, we could partition the state space into multiple regions, in each of which a linear or bilinear model is established. When the trajectory of the system switches among the state space sub-regions, it is actually switching among different working zones. The method in this paper is to design feedback controllers to stabilize the nonlinear system during the switchings of the working zones.

## 3. PRELIMINARIES

According to the similarity between bilinear systems and linear systems, specific controllers were designed for bilinear systems, which provide state-based linear controllers for the divided bilinear system [28]. When only one subsystem in (2), the system turns into a bilinear system. Based on [15], a division controller can be designed to stabilize the bilinear system as follows. Let $e_j(x) = c_j^{\mathrm{T}} x + 1 (j \in M)$, and the state space is divided into the following regions:

$$\begin{cases} S_j^+ = \{x | e_j(x) \geq \varepsilon\} \\ S_j^0 = \{x | -\varepsilon \leq e_j(x) \leq \varepsilon\} \\ S_j^- = \{x | e_j(x) \leq -\varepsilon\}, \end{cases} \tag{5}$$

where $\varepsilon$ is a small positive scalar. Then, the division controller can be expressed for these regions as:

$$u = \begin{cases} \frac{k_j^+ x + u_j^{\text{ref}}}{e_j(x)}, & x \in S_j^+ \\ 0, & x \in S_j^0 \\ \frac{k_j^- x + u_j^{\text{ref}}}{e_j(x)}, & x \in S_j^-, \end{cases} \tag{6}$$

with $k_j^+$ and $k_j^-$ as the $1 \times n$ state-feedback gain, and $u_j^{\text{ref}}$ the input reference for specifying the equilibrium point.

The bilinear system can be stabilized by the designed division controller for the predefined equilibrium state.

Before the start of the design, Finsler Lemma is presented here [19]:

**Lemma 1.** *Let $x \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$ such that rank$(B) < n$. The following statement are equivalent:*

- $x^{\mathrm{T}} Q x < 0$, $\forall x \neq 0$ *such that* $Bx = 0$*;*
- $B_\perp^{\mathrm{T}} Q B_\perp < 0$*;*
- $\exists \lambda \in \mathbb{R}$*:* $Q - \lambda B^{\mathrm{T}} B < 0$*;*
- $\exists \zeta \in \mathbb{R}^{n\times m}$*:* $Q + \zeta B + B^{\mathrm{T}} \zeta^{\mathrm{T}} < 0$.

*where $B_\perp$ is a basis for the null space of $B$.*

The lemma is used in the proof of the stability of the controller design.

## 4. SBLS Stabilizing Controller Design

For the switching bilinear system described in (2), a method is proposed to design a stabilizing controller for the SBLS based on the aforementioned division controllers. The controller design can be carried on step by step: 1) deriving the corresponding Switching Linear System (SLS) from the SBLS; 2) designing a stabilizing state-feedback controller for the derived SLS; 3) obtaining the stabilizing controller for the SBLS.

### *4.1. Obtain Corresponding Switching Linear System*

By designing stabilizing switching division controllers for each bilinear subsystem $i \in \Lambda$, we could turn the switching bilinear systems into the corresponding switching linear systems, by partitioning the state-space polyhedron $\Omega_i$ into multiple subregions. When the control input is $u_{i,j} (i \in \Lambda, j \in M_i)$ with $M_i = \{1, \ldots, m_i\}$ for sub-bilinear system $i \in \Lambda$, then two state-feedback controllers can be designed for each control input $u_{i,j}$ as

$$\begin{cases} S_{i,j}^+ = \{x | e_{i,j}(x) \geq 0\} \\ S_{i,j}^- = \{x | e_{i,j}(x) \leq 0\}, \end{cases} \tag{7}$$

where $e_{i,j}(x) = c_{i,j}^{\mathrm{T}} x + 1$ $(j \in M)$, and therefore there will be at most $2^{m_i}$ state subspace partitions in $\Omega_i$. Let's define the set of state subspace partitions of $\Omega_i$ is $\Gamma_i$. For region $S_j^0$ in (5), the switching dynamic depends only on the dynamics of the two linear systems on the both sides, thus we could temporally omit the $\varepsilon$. The polyhedral partition of $\Omega_i$ $(i \in \Lambda)$ for bilinear subsystem $i$ can be defined as $\{\Omega_{i,l}\}_{i\in\Lambda, l\in\Gamma_i}$, where $\cup_{l\in\Gamma_i} \Omega_{i,l} = \Omega_i, \Omega_{i,l} \neq \emptyset, \Omega_{i,l_1} \cap \Omega_{i,l_2} \neq \emptyset, \forall l_1 \neq l_2, l_1, l_2 \in \Gamma_i$.

According to the polyhedral partition of the state space, the controller can be designed for each polyhedron $\Omega_{i,l}$ as

$$U_{i,l} = [u_{i,l,1}\ u_{i,l,2} \cdots u_{i,l,m_i}], \quad i \in \Lambda, l \in \Gamma_i, m_i \in M_i, \tag{8}$$

with the equilibrium as the origin, where

$$u_{i,l,j} = \frac{k_{i,l,j} x}{e_{i,j}(x)}, \quad \text{if } x \in \Omega_{i,l}, j \in M_i, l \in \Gamma_i, i \in \Lambda. \tag{9}$$

If (9) is substituted into (2), then the bilinear terms are eliminated, and a close-loop linear system is obtained. Consequently, the bilinear system in (2) becomes a switching linear system, i.e. the corresponding SLS of the SBLS. With

(9), the parameter of the control input, which is related to the system state, will be eliminated in (2). In other words, the bilinear term in the switching bilinear system turns into a linear term. However, if the denominator is 0, the control input will explode due to the singularity point. For this reason, the state space in the corresponding switching linear system is further divided into more subspace partitions by letting $e_{i,j}(x) = c_{i,j}^{\mathrm{T}} x + 1 = 0$.

To control the SBLS, the state-feedback controller can be first designed to stabilize the following corresponding SLS:

$$\dot{x} = (A_i + \sum_{j=1}^{m_i} b_{i,j} k_{i,l,j})x, \quad \text{if } x \in \Omega_{i,l}, j \in M_i, l \in \Gamma_i, i \in \Lambda. \tag{10}$$

Define

$$\begin{aligned} B_i &= [b_{i,1}\ b_{i,2} \cdots b_{i,m_i}], \\ K_{i,l} &= [(k_{i,l,1})^{\mathrm{T}}\ (k_{i,l,2})^{\mathrm{T}} \cdots (k_{i,l,m_i})^{\mathrm{T}}]^{\mathrm{T}}, \end{aligned} \tag{11}$$

then the equations of the corresponding SLS system can be described as

$$\dot{x} = (A_i + B_i K_{i,l})x, \quad \text{if } x \in \Omega_{i,l}, j \in M_i, l \in \Gamma_i, i \in \Lambda. \tag{12}$$

*4.2. State-feedback Control Design for SLS*

Now, we will design switching state-feedback control laws to stabilize SLS in (12), and asymptotically steer the system state to the predefined equilibrium (i.e. the origin) using an extension of the approach in [13, 16, 23]. After deriving the correspond SLS system, stabilizing controllers will be designed based on the results in [16]. But, this paper aims at designing stabilizing controller for SBL systems that are often found in flow systems. To address the specific problem, we mainly focuses on the state-based switchings, and state feedback controllers are designed for the switching bilinear systems. In addition, in this paper, a relaxed switching condition is proposed to improve the feasible region of the control input, and it guarantees the reduction of the overall Lyapunov function along the switchings of the subsystems at the same time. This is realized by designing the relaxing LMI constraints. The effect of the relaxation is verified in the experiment.

We can describe each polyhedral region $\Omega_{i,l}$ as a series of linear inequalities, where $F_{i,l}$ is the parameter matrix, and $f_{i,l}$ is the constant vector of the linear inequalities of the polyhedral region $\Omega_{i,l}$,

$$\underbrace{[F_{i,l} \quad f_{i,l}]}_{\bar{F}_{i,l}} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0, \quad \text{if } x \in \Omega_{i,l}, \tag{13}$$

and so we can use an equality and inequality to represent the boundary hyperplane for two neighboring regions $\Omega_{i,l}$ and $\Omega_{i',l'}$ as

$$\underbrace{[\Psi_{ii',ll'} \quad \psi_{ii',ll'}]}_{\bar{\Psi}_{ii',ll'}} \begin{bmatrix} x \\ 1 \end{bmatrix} = 0, \text{ and } \underbrace{[\Phi_{ii',ll'} \quad \phi_{i',ll'}]}_{\bar{\Phi}_{ii',ll'}} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0,$$
$$\forall\, x \in \Omega_{i,l} \cap \Omega_{i',l'}. \tag{14}$$

The linear equalities represent the hyperplane of the boundary, and the linear inequalities indicate on which region of the hyperplane the boundary is active. $\Psi_{ii',ll'}$ is the parameter matrix, and $\psi_{ii',ll'}$ is the constant vector of the linear equalities of the boundary hyperplane between the two neighboring regions $\Omega_{i,l}$ and $\Omega_{i',l'}$. $\Phi_{ii',ll'}$ is the parameter matrix, and $\phi_{ii',ll'}$ is the constant vector of the linear inequalities of the boundary hyperplane between the two neighboring regions $\Omega_{i,l}$ and $\Omega_{i',l'}$.

Lyapunov functions are described for each polyhedral region $\Omega_{i,l}$ $(l \in \Gamma_i, i \in \Lambda)$ as

$$V_{i,l}(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix} P_{i,l} & \star \\ s_{i,l}^{\mathrm{T}} & r_{i,l} \end{bmatrix}}_{\bar{P}_{i,l}} \underbrace{\begin{bmatrix} x \\ 1 \end{bmatrix}}_{\bar{x}}, \quad \forall\, l \in \Gamma_i, i \in \Lambda, x \in \Omega_{i,l}, \tag{15}$$

$$\begin{bmatrix} \bar{Q}_{i,l} & \star \\ \bar{F}_{i,l}\bar{Q}_{i,l} & R_{i,l} \end{bmatrix} > 0, \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{19}$$

$$\begin{bmatrix} Q_{i,l} & \star \\ F_{i,l}Q_{i,l} & R_{i,l} \end{bmatrix} > 0,\ \bar{Q}_{i,l} = \begin{bmatrix} Q_{i,l} & \star \\ 0 & q_{i,l} \end{bmatrix} > 0, \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \in \Omega_{i,l}\}, \tag{20}$$

$$\begin{bmatrix} \bar{A}_i\bar{Q}_{i,l} + \bar{Q}_{i,l}\bar{A}_i^{\mathrm{T}} + \bar{B}_iN_{i,l} + N_{i,l}^{\mathrm{T}}\bar{B}_i^{\mathrm{T}} & \star \\ \bar{F}_{i,l}\bar{Q}_{i,l} & -M_{i,l} \end{bmatrix} < 0, \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{21}$$

$$\begin{bmatrix} A_iQ_{i,l} + Q_{i,l}A_i^{\mathrm{T}} + B_iN_{i,l} + N_{i,l}^{\mathrm{T}}B_i^{\mathrm{T}} & \star \\ F_{i,l}Q_{i,l} & -M_{i,l} \end{bmatrix} < 0, \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \in \Omega_{i,l}\}, \tag{22}$$

$$\begin{bmatrix} \bar{Q}_{i,l} & \star & \star & \star \\ \bar{Q}_{i,l} & \bar{Q}_{i',l'} & \star & \star \\ \bar{\Psi}_{ii',ll'}\bar{Q}_{i,l} & 0 & -\lambda_{ii',ll'} & \star \\ \bar{\Phi}_{ii',ll'}\bar{Q}_{i,l} & 0 & 0 & -\Theta_{ii',ll'} \end{bmatrix} \geq 0, \quad \forall\, i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} : \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset \tag{23}$$

$$\begin{bmatrix} \bar{Q}_{i',l'} & \star & \star & \star \\ \bar{Q}_{i',l'} & \bar{Q}_{i,l} & \star & \star \\ \bar{\Psi}_{ii',ll'}\bar{Q}_{i',l'} & 0 & \lambda_{ii',ll'} & \star \\ \bar{\Phi}_{ii',ll'}\bar{Q}_{i',l'} & 0 & 0 & \Theta_{ii',ll'} \end{bmatrix} \geq 0, \quad \forall\, i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} : \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset \tag{24}$$

where $\bar{x} = [x\ 1]^{\mathrm{T}}$, $P_{i,l} \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $s_{i,l}$ is an $n \times 1$ dimensional vector, and $r_{i,l} \in \mathbb{R}$.

Based on the definitions above, a sufficient condition can be derived to design switched state-feedback control laws for the SLS in (12), which can asymptotically steer the system state to the origin (i.e. the equilibrium for at least one of the subsystems).

In order to describe in a more compact way, we use the augmented system matrices defined as follows in the following theorems:

$$\bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix},\ \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}. \tag{16}$$

**Theorem 2.** *The SLS system in* (12) *can be asymptotically stabilized, on condition that feasible solutions are founded satisfying* (19)-(24)*, with positive definite matrices $\bar{Q}_{i,l}$, $Q_{i,l}$, $R_{i,l}$, and $M_{i,l}$, and matrices $\Theta_{i,l}$ and scalars $\lambda_{i,l}$, and deriving the state-based feedback control laws with gains for different regions as*

$$\bar{K}_{i,l} = N_{i,l}\bar{Q}_{i,l}^{-1} = N_{i,l}\bar{P}_{i,l} \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{17}$$

*and*

$$K_{i,l} = N_{i,l}Q_{i,l}^{-1} = N_{i,l}P_{i,l} \quad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \in \Omega_{i,l}\}. \tag{18}$$

*Proof*: First, we apply the Schur complement on (19) to the second row and column. By multiplying from both sides by $\bar{Q}_{i,l}^{-1} = \bar{P}_{i,l}$, the result yields

$$\bar{P}_{i,l} - \bar{F}_{i,l}^{\mathrm{T}}R_{i,l}^{-1}\bar{F}_{i,l} > 0, \qquad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{25}$$

which makes sure that the Lyapunov function is positive on each state polyhedron, that is

$$V_{i,l} > 0, \quad \text{if } \bar{F}_{i,l}\bar{x}_{i,l} \geq 0 \text{ and } \bar{x}_{i,l} \neq 0, \qquad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}. \tag{26}$$

Second, we apply the Schur complement again on (21) with respect to the second row and column. By multiplying from both sides by $\bar{Q}_{i,l}^{-1} = \bar{P}_{i,l}$, and using (17). then the result turns into

$$\bar{P}_{i,l}(\bar{A}_i + \bar{B}_i\bar{K}_{i,l}) + (\bar{A}_i + \bar{B}_i\bar{K}_{i,l})^{\mathrm{T}}\bar{P}_{i,l} + \bar{F}_{i,l}^{\mathrm{T}}M_{i,l}^{-1}\bar{F}_{i,l} < 0, \qquad \forall\,(i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}, \tag{27}$$

which makes sure that the derivative of the Lyapunov function is negative on each state polyhedron, i.e.

$$\dot{V}_{i,l} < 0, \quad \text{if } \bar{F}_{i,l}\bar{x}_{i,l} \geq 0 \text{ and } \bar{x}_{i,l} \neq 0, \qquad \forall\, (i,l) \in \{(i,l) \mid i \in \Lambda, l \in \Gamma_i, 0 \notin \Omega_{i,l}\}. \tag{28}$$

In particular, when the subsystem contains the origin, i.e. for the polyhedron with $0 \in \Omega_{i,l}$, the LMIs in (20) and (22) are applied to guarantee the Lyapunov function is positive and the derivative of Lyapunov function is negative on the region. In order to guarantee that the derivative of the Lyapunov function $\dot{V}_{i,l}$ will only become zero if the state $x$ is zero, the conditions in (19) and (21) need to be satisfied, where the row and column corresponding to the augmented variable are removed. In (20), the augmented $\bar{Q}_{i,l}$ is defined for the polyhedron $\Omega_{i,l}$ with the origin, which makes it comparable with the augmented $\bar{Q}_{i,l}$ for the polyhedron $\Omega_{i,l}$ without the origin in the boundary conditions (23) and (24).

At last, the Schur complement is carried out on (23) 3 times, every time with respect to the last row and the last column. Again, adopting (17), the result is multiplied from both sides by $\bar{Q}_{i,l}^{-1} = \bar{P}_{i,l}$. Then we derive

$$\bar{P}_{i,l} - \bar{P}_{i',l'} + \lambda_{ii',ll'}^{-1}\bar{\Psi}_{ii',ll'}^{\mathrm{T}}\bar{\Psi}_{ii',ll'} + \bar{\Phi}_{ii',ll'}^{\mathrm{T}}\Theta_{ii',ll'}^{-1}\bar{\Phi}_{ii',ll'} \geq 0, \qquad \forall\, i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} : \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset. \tag{29}$$

An equality and inequality is defined to describe the states on the boundary of the polyhedra $\Omega_{i,l}$ and $\Omega_{i',l'}$ as

$$\mathscr{S}_{ii',ll'} = \{\bar{x} \mid \bar{\Psi}_{ii',ll'}\bar{x} = 0 \wedge \bar{\Phi}_{ii',ll'}\bar{x} \geq 0\}. \tag{30}$$

Applying Finsler's Lemma [19], multiplying the augmented states on both sides, we obtain

$$V_{i,l} \geq V_{i',l'}, \quad \text{if } \bar{x} \in \mathscr{S}_{ii',ll'}, \qquad \forall\, i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} : \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset. \tag{31}$$

For (24), we do the same way and yield

$$\bar{P}_{i,l} - \bar{P}_{i',l'} + \lambda_{ii',ll'}^{-1}\bar{\Psi}_{ii',ll'}^{\mathrm{T}}\bar{\Psi}_{ii',ll'} + \bar{\Phi}_{ii',ll'}^{\mathrm{T}}\Theta_{ii',ll'}^{-1}\bar{\Phi}_{ii',ll'} \leq 0, \qquad \forall\, i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} : \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset. \tag{32}$$

Multiply the augmented states on both sides, and then applying Finsler's Lemma in Lemma 1, we derive

$$V_{i,l} \leq V_{i',l'}, \quad \text{if } \bar{x} \in \mathscr{S}_{ii',ll'}, \qquad \forall\, i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} : \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset. \tag{33}$$

Equations (31) and (33) guarantee the values of the Lyapunov functions are equal to each other ($V_{i,l} = V_{i',l'}$), on the boundary of neighboring polyhedra $\Omega_{i,l}$ and $\Omega_{i',l'}$.

Consequently, by applying the following state feedback control laws to all of the polyhedral regions

$$U_{i,l} = \bar{K}_{i,l}\bar{x} \quad \forall\, \bar{x} \in \{\bar{x} \mid x \in \Omega_{i,l}, 0 \notin \Omega_{i,l}, i \in \Lambda, l \in \Gamma_i\}, \tag{34}$$

and

$$U_{i,l} = K_{i,l}x \quad \forall\, x \in \{x \mid x \in \Omega_{i,l}, 0 \in \Omega_{i,l}, i \in \Lambda, l \in \Gamma_i\}, \tag{35}$$

it is possible to derive a positive decreasing overall Lyapunov function with continuous values on the boundaries between the switching of polyhedral regions. Therefore, the SLS system in (12) can be asymptotically stabilized by the designed control laws. □

For constraints (23) and (24), they require the Lyapunov functions need to be equal on the boundaries of the state regions, which could be too conservative in the controller design. In some situation, there may not have feasible solutions to satisfy all the constraints, from (19) to (24). Consequently, in following theorem, we remove constraint (23) and (24), and relax the condition by allowing the Lyapunov functions jumping on the boundaries of the state polyhedra.

Define $x_{i,l}$ as the point in the state polyhedron $\Omega_{i,l}$ that is closest to the origin, and $d_{i,l}$ as the distance from $x_{i,l}$ to the origin.

**Theorem 3.** *If positive definite matrices $\bar{Q}_{i,l}$, $Q_{i,l}$, $R_{i,l}$, and $M_{i,l}$, and matrices $\Theta_{i,l}$ and scalars $\lambda_{i,l}$ exist and satisfy inequalities* (19)-(22)*, and* (39)*, then taking* (15) *as the state-based Lyapunov functions, the state feedback control laws with gains given in* (17) *and* (18) *can stabilize the SLS system in* (12) *asymptotically.*

$$\begin{bmatrix} \bar{Q}_{i,l} & \star & \star & \star \\ \bar{Q}_{i,l} & \bar{Q}_{i',l'} & \star & \star \\ \bar{\Psi}_{ii',ll'}\bar{Q}_{i,l} & 0 & -\lambda_{ii',ll'} & \star \\ \bar{\Phi}_{ii',ll'}\bar{Q}_{i,l} & 0 & 0 & -\Theta_{ii',ll'} \end{bmatrix} > 0, \quad \text{if } d_{i,j} > d_{i',j'}, \text{ and } \{\Omega_{i,l} \cap \Omega_{i',l'}\} \neq \emptyset, \forall i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'} \quad (39)$$

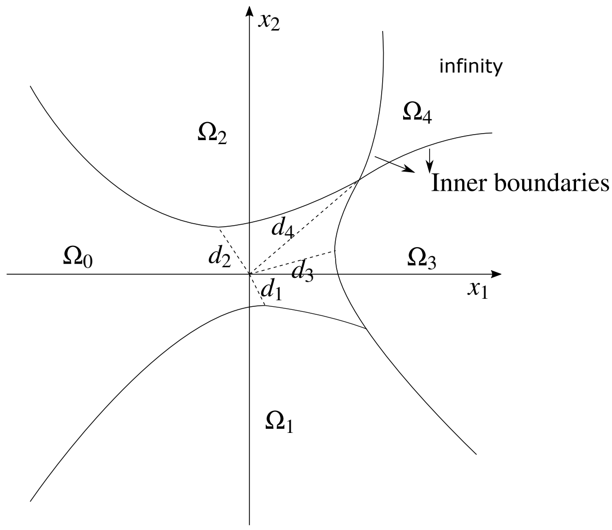



Figure 1: Illustration for sequence of state space partition

*Proof*: To prove the conclusion, we need to prove that the smallest Lyapunov function value on a state polyhedron is always scaled on the inner boundaries of the polyhedron that does not contain the origin (i.e. the Lyapunov function on the polyhedron that does not contain the origin, will always increase when the states tend to infinity); then, based on the inequality constraints on the boundaries of the polyhedra, a decreasing sequence of Lyapunov functions with respect to the sequence of state polyhedral switchings is proved to exist, which can lead the states of the system to converge to the origin.

First, we can prove that, the smallest Lyapunov function on a state polyhedron is always scaled on the inner boundaries of the polyhedron, except for the polyhedron with origin. Taking the state space partition in Fig. 1 for example, the inner boundaries of state polyhedron $\Omega_4$ are the boundary lines connected with its neighboring state polyhedra. Suppose the smallest Lyapunov function is inside the polyhedron $\Omega_{i,l}$ that does not contain the origin, then we can prove that there must exist a $x^*_{i,l}$ allowing $\dot{V}_{i,l}(x^*_{i,l}) = 0$. This is contradicting with the condition $\dot{V}_{i,l} < 0$ on $\Omega_{i,l}$. In addition, if the smallest value of Lyapunov function is scaled on the infinite boundary of the polyhedron $\Omega_{i,l}$ (see Fig. 1), because we have $V_{i,l} \geq 0$, then it is straightforward that $V_{i,l}(x^*_{i,l}) \mid_{x^*_{i,l} \to \infty} = 0$. This means $\dot{V}_{i,l}(x^*_{i,l}) \mid_{x^*_{i,l} \to \infty} = 0$, which is contradicting with the condition $\dot{V}_{i,l} < 0$. Consequently, we prove that the smallest Lyapunov function value on a state polyhedron is always scaled on the inner boundaries of the polyhedron without the origin.

Second, if we can make sure the boundary condition (39) is satisfied, the following inequalities are guaranteed:

$$\begin{aligned} &\bar{P}_{i,l} - \bar{P}_{i',l'} + \lambda^{-1}_{ii',ll'} \bar{\Psi}^{\mathrm{T}}_{ii',ll'} \bar{\Psi}_{ii',ll'} + \bar{\Phi}^{\mathrm{T}}_{ii',ll'} \Theta^{-1}_{ii',ll'} \bar{\Phi}_{ii',ll'} > 0, \\ &\quad \text{if } d_{i,j} > d_{i',j'}, \text{ and } \Omega_{i,l} \cap \Omega_{i',l'} \neq \emptyset, \\ &\quad \forall i,i' \in \Lambda, l \in \Gamma_i, l' \in \Gamma_{i'}, \end{aligned} \quad (40)$$

which ensures that $V_{i,l} \geq V_{i',l'}$ for all the states $\bar{x} \in \mathscr{S}_{ii',ll'}$ on the boundary of $\Omega_{i,l}$ and $\Omega_{i',l'}$.

Third, because the smallest Lyapunov function value is always scaled on the inner boundaries of a polyhedron that does not contain the origin, we know that the smallest Lyapunov function value of polyhedron $\Omega_{i,l}$ must scale on the boundary of $\Omega_{i,l}$ and $\Omega_{i',l'}$, i.e. $x^*_{i,l} \in \Omega_{i,l} \cap \Omega_{i',l'}$. In addition, since we have $V_{i,l}(x) > V_{i',l'}(x)$ on $\mathscr{S}_{ii',ll'}$, we have

$V_{i,l}(x^*_{i,l}) > V_{i',l'}(x)$ on $\mathscr{S}_{ii',ll'}$. Moreover, because the value of the Lyapunov function on the state polyhedron $\Omega_{i',l'}$ is always larger equal than the minimum Lyapunov function value, so it is obvious that $V_{i',l'}(x) \geq V_{i',l'}(x^*_{i',l'})$ on $\Omega_{i',l'}$. Therefore, we obtain $V_{i,l}(x^*_{i,l}) > V_{i',l'}(x^*_{i',l'})$, which means that the minimum Lyapunov function values decrease when the state switches from $\Omega_{i,l}$ to $\Omega_{i',l'}$.

Let's define a state polyhedron distance as the shortest distance from the origin to the state polyhedron, such as $d_x$ in Fig. 1. Finally, if a feasible solution exists for (19)-(22) and (39), there must exist a sequence of reducing state polyhedron distances connecting all the state polyhedra in $\Omega$ to form a state polyhedron switching path getting near and near to the origin, until to the origin. The sequence of state polyhedron distances satisfies

$$d_p \geq d_{p-1} \geq \cdots \geq d_1 \geq 0, \tag{41}$$

with $p$ as the index of polyhedron $\Omega_{i,l}$, $\forall\ i \in \Lambda, l \in \Gamma_i$ in $\Omega$, which is corresponding to a sequence of decreasing minimum Lyapunov function values for all the polyhedra as

$$V_p(x^*_p) \geq V_{p-1}(x^*_{p-1}) \geq \cdots \geq V_1(x^*_1) \geq 0, \tag{42}$$

that guarantees to steer the state to asymptotically converge to the origin, from an initial state $x_0$ within any of the polyhedra in $\Omega$.

In such case, if a Zeno behavior exists on the boundary of two state subspaces, the Lyapunov function will bouncing on the boundary, then the value of Lyapunov function will reduce and then increase without stopping. Consequently, if a feasible solution can be found, it is not possible to have Zeno behaviors on the switching boundaries. □

Taking the state space partition in Fig. 1 for example, if a feasible solution exists for (19)-(22) and (39), then we have a sequence of state polyhedron distances

$$d_4 \geq d_3 \geq d_2 \geq d_1 \geq 0, \tag{43}$$

corresponding to a sequence of decreasing minimum Lyapunov function values

$$V_4(x^*_4) \geq V_3(x^*_3) \geq V_2(x^*_2) \geq V_1(x^*_1) \geq V_0(x^*_0), \tag{44}$$

for all the polyhedra $\Omega_0$, $\Omega_1$, $\Omega_2$, $\Omega_3$, and $\Omega_4$, so that we could use the state feedback control laws with gains given in (17) and (18) to steer the state to asymptotically converge to the origin.

### *4.3. Controller Design for Stabilizing SBLS*

For the SBLS, when $e_{i,j}(x) = 0$ on some of the boundaries of the subregions, the value of the controller in (9) becomes infinite, which is not feasible. Based on the result of stabilizing state-feedback control design for the corresponding SLS of the SBLS, the feasibility on the boundaries of the SBLS need to be further checked to make the whole system stabilized. In order to avoid the explosion in computing near the boundary when $e_{i,j}(x) = 0$, we define a small positive value $\varepsilon_{i,j}$ in the following equations. So, we could check the stability of the switching near the boundary and its small vicinity with the method in this section. Thus, according to Sec. 3, the following adaptation is made to adjust the division of the system, as

$$\begin{cases} S^+_{i,j} = \{x | e_{i,j}(x) \geq \varepsilon_{i,j}\} \\ S^0_{i,j} = \{x || e_{i,j}(x)| \leq \varepsilon_{i,j}\} \\ S^-_{i,j} = \{x | e_{i,j}(x) \leq \varepsilon_{i,j}\}, \end{cases} \quad \forall i \in \Lambda, j \in M_i, \tag{45}$$

with $\varepsilon_{i,j}$ is a very small positive value. According to the state space partition in Sec. 4.1, we can write the controller design of the SBLS as

$$\begin{cases} u_{i,l,j} = \frac{k_{i,l,j}x}{e_{i,j}(x)},\ \forall\, x \in \Omega_{i,l} \setminus S^0_{i,j} :\ l, l' \in \Gamma_i, & j \in M_i, i \in \Lambda, \\ u_{i,ll',j} = 0,\ \forall\, x \in (\Omega_{i,l} \cap S^0_{i,j}) \cup (\Omega_{i,l'} \cap S^0_{i,j}) : & j \in M_i, l \neq l', l, l' \in \Gamma_i, i \in \Lambda, \end{cases} \tag{46}$$

where the state-feedback controllers designed for the corresponding SLS are taken as the controllers for the SBLS within the state polyhedra, however when $e_{i,j}(x)=0$, to make sure the designed controllers are feasible on the state boundaries of some of the switchings, the control inputs are taken as 0. It means that no control input can influence the system states on the state boundaries of some of the switchings, so only an autonomous system exists on these boundaries of the bilinear subsystems. If along the switching sequence of state polyhedra obtained in Theorem 3, the system state can cross these predefined switching boundaries with autonomous system movement, then the SBLS state can converge to the origin and can be stabilized. This can be checked through the following Theorem.

**Theorem 4.** *Considering an SBLS expressed by* (1)*, suppose it has a corresponding SLS with the form of* (12)*, and state-feedback controller is obtained based on solving the problem in Theorem 3 for the corresponding SLS system, then the SBLS can be asymptotically stabilized by the controller described in* (46)*, if along the state-based switching sequence, at each switching from polyhedron $\Omega_{i,l}$ to $\Omega_{i,l'}$ for any bilinear subsystem i, when $u_{i,ll',j}=0,\ \forall\ j\in M_i$, it holds that*

$$\exists\, x\in(\Omega_{i,l}\cap S^0_{i,j})\cup(\Omega_{i,l'}\cap S^0_{i,j}),\quad e^{A_i t}x\in\Omega_{i,l'}\setminus S^0_{i,j}, \tag{47}$$

*which means that, with its autonomous dynamics as in* (47)*, the system state can get over the boundary of $\Omega_{i,l}$ and $\Omega_{i,l'}$ without control inputs (i.e. $u_{i,ll',j}=0$).*

*Proof*: 1) We can obtain the corresponding SLS for the SBLS as in 4.1; 2) According to Theorem 3, we can find a switching sequence of state polyhedra, and obtain state feedback controllers for each state polyhedron of the SLS to steer the system state to asymptotically converge to the origin; 3) We can use Theorem 4 to make sure that the state of SBLS can transit from the previous state polyhedron to the next state polyhedron along the previous derived switching sequence of the state polyhedra, until to the origin. Therefore, the SBLS is asymptotically stabilized. □

### 5. Examples

In this section, we evaluate the performance of the proposed method through two examples: one is to design stabilizing controllers for a SBLS with two second-order subsystems; the other is to design traffic signal controllers for an urban traffic network with two roads.

*5.1. Example 1:*

For the first example, the conditions presented in Theorem 4 is applied to design stabilizing control laws for a second-order system. We directly use the SBLS model in (2) to describe the SBLS, and the vectors and matrices in the model are as follow:

$$A_1=\begin{bmatrix}-1 & 4\\ 5 & -8\end{bmatrix},\ b_{1,1}=\begin{bmatrix}1\\ 0\end{bmatrix},\ c_{1,1}=\begin{bmatrix}-1\\ 0\end{bmatrix}$$
$$A_2=\begin{bmatrix}-5 & -3\\ 2 & -1\end{bmatrix},\ b_{2,1}=\begin{bmatrix}0\\ -1\end{bmatrix},\ c_{2,1}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

The two bilinear subsystems are separated by $x_1+x_2=0$. According to Sec. 4.1, we can partition the state space into 4 regions with $\Lambda=\{1,2\}$ and $\Gamma_1=\{1,2\},\Gamma_2=\{1,2\}$. Therefore, we can derive the parameters for the corresponding SLS given in (12) as:

$$\begin{aligned}
&A_1=\begin{bmatrix}-1 & 4\\ 5 & -8\end{bmatrix},\ B_1=\begin{bmatrix}1\\ 0\end{bmatrix},\\
&F_{1,1}=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix},\ \bar{F}_{1,2}=\begin{bmatrix}1 & 0 & -1\\ -1 & -1 & 0\end{bmatrix},\\
&\bar{\Psi}_{11,12}=\begin{bmatrix}-1 & 0 & 1\end{bmatrix},\ \bar{\Phi}_{11,12}=\begin{bmatrix}0 & -1 & -1\end{bmatrix},\\
&A_2=\begin{bmatrix}-5 & -3\\ 2 & -1\end{bmatrix},\ B_2=\begin{bmatrix}0\\ -1\end{bmatrix},\\
&F_{2,1}=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix},\ \bar{F}_{2,2}=\begin{bmatrix}0 & -1 & -1\\ 1 & 1 & 0\end{bmatrix},
\end{aligned}$$

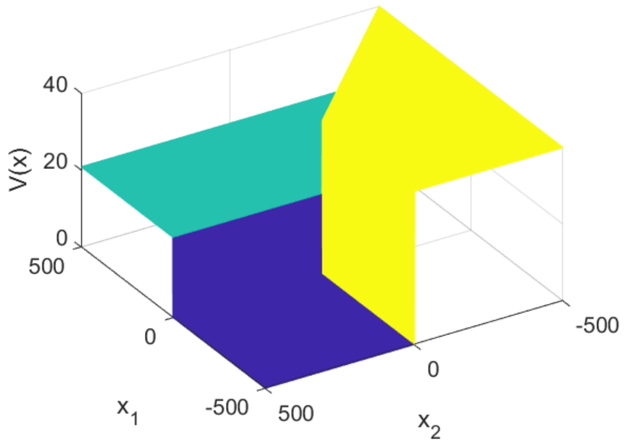



Figure 2: Illustration for the overall Lyapunov function

$$\bar{\Psi}_{22,12} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \bar{\Phi}_{22,12} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

Using the Yalmip toolbox (with SeDuMi solver) to solve the feasibility problems in Section 4.1. If we use Theorem 2 and solve the problems from (19)-(24), no feasible solution can be found. If we use Theorem 3 and solve the problems from (19)-(22) and (39), then we can obtain an decreasing overall Lyapunov function as in Fig. 2, where the Lyapunov functions decrease from polyhedron to polyhedron, from far away from the origin to the vicinity of the origin. As a result, the controllers are obtained as

$$U_{1,1} = \frac{K_{1,1}x}{-x_1+1},\ U_{1,2} = \frac{\bar{K}_{1,2}\bar{x}}{-x_1+1},\ U_{1,12} = 0,$$
$$U_{2,1} = \frac{K_{2,1}x}{x_2+1},\ U_{2,2} = \frac{\bar{K}_{2,2}\bar{x}}{x_2+1},\ U_{2,12} = 0,$$

where the feedback control laws for all the regions are

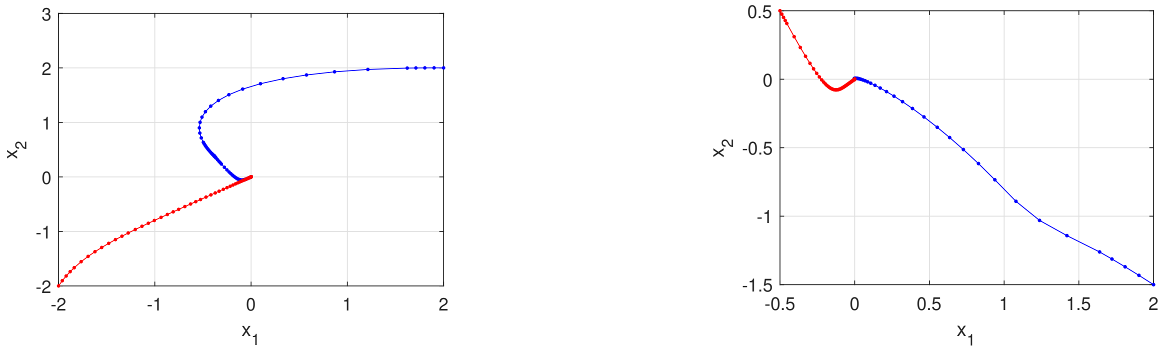
$$K_{1,1} = [-2.7384\ -1.3923],\ \bar{K}_{1,2} = [-4.5474\ -2.1275\ 1.2429];$$
$$K_{2,1} = [-0.3051\ -0.6086],\ \bar{K}_{2,2} = [-0.6752\ -1.9942\ -3.2427],$$

which is able to steer state to the origin for different initial conditions, as shown in Fig. 3(a) and 3(b). The designed state feedback control laws are illustrated in Fig. 4(a) and 4(b) for the initial states $[2,2]$ and $[-2,-2]$, where the switchings of the controllers can be seen.

*5.2. Example 2:*

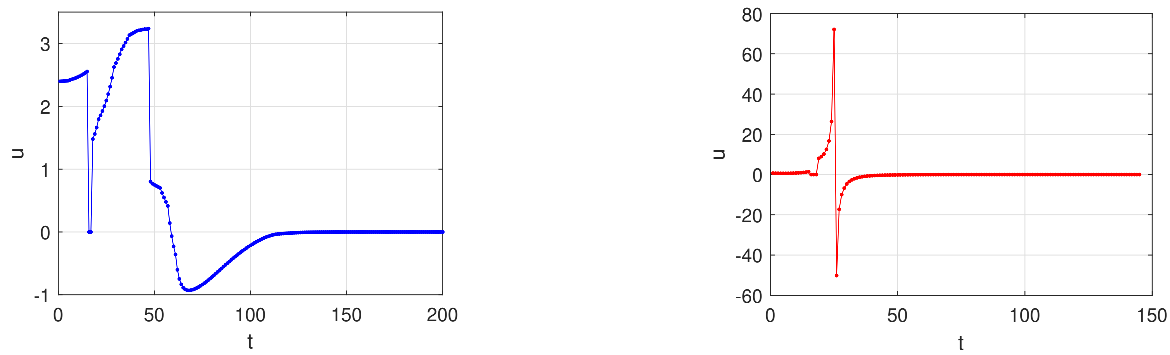
For the second example, we consider urban traffic flow control problem with traffic signals, and apply the switching link flow model proposed in [24] to design stabilized traffic light controllers. The traffic model proposed in [24] describes the nonlinear traffic flow property by dividing it into several working zones, and in each working zone a bilinear model is established. A bilinear function is used to build the model is because the traffic flow on each link of the traffic network depends not only on the flow of itself, but also relays on the traffic flows on the upper stream links and the down stream links. In addition, the impact of the flows on the upper stream links and the down stream links is coupled with the traffic signal variables (i.e. control input variables), which forms the bilinear term of the model. In the model, for simplicity, we consider 2 link state modes for each link, i.e. Free-flow mode (F), and Congestion mode (C). The state of a road link may switch between the two link state modes $M = \{F, C\}$.

For each of the link state mode in $M$, a time-variant model is formulated to update the traffic state on the link based on the averaged entering and leaving flows, influenced by the traffic signals. The dynamic evolution of the




(a) (b)

Figure 3: Closed-loop trajectories with initial states (a) $[2,2]$ and $[-2,-2]$, (b) $[0.5,0.5]$ and $[2,-1.5]$




(a) (b)

Figure 4: Feedback control inputs for initial states (a) $[2,2]$ and (b) $[-2,-2]$

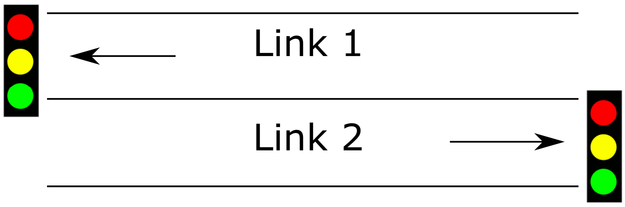



Figure 5: A two-link urban traffic system with traffic lights

traffic density at time step $k$ is

$$\rho(k+1) = A\rho(k) + B_1\rho(k)U(k) + B_2U(k) + Cd(k) + F, \tag{48}$$
$$U(k) = \beta^{\mathrm{T}}(k)r(k), \tag{49}$$

where $\rho(k)$ is the vector of link densities, $U(k)$ is the scalar control input at time step $k$, $\beta(k)$ is the vector of turning ratios in front of the traffic signals on a link at time step $k$, $r(k)$ is the vector of green time splits on a link at time step $k$, $d(k)$ is a vector containing the demand from upstream links (input flows provided by upstream links), and the supply from the downstream links (available flows that can be accepted by downstream links).

In this context, the dynamic model can be further written explicitly for the two modes. For the link state mode F, the upstream and downstream of the link are all having free flows, thus the dynamic model for updating the density becomes

$$\rho(k+1) = a\rho(k) + b\rho(k)U(k) + cd(k), \tag{50}$$

where $a = 1 + \beta_{\mathrm{r}} b$, $b = -\frac{T_{\mathrm{s}}}{l} v_{\mathrm{f}}$, $c = [\frac{T_{\mathrm{s}}}{l}\ 0]$, and $l$ is the length of the link. For the link state mode C, the upstream and the downstream link are all congested, thus the dynamic model for updating the density becomes

$$\rho(k+1) = a\rho(k) + cd(k) + f, \tag{51}$$

where $a = 1 + \frac{T_{\mathrm{s}}}{l} w_{\mathrm{c}}$, $c = [0\ -\frac{T_{\mathrm{s}}}{l}]$, and $f = -\frac{T_{\mathrm{s}}}{l} w_{\mathrm{c}} \rho_{\mathrm{J}}$. For more details about the modeling, please refer to [24]. In fact, as shown in (51), the traffic state in congestion mode cannot be affected by the local traffic signals on this link, which means, in mode C, congestion cannot be removed by only adjusting the local traffic signals, additional actions like adjusting upstream and downstream traffic signals are required to regulate the input and output traffic flows of the link. Therefore, we will focus on designing a local controller for the free-flow region.

We consider an example with two links as Fig. 5 shows. When both Link 1 and Link 2 are in F mode, the model of the links can be written in a matrix expression as

$$\rho(k+1) = A\rho(k) + \sum_{j=1}^{2} B_j \rho(k) U_j(k) + Cd(k) + F, \tag{52}$$

where the matrices are

$$A = \begin{bmatrix} a_{\mathrm{F}} & 0 \\ 0 & a_{\mathrm{F}} \end{bmatrix},\ B_1 = \begin{bmatrix} b_{\mathrm{F}} & 0 \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 0 \\ 0 & b_{\mathrm{F}} \end{bmatrix},$$
$$C = \begin{bmatrix} c_{\mathrm{F}} & 0 \\ 0 & c_{\mathrm{F}} \end{bmatrix},\ F = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

Let us define the parameters of the two links as $T_{\mathrm{s}} = 90$ s, $l = 800$ m, $v_{\mathrm{f}} = 60$ veh/h, $w_{\mathrm{c}} = -30$ veh/h, $\rho_{\mathrm{c}} = 60$ veh/km, $\rho_{\mathrm{J}} = 180$ veh/km, $\beta_{\mathrm{r}} = 0.2$, and $Q_{\mathrm{C}} = 3600$ veh/h. According to the link model described above, we have $b_{\mathrm{F}} = -T_{\mathrm{s}} v_{\mathrm{f}}/l$, $a_{\mathrm{F}} = 1 + \beta_{\mathrm{r}} b_{\mathrm{F}}$, $a_{\mathrm{C}} = 1 + T_{\mathrm{s}} w_{\mathrm{c}}/l$, $c_{\mathrm{F}} = T_{\mathrm{s}}/l$, $c_{\mathrm{C}} = -T_{\mathrm{s}}/l$, $f_{\mathrm{F}} = 0$, and $f_{\mathrm{C}} = -T_{\mathrm{s}} w_{\mathrm{c}} \rho_{\mathrm{J}}/l$, which are the elements of the matrices (52).

We want to control the link densities to their reference densities, i.e. $\rho_{\mathrm{r}} = 30$ veh/km. Therefore, we define a new link state as the error between the link density and the link density reference, i.e. $e(k) = \rho(k) - \bar{\rho}_{\mathrm{r}}$ and $\bar{\rho}_{\mathrm{r}} = [\rho_{\mathrm{r}}\ \rho_{\mathrm{r}}]^{\mathrm{T}}$.

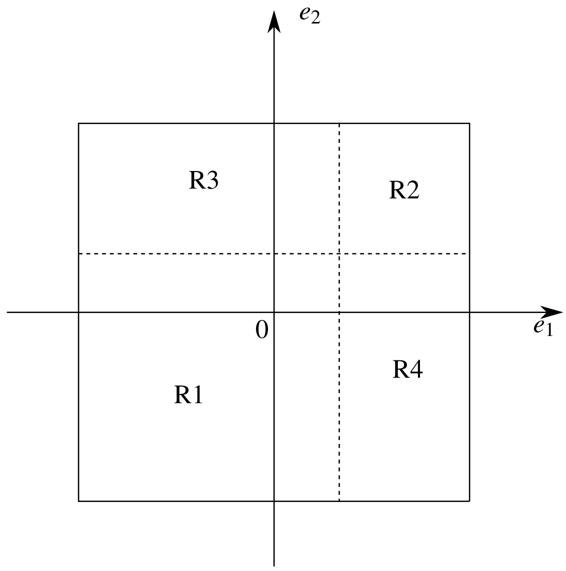


Figure 6: State space partitioned by dotted lines for the two-link bilinear system

Then, the bilinear model for the two-link system can be further written as a bilinear model on system errors, i.e.

$$e(k+1) = Ae(k) + \sum_{j=1}^{2} B_j(e(k) + \rho_{r,j})U_j(k) + H, \tag{53}$$

where $\rho_{r,1} = [\rho_r \ 0]^T$, $\rho_{r,2} = [0 \ \rho_r]^T$. According to (2), the model can be further written as

$$e(k+1) = Ae(k) + \sum_{j=1}^{2} \rho_{r,j}(G_j^T e(k) + 1)U_j(k) + H, \tag{54}$$

where

$$G_1 = [\frac{b_F}{\rho_r} \ 0]^T, \tag{55}$$

$$G_2 = [0 \ \frac{b_F}{\rho_r}]^T, \tag{56}$$

$$H = \beta_r b_F \bar{\rho}_r + Cd(k) + F. \tag{57}$$

Consequently, the two-link system model is formulated as a bilinear model, based on which we could design stabilizing controllers according to the method in Section 4. According to Section 4.1, the state space is partitioned into 4 regions by the dotted lines as shown in Fig. 6. The demand of the links (input flows) is given as $d(k) = [288 \ 288]^T$ veh/h. By solving the feasibility problem (19)-(22) and (39) for the 4 regions, a decreasing overall Lyapunov function is obtained as shown in Fig. 7, where the Lyapunov functions reduce until to the origin (i.e. the error of the link densities with respect to the reference density $e = 0$). As shown in Fig. 6, the origin (equilibrium point) is in Region 1. As a result, the controllers are obtained for Region $i$ as

$$U_{i,1} = \frac{K_{i,1}e}{\frac{b_{1,F}}{\rho_r}e_1 + 1}, \ U_{i,2} = \frac{K_{i,2}e}{\frac{b_{2,F}}{\rho_r}e_2 + 1}, \quad i = 1$$

$$U_{i,1} = \frac{\bar{K}_{i,1}\bar{e}}{\frac{b_{1,F}}{\rho_r}e_1 + 1}, \ U_{i,2} = \frac{\bar{K}_{i,2}\bar{e}}{\frac{b_{2,F}}{\rho_r}e_2 + 1}, \quad i = 2,3,4$$

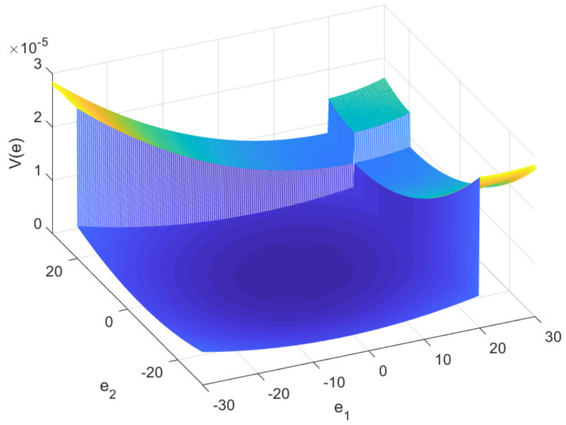
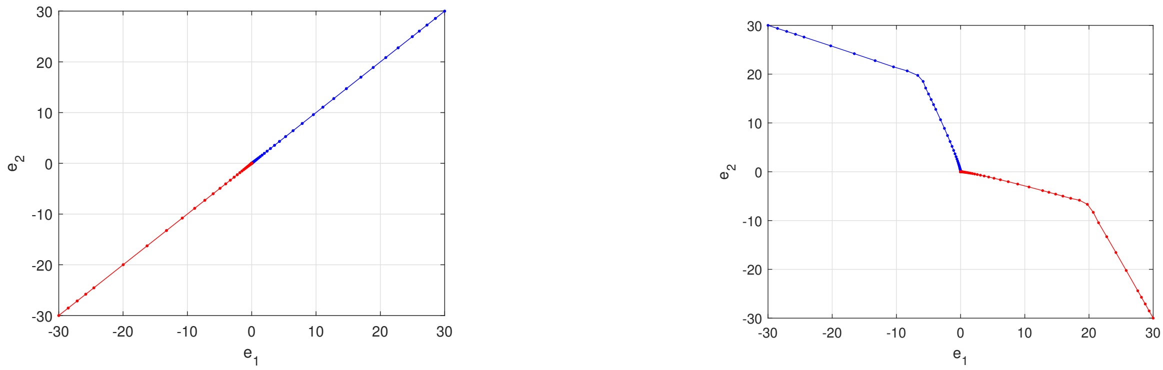



Figure 7: Illustration for the overall Lyapunov function for the two-link system




(a)

(b)

Figure 8: Closed-loop trajectories with initial states (a) $[-30,-30]$ and $[30,30]$, (b) $[-30,30]$ and $[30,-30]$ of two-link system

where the feedback control laws for all the regions are

$$
\begin{aligned}
&K_{1,1} = [-0.0426\ 0.0030],\ K_{1,2} = [0.0030\ -0.0426];\\
&\bar{K}_{2,1} = [-0.0938\ -0.0050\ 1.5090],\\
&\bar{K}_{2,2} = [-0.0050\ -0.0938\ 1.5090];\\
&\bar{K}_{3,1} = [-0.0623\ -0.0030\ 0.8380],\\
&\bar{K}_{3,2} = [-0.0030\ -0.0623\ 0.8380];\\
&\bar{K}_{4,1} = [-0.0623\ -0.0030\ 0.8380],\\
&\bar{K}_{4,2} = [-0.0030\ -0.0623\ 0.8380],
\end{aligned}
$$

which are able to control the link densities to their reference densities (the origin in the error state space) for different initial conditions, as shown in Fig. 8(a) and 8(b).

## 6. Conclusions

In practice, there are some complex nonlinear systems that can be approximated by switching bilinear systems. Designing stabilizing controller for switching bilinear systems makes it possible to better control these kind of nonlinear systems. In this paper, a method is proposed for designing controllers to stabilize state-based switching bilinear systems. By considering the similarity between linear systems and bilinear systems, feedback control laws are designed for the switching bilinear systems to turn the close-loop systems into corresponding state-based switching linear systems. Instead of studying the method to stabilize the switching bilinear system directly, we propose a theorem to asymptotically stabilize the close-loop switching linear system based on multiple Lyapunov functions. In addition, we further relax the conditions by allowing the Lyapunov functions to jump on the boundary of the neighboring state regions. A relaxed conditions are proved to be able to asymptotically stabilize the corresponding switching linear systems, and is less conservative.

The results are verified with two examples. First, the proposed method is used to design stabilizing state-based controllers for a standard second-order switching bilinear system. The results show that when no feasible solution can be found for the second-order switching bilinear system with the conservative conditions in Theorem 2, we are able to find a feasible solution by the relaxed conditions in Theorem 3 to stabilize the system. Then, the method is applied to design traffic light controllers for a two-link urban traffic system. The results show that the controllers are able to regulate the link densities to the desired reference densities for different initial conditions.

In the future, we will use the proposed method to design traffic signal controllers for larger traffic networks, and will make it possible to apply the method in real traffic. In addition, we want to further design controllers that not only stabilize the system, but also optimize the future performance. Methods need to be further considered to solve nonlinear matrix inequalities, e.g. the Chang-Yang decoupling approach proposed in [6, 7].

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