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Model Reference Adaptive Stabilizing Control with Application to Leaderless Consensus

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***Abstract*—This paper describes an extension of the well-known model reference adaptive control (MRAC) approach. The extension relies on explicitly involving the tracking error in the feedback control law: it is shown that including this term along with its appropriate extra adaptive gain allows to handle possibly unstable reference dynamics. Due to its stabilizing nature, the proposed framework is referred to as model reference adaptive stabilizing control (MRASC). Such an extension turns out to be particularly useful in leaderless consensus of heterogeneous uncertain agents, since the literature has discussed that leaderless adaptation may not avoid unstable closed-loop dynamics. In such consensus setting, the framework, referred to as model reference adaptive stabilizing consensus (MRASCon), generalizes existing MRAC-based consensus schemes and can achieve consensus when state-of-the-art MRAC-based schemes may fail.**

***Index Terms*—MRAC, adaptive stabilization, consensus, directed spanning tree, multiagent systems.**

I. INTRODUCTION

Abbreviations:	
MRAC	Model Reference Adaptive Control
MRACon	Model Reference Adaptive Consensus
CMRAC	Closed-loop Model Reference Adaptive Control
CMRACon	Closed-loop Model Reference Adaptive Consensus
MRASC	Model Reference Adaptive Stabilizing Control
MRASCon	Model Reference Adaptive Stabilizing Consensus

Model Reference Adaptive Control (MRAC) is a long-standing and powerful tool for controlling uncertain systems, with the goal of tracking a reference model specified by the designer. As sketched in Fig. 1 (left), the idea of MRAC is to tune the behavior of the closed-loop system so as to match that of the reference model [1]–[3].

Recent years have seen increasing research devoted to improving MRAC. One modification of MRAC is recognized as closed-loop MRAC (CMRAC) [4], [5], sketched in Fig. 1 (right): this comes from the fact that the classic (open-loop) reference model is modified into a closed-loop reference model by explicitly involving the tracking error in the reference dynamics. In CMRAC, the tracking problem is separated into two objectives, i.e., the tracking of the closed-loop reference, and the convergence of the closed-loop reference to the original open-loop reference. These two objectives arise because the closed-loop reference model plays the role of an observer. This observer introduces a new degree of freedom in CMRAC, which can provide better transient behavior than MRAC under suitable observer gains [4]. Nevertheless, due to its separation nature, CMRAC may tend to deviate from the target open-loop reference model (cf. our simulations in Section IV).

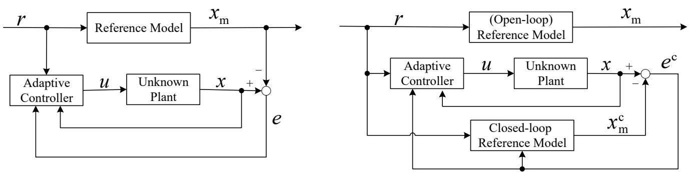

Fig. 1. Block diagrams of MRAC (left) and CMRAC (right). CMRAC modifies the original reference model: the modified closed-loop reference model plays a role of an observer of the original reference model.

When MRAC is applied to multiagent systems, several interesting results have been reported in recent years. For linear heterogeneous uncertain agents, it has been shown that leader-follower tracking is attainable via hierarchical distributed model reference adaptation [6], or via MRAC-based adaptation in both feedback gains and coupling gains [7]. Meanwhile, the idea of CMRAC has also been adopted in a multiagent setting to address leader-follower tracking problems [8], [9].

The problem of leaderless consensus is essentially different from that of leader-follower tracking: without a leader, the agents have to reach consensus by purely collaborating with each other. It has been shown, for a network of linear heterogeneous harmonic oscillators with unknown frequencies, that leaderless consensus is attainable by adaptively learning an a priori unknown group model [10]. More recently, the MRACon framework has been proposed for leaderless consensus [11], assuming that the communication graph among the agents is directed (asymmetric) with a spanning tree structure: this is widely known as a general assumption in the field of multiagent systems [12]–[16].

The main contribution of this paper is twofold:

1) A novel extension of MRAC is proposed, which is called model reference adaptive stabilizing control (MRASC) and is sketched in Fig. 2. Partly inspired by CMRAC [4], [5], MRASC also involves the tracking error for feedback. However, differently from CMRAC, MRASC feeds the tracking error directly into the control loop. As

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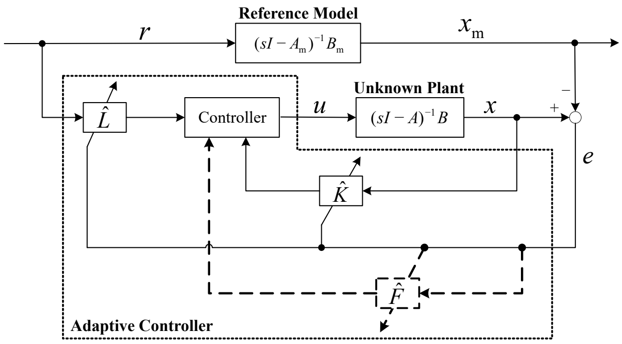

Fig. 2. Diagram of the proposed MRASC approach. MRASC modifies the adaptive controller: the tracking error is explicitly involved for feedback control along with its appropriate extra adaptive gain.

such, MRASC does not modify the target reference model to track. Compared with MRAC/CMRAC, MRASC can exhibit better transient behavior and, more importantly, can handle marginally stable or unstable reference models thanks to a properly designed adaptive stabilization term.

2) Motivated by its ability to track unstable reference models, MRASC is suitably adopted in leaderless multiagent systems, resulting in a new framework named model reference adaptive stabilizing consensus (MRASCon). The novelty comes from the result known in the literature [17] that the consensus manifold in MRAC-based leaderless adaptation cannot be specified a priori and might even be divergent. MRASCon allows to specify the consensus manifold by selecting proper reference dynamics. We show that when unstable closed-loop dynamics arise, embedding state-of-the-art MRAC within the framework of MRACon [11] may fail to solve the consensus problem.

Notations: Denote $\mathbb{R}, \mathbb{C}$ as the real and complex space of numbers, respectively. For a $\lambda \in \mathbb{C}$, denote λ^H as its complex conjugate and $\Re(\lambda)$ as its real part. For a vector a, $\mathrm{span}(a)$ is the real space spanned by a, i.e., $\{\kappa a | \kappa \in \mathbb{R}\}$. For a matrix A, let $\mathrm{null}(A)$ be its zero space; if A is square, let $\lambda_i(A)$ with some subscripts i be its eigenvalues. Let I be the identity matrix and $\mathbf{1}$ be the vector with each element being 1, where the dimensions are omitted when clear from the context. The operator $\otimes$ stands for the Kronecker product.

Basics of graph theory: A directed graph (or simply *digraph*) $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ consists of a node set $\mathcal{V} = \{1, 2, \cdots, N\}$, an edge set $\mathcal{E} = \{\mathcal{E}_{ij} | i \to j, i \neq j\}$, and an weighted adjacency matrix $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N \times N}$ such that $a_{ij} > 0$ if $\mathcal{E}_{ji} \in \mathcal{E}$; and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = (l_{ij}) \in \mathbb{R}^{N \times N}$ associated with $\mathcal{G}$ consists of $l_{ij} = -a_{ij}$ for $i \neq j$, and $l_{ii} = \sum_{j=1}^{N} a_{ij}$. For $\mathcal{E}_{ij} \in \mathcal{E}$, i is called an in-neighbor of j and j an out-neighbor of i in return: $i \in \mathcal{N}_{\text{in}}(j)$ and $j \in \mathcal{N}_{\text{out}}(i)$. A path is a sequence of edges connecting a pair of nodes, which respects the edge directions. A directed spanning tree $\bar{\mathcal{G}}(\mathcal{V}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ of $\mathcal{G}$ is a subgraph with the same nodes and selected edges from $\mathcal{G}$, such that there exists a root (i.e., a node that has no in-neighbors) and one can find a unique path from the root to every other node. Let p_k denote the unique in-neighbor (parent) of node $k+1$ in $\bar{\mathcal{G}}(\mathcal{V}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$, $k = 1, \cdots, N-1$.

II. MRASC: Model Reference Adaptive Stabilizing Control

A. Problem formulation

Consider an LTI system with dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

in which $x \in \mathbb{R}^n$ is the state, and $u \in \mathbb{R}$ is the control input. The matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^n$ are assumed to be *unknown*.

The control objective of a standard MRAC framework is to design u such that the plant state follows the state $x_{\mathrm{m}} \in \mathbb{R}^n$ of a reference model specified by the LTI system

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}} x_{\mathrm{m}}(t) + B_{\mathrm{m}} r(t) \tag{2}$$

where the pair $(A_{\mathrm{m}}, B_{\mathrm{m}})$ has the same dimension as the pair (A, B), and the reference input $r \in \mathbb{R}$ is continuous. The reference model (2) is open-loop as its dynamics will not be directly influenced by the tracking error $e = x - x_{\mathrm{m}}$.

In line with the standard MRAC formulation (see e.g. [2, Ch. 3.3.2] and [3, Ch. 6.2]), the following condition is assumed.

Assumption 1 (Matching condition): There exist ideal gains $K^* \in \mathbb{R}^{1 \times n}$ and $L^* \in \mathbb{R}$ such that

$$A + BK^* = A_{\mathrm{m}}, \quad BL^* = B_{\mathrm{m}} \tag{3}$$

where K^*, L^* are unknown but $\mathrm{sgn}(L^*)$ is a priori known.

However, differently from standard MRAC, the matrix A_{m} is not necessarily stable. Instead, we make the following more general assumption.

Assumption 2 (Stabilizability): The pair $(A_{\mathrm{m}}, B_{\mathrm{m}})$ is stabilizable.

Remark 1: It is known in the MRAC literature that Assumption 1 amounts to imposing some structural requirements so that the unknown plant (1) can match the behavior of the reference model (2) for some unknown gains. Notice that the single-input case is considered here as in the standard MRAC formulation. Addressing multiple inputs is possible provided that other structural conditions are satisfied.

In the following, let us show how the standard MRAC needs to be modified in such a way as to handle the more general Assumption 2.

B. MRASC: design and stability

Under Assumption 2, it follows from [18, Th. 3] that there exists a unique solution $P > 0$ of the algebraic Riccati equation[1]:

$$A_{\mathrm{m}}^T P + P A_{\mathrm{m}} - P B_{\mathrm{m}} B_{\mathrm{m}}^T P + I = 0. \tag{4}$$

Furthermore, upon defining

$$G = -B_{\mathrm{m}}^T P, \tag{5}$$

the matrix $A_{\mathrm{m}} + B_{\mathrm{m}} G$ is asymptotically stable, i.e., G is a stabilizing gain of $(A_{\mathrm{m}}, B_{\mathrm{m}})$.

[1]This holds as (I, A_{m}) is detectable. Equation (4) may be generalized to $A_{\mathrm{m}}^T P + P A_{\mathrm{m}} - P B_{\mathrm{m}} B_{\mathrm{m}}^T P + C^T C = 0$ for a detectable pair (C, A_{m}). For the sake of simplicity, we choose $C = I$.

Consider the following feedback adaptive law, which we call model reference adaptive stabilizing control (MRASC) law:

$$\begin{aligned} u &= \hat{K}(t)x + \hat{L}(t)r + \hat{F}(t)e \\ \dot{\hat{K}} &= -\gamma \operatorname{sgn}(L^*) B_{\mathrm{m}}^T P e x^T \\ \dot{\hat{L}} &= -\gamma \operatorname{sgn}(L^*) B_{\mathrm{m}}^T P e r^T \\ \dot{\hat{F}} &= -\gamma \operatorname{sgn}(L^*) B_{\mathrm{m}}^T P e e^T \end{aligned} \tag{6}$$

where $\hat{K}(t), \hat{L}(t)$ are the estimates of K^*, L^* in (3), respectively, and $\hat{F}(t)$ is the estimate of $F^* \triangleq L^* G$ with G as in (5). Moreover, $\gamma \in \mathbb{R}^+$ is a constant adaptation gain. It is clear that, as compared to the standard MRAC scheme, the proposed scheme (6) includes an explicit feedback gain from the tracking error (cf. Fig. 2), leading to an extra adaptive law. Define $\tilde{K} = \hat{K} - K^*$, $\tilde{L} = \hat{L} - L^*$, and $\tilde{F} = \hat{F} - F^*$ as the parameter estimation errors.

Remark 2: The additional term $\hat{F}e$ can be seen as a consensus term between the reference model (2) and the system (1). The idea of introducing a consensus between the reference model and the system was introduced in closed-loop MRAC (CMRAC) [4], [5]. However, in CMRAC, the consensus term is introduced as a closed-loop action in the reference model (called, for this reason, closed-loop reference model), which modifies its original open-loop dynamics. In (6), the consensus term is introduced in the controller, without modifying the reference dynamics.

We have the following result for MRASC.

Theorem 1: Under Assumptions 1-2, consider the MRASC (6) law applied to the plant (1) with reference model (2). Then, the tracking error e converges to zero asymptotically, and the parameter estimation errors $\tilde{K}, \tilde{L}, \tilde{F}$ are globally uniformly bounded.

Proof. In the ideal case that K^*, L^* are known, so that F^* would also be known, the ideal controller $u^* = K^* x + L^* r + F^* e$ would result in the stable error dynamics

$$\dot{e} = (A_{\mathrm{m}} + B_{\mathrm{m}} G)e, \tag{7}$$

which is stable by the design of G.

However, K^*, L^* are unknown since the pair (A, B) is unknown, which prevents us from using the ideal controller u^*. Nevertheless, u^* can be used for stability analysis by adding and subtracting Bu^* in (1), which leads to the closed-loop plant x as

$$\begin{aligned} \dot{x} =& Ax + B\hat{K}x + B\hat{L}r + B\hat{F}e + BK^*x \\ &+ BL^*r + BF^*e - BK^*x - BL^*r - BF^*e \\ =& A_{\mathrm{m}}x + B_{\mathrm{m}}r + B_{\mathrm{m}}Ge + B(\tilde{K}x + \tilde{L}r + \tilde{F}e). \end{aligned} \tag{8}$$

Then, the error dynamics becomes

$$\begin{aligned} \dot{e} =& (A_{\mathrm{m}} + B_{\mathrm{m}}G)e + B(\tilde{K}x + \tilde{L}r + \tilde{F}e) \\ =& (A_{\mathrm{m}} + B_{\mathrm{m}}G)e + B_{\mathrm{m}}{L^*}^{-1}(\tilde{K}x + \tilde{L}r + \tilde{F}e). \end{aligned} \tag{9}$$

The equation (9) relates the tracking error with the parameter estimation errors, and can be used to prove stability. To proceed, consider the Lyapunov candidate

$$V = e^T P e + \gamma^{-1}\Big(\mathrm{tr}(\tilde{K}^T \Gamma \tilde{K}) + \mathrm{tr}(\tilde{L}^T \Gamma \tilde{L}) + \mathrm{tr}(\tilde{F}^T \Gamma \tilde{F})\Big) \tag{10}$$

where P is defined in (4) and $\Gamma \triangleq {L^*}^{-1} \operatorname{sgn}(L^*)$. Note that both P and Γ are positive definite.

The derivative of (10) along the trajectory of (9) is given by

$$\begin{aligned} \dot{V} =& 2e^T P(A_{\mathrm{m}} + B_{\mathrm{m}}G)e + 2e^T P B_{\mathrm{m}} {L^*}^{-1}(\tilde{K}x + \tilde{L}r + \tilde{F}e) \\ &+ 2\gamma^{-1}\Big(\mathrm{tr}(\tilde{K}^T \Gamma \dot{\hat{K}}) + \mathrm{tr}(\tilde{L}^T \Gamma \dot{\hat{L}}) + \mathrm{tr}(\tilde{F}^T \Gamma \dot{\hat{F}})\Big). \end{aligned} \tag{11}$$

Substituting (4) and (5) leads to

$$\begin{aligned} \dot{V} =& -e^T(I + PB_{\mathrm{m}}B_{\mathrm{m}}^T P)e \\ &+ 2e^T P B_{\mathrm{m}} {L^*}^{-1}(\tilde{K}x + \tilde{L}r + \tilde{F}e) \\ &+ 2\gamma^{-1}\Big(\mathrm{tr}(\tilde{K}^T \Gamma \dot{\hat{K}}) + \mathrm{tr}(\tilde{L}^T \Gamma \dot{\hat{L}}) + \mathrm{tr}(\tilde{F}^T \Gamma \dot{\hat{F}})\Big). \end{aligned} \tag{12}$$

Note that

$$\begin{aligned} e^T P B_{\mathrm{m}} {L^*}^{-1} \tilde{K} x =& x^T \tilde{K}^T {L^*}^{-1} B_{\mathrm{m}}^T P e \\ =& \mathrm{tr}(\tilde{K}^T {L^*}^{-1} B_{\mathrm{m}}^T P e x^T) \end{aligned} \tag{13}$$

where the second equality holds since the trace operator is invariant under cyclic permutations. Clearly, with the definition of Γ and the designed $\dot{\hat{K}}$ in (6),

$$\begin{aligned} & e^T P B_{\mathrm{m}} {L^*}^{-1} \tilde{K} x + \gamma^{-1} \mathrm{tr}(\tilde{K}^T \Gamma \dot{\hat{K}}) \\ =& e^T P B_{\mathrm{m}} {L^*}^{-1} \tilde{K} x - \mathrm{tr}(\tilde{K}^T {L^*}^{-1} B_{\mathrm{m}}^T P e x^T) = 0. \end{aligned} \tag{14}$$

Similar analysis as (13)-(14) can be performed for the other terms related with $\tilde{L}$ and $\tilde{F}$. Then, it follows from (12) that

$$\dot{V} = -e^T(I + PB_{\mathrm{m}}B_{\mathrm{m}}^T P)e \leq 0. \tag{15}$$

Let us denote the right-hand side of equation (9) as $\dot{e} = f(e, t)$. Since both f and $\frac{\partial f}{\partial e}$ are continuous, $f(e, t)$ is locally Lipschitz in e uniformly in t [19, Lem. 3.2]. Then, applying the LaSalle-Yoshizawa Theorem [1, Th. 2.1], all signals e, $\tilde{K}$, $\tilde{L}$, $\tilde{F}$ are globally uniformly bounded and

$$\lim_{t\to\infty} e^T(I + PB_{\mathrm{m}}B_{\mathrm{m}}^T P)e = 0 \Rightarrow \lim_{t\to\infty} e = 0. \tag{16}$$

This completes the proof. ∎

Remark 3: In MRAC and CMRAC, the matrix A_{m} of the reference model must be stable. The proof of Theorem 1 shows that the novel adaptive stabilization term in (6) makes the tracking error converge to zero even without the condition that A_{m} is stable. This feature endows MRASC with the power to track an unstable reference model with asymptotic convergent errors: though possibly resulting in unbounded control signals at infinity, in practice one may exploit such a feature in a short time interval to generate certain patterns in the state response, e.g., spikes, as discussed in [20].

Remark 4: As known from classical MRAC results, there is no guarantee in general that the estimates $\hat{K}$, $\hat{L}$, $\hat{F}$ converge to their ideal values (cf. [3, Ch. 6.4] and recent relaxed persistence of excitation conditions to guarantee convergence of the estimation errors to zero [21], [22]). With this in mind, we will address a consensus problem in the following section, where the proposed MRASC law plays an important role.

III. MRASCon: Model Reference Adaptive Stabilizing Consensus

A. Problem formulation

Consider a network of N *heterogeneous* LTI agents where the dynamics of the i-th ($i \in \{1, 2, \cdots, N\}$) agent follows

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) \tag{17}$$

in which $x_i \in \mathbb{R}^n$ is the state, and $u_i \in \mathbb{R}$ is the control input. The matrices $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^n$ are assumed to be *unknown*. The interaction graph between the agents (17) is a digraph denoted by $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ or simply $\mathcal{G}$.

The control objective is to design u_i such that the agents reach consensus, i.e., $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, $\forall i, j \in \mathcal{V}$. Along with the previous Assumption 2, the following assumptions are made.

Assumption 3 (Decentralized matching condition): There exists a family of homogenization gains $K_i^* \in \mathbb{R}^{1\times n}$ and $L_i^* \in \mathbb{R}$ such that

$$A_i + B_i K_i^* = A_{\rm m}, \quad B_i L_i^* = B_{\rm m} \tag{18}$$

where K_i^*, L_i^* are unknown but $\operatorname{sgn}(L_i^*)$ is a priori known, $\forall i \in \mathcal{V}$.

Assumption 4 (Connectivity): The communication digraph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ contains a directed spanning tree $\bar{\mathcal{G}}(\mathcal{V}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ (the specific structure of the tree is not necessarily known).

Remark 5: Assumption 3 extends Assumption 1 from a single plant to a multiagent network. Assumption 4 is standard in the field of multiagent systems [12]–[16]. In fact, undirected or directed acyclic communication graphs are often required in state-of-the-art distributed MRAC [6]–[10], which is more conservative than Assumption 4.

B. A technical lemma

Under Assumption 4, let us construct two matrices based on $\bar{\mathcal{G}}(\mathcal{V}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$. Define $\Xi \in \mathbb{R}^{(N-1)\times N}$ as

$$\Xi_{kj} = \begin{cases} -1, & \text{if } j = k+1, \\ 1, & \text{if } j = \mathrm{p}_k, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

In fact, Ξ is the difference matrix along the tree: $\Xi x = (x_{\mathrm{p}_1} - x_2, x_{\mathrm{p}_2} - x_3, \cdots, x_{\mathrm{p}_{N-1}} - x_N)^T$ for any vector x. Define $Q \in \mathbb{R}^{(N-1)\times(N-1)}$ as

$$Q_{kj} = \sum_{c \in \bar{\mathcal{V}}_{j+1}} (\mathcal{L}_{k+1,c} - \mathcal{L}_{\mathrm{p}_k,c}), \tag{20}$$

where $\bar{\mathcal{V}}_{j+1}$ represents the vertex set of the subtree of $\bar{\mathcal{G}}$ rooting at node $j+1$. Note that $\mathcal{L}$ is the Laplacian matrix of the communication digraph $\mathcal{G}$.

Lemma 1 ([15], [16]): Under Assumption 4, the following statements hold for $\mathcal{L}$ (of $\mathcal{G}$), and Ξ, Q defined above:

1) $0 = \lambda_1(\mathcal{L}) < \Re(\lambda_2(\mathcal{L})) \le \Re(\lambda_3(\mathcal{L})) \le \cdots \le \Re(\lambda_N(\mathcal{L}))$. Moreover, $\operatorname{null}(\mathcal{L}) = \operatorname{span}(\mathbf{1}_N)$.
2) $\Xi\mathcal{L} = Q\Xi$. Moreover, $\operatorname{null}(\Xi) = \operatorname{span}(\mathbf{1}_N)$.
3) $\lambda_i(Q) = \lambda_{i+1}(\mathcal{L})$, $i = 1, \cdots, N-1$.

C. MRASCon: design and stability

Inspired by MRACon [11], we propose a novel MRASCon framework in this section. The idea of MRASCon is to separate the consensus problem into two parts: the decentralized tracking of each agent to a local reference model by MRASC established in the previous section, and the consensus over the reference models by manipulating their external inputs. This results in a reference model for each agent $i \in \mathcal{V}$ as

$$\begin{aligned} \dot{x}_{\mathrm{m},i}(t) &= A_{\rm m} x_{\mathrm{m},i}(t) + B_{\rm m} r_i(t), \\ r_i(t) &= r(t) + cG \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) \end{aligned} \tag{21}$$

where $A_{\rm m}, B_{\rm m}$ are defined in (18), a_{ij} is from the adjacency matrix $\mathcal{A}$, and $r(t)$ is a user-designed common reference input playing a similar role as the reference model input in (2). Here, $c \in \mathbb{R}^+$ is the coupling gain of the network and $G \in \mathbb{R}^{1\times n}$ is the stabilizing gain defined as in (5). The reference models (21) are open-loop as they are not affected by the tracking errors $e_i = x_i - x_{\mathrm{m},i}$, $i \in \mathcal{V}$.

Consider the following feedback adaptive law, which we call model reference adaptive stabilizing consensus (MRASCon) law:

$$\begin{aligned} u_i &= \hat{K}_i(t) x_i + \hat{L}_i(t) r_i + \hat{F}_i(t) e_i \\ \dot{\hat{K}}_i &= -\gamma_i \operatorname{sgn}(L_i^*) B_{\rm m}^T P e_i x_i^T \\ \dot{\hat{L}}_i &= -\gamma_i \operatorname{sgn}(L_i^*) B_{\rm m}^T P e_i r_i^T \\ \dot{\hat{F}}_i &= -\gamma_i \operatorname{sgn}(L_i^*) B_{\rm m}^T P e_i e_i^T \end{aligned} \tag{22}$$

where $\hat{K}_i(t), \hat{L}_i(t)$ are the estimates of K_i^*, L_i^* in (18), respectively, and $\hat{F}_i(t)$ is the estimate of $F_i^* \triangleq L_i^* G$. Moreover, $\gamma_i \in \mathbb{R}^+$ is the constant gain for adaptation.

Theorem 2: Under Assumptions 2-4, consider the MRASCon law (22) applied to the multiagent system (17) with reference models (21). Then, for each agent $i \in \mathcal{V}$, all tracking errors e_i converge to zero asymptotically, and all parameter estimation errors $\tilde{K}_i, \tilde{L}_i, \tilde{F}_i$ are globally uniformly bounded. Furthermore, if $c \ge \frac{1}{2\Re(\lambda_2(\mathcal{L}))}$, the multiagent system (17) reaches consensus asymptotically.

Proof. Following similar steps as in the proof of Theorem 1, the dynamics of e_i can be obtained as

$$\dot{e}_i = (A_{\rm m} + B_{\rm m} G) e_i + B_i(\tilde{K}_i x_i + \tilde{L}_i r_i + \tilde{F}_i e_i). \tag{23}$$

Similarly to (10), consider the Lyapunov candidate

$$V_i = e_i^T P e_i + \gamma_i^{-1}\big(\mathrm{tr}(\tilde{K}_i^T \Gamma_i \tilde{K}_i) + \mathrm{tr}(\tilde{L}_i^T \Gamma_i \tilde{L}_i) + \mathrm{tr}(\tilde{F}_i^T \Gamma_i \tilde{F}_i)\big) \tag{24}$$

where P is defined in (4) and $\Gamma_i \triangleq L_i^{*-1} \operatorname{sgn}(L_i^*) > 0$. Along similar lines as the proof of Theorem 1, one can conclude that the closed-loop error signals $e_i, \tilde{K}_i, \tilde{L}_i, \tilde{F}_i$ are globally uniformly bounded, and the tracking errors e_i converge to zero asymptotically.

Let $\mathbf{x}_{\rm m}, \mathbf{x}, \mathbf{e}$ be the stacked vectors of $x_{{\rm m},i}, x_i, e_i$, respectively (e.g., $\mathbf{x}_{\rm m} = (x_{{\rm m},1}^T, x_{{\rm m},2}^T, \cdots, x_{{\rm m},N}^T)^T$). It follows from (21) that

$$\begin{aligned}\dot{\mathbf{x}}_{\rm m} &= (I \otimes A_{\rm m})\mathbf{x}_{\rm m} + c(\mathcal{L} \otimes B_{\rm m}G)\mathbf{x} + (\mathbf{1}_N \otimes B_{\rm m}r) \\ &= (I \otimes A_{\rm m} + c\mathcal{L} \otimes B_{\rm m}G)\mathbf{x}_{\rm m} \\ &\quad + c(\mathcal{L} \otimes B_{\rm m}G)\mathbf{e} + (\mathbf{1}_N \otimes B_{\rm m}r). \end{aligned} \tag{25}$$

Let $\bar{\mathbf{x}}_{\rm m} = (\Xi \otimes I)\mathbf{x}_{\rm m}$ where Ξ is defined as in (19). Since $\text{null}(\Xi) = \text{span}(\mathbf{1}_N)$ (see Lemma 1), one has $\Xi\mathbf{1}_N = 0$, and $\bar{\mathbf{x}}_{\rm m} = 0$ if and only if $x_{{\rm m},i} = x_{{\rm m},j}$ for any $i, j \in \mathcal{V}$. Based on statement 2 of Lemma 1, we have

$$\dot{\bar{\mathbf{x}}}_{\rm m} = (I \otimes A_{\rm m} + cQ \otimes B_{\rm m}G)\bar{\mathbf{x}}_{\rm m} + c(\Xi\mathcal{L} \otimes B_{\rm m}G)\mathbf{e} \tag{26}$$

where Q is defined as in (20).

Next, we claim that the first part of the dynamics in (26) is stable, i.e., the matrix $I \otimes A_{\rm m} + cQ \otimes B_{\rm m}G$ is Hurwitz. Note that any square matrix is unitarily similar to an upper triangular matrix with diagonal entries being its eigenvalues. When this fact is applied to Q and noticing statement 3) of Lemma 2, it is sufficient to show that the upper triangular matrix

$$\begin{pmatrix} A_{\rm m} + c\lambda_2(\mathcal{L})B_{\rm m}G & & \star \\ & \ddots & \\ 0 & & A_{\rm m} + c\lambda_N(\mathcal{L})B_{\rm m}G \end{pmatrix}$$

is Hurwitz, which is then equivalent to show that the blocks on the main diagonal are Hurwitz. In fact, based on (4) and (5),

$$\begin{aligned} &(A_{\rm m} + c\lambda_i(\mathcal{L})B_{\rm m}G)^H P + P(A_{\rm m} + c\lambda_i(\mathcal{L})B_{\rm m}G) \\ =&A_{\rm m}^T P + PA_{\rm m}^T - c\lambda_i(\mathcal{L})^H PB_{\rm m}B_{\rm m}^T P - c\lambda_i(\mathcal{L})PB_{\rm m}B_{\rm m}^T P \\ =&A_{\rm m}P + PA_{\rm m}^T - 2c\Re(\lambda_i(\mathcal{L}))PB_{\rm m}B_{\rm m}^T P \\ =&-I + (1 - 2c\Re(\lambda_i(\mathcal{L})))PB_{\rm m}B_{\rm m}^T P. \end{aligned} \tag{27}$$

for any $i \in \{2, 3, \cdots, N\}$. Then, provided $c \geq \frac{1}{2\Re(\lambda_2(\mathcal{L}))}$, the blocks $A_{\rm m} + c\lambda_i(\mathcal{L})B_{\rm m}G$ are indeed Hurwitz.

Since $\lim_{t\to\infty} \mathbf{e} = 0$ and the dynamics of $\bar{\mathbf{x}}_{\rm m}$ in (26) is internally stable, we have $\lim_{t\to\infty} \bar{\mathbf{x}}_{\rm m} = 0$, which implies the consensus over $x_{{\rm m},i}$.

Now since $x_i \to x_{{\rm m},i}$ and $x_{{\rm m},i} \to x_{{\rm m},j}$ as $t \to \infty$, we conclude that $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, $\forall i, j \in \mathcal{V}$. This completes the proof. ■

Remark 6: Similarly to Theorem 1, Theorem 2 confirms that MRASCon drives the consensus error to zero asymptotically, even in the case that the homogeneous dynamics matrix $A_{\rm m}$ not necessarily stable. As known from [17], MRAC-based leaderless consensus does not allow to specify the consensus manifold a priori since the consensus manifold is completely unknown and it can be either bounded or divergent depending on the initial conditions and the system dynamics. MRASCon allows to specify rich classes of consensus manifolds a priori via proper design of the reference dynamics matrix $A_{\rm m}$ and the reference input $r(t)$.

Remark 7: The reference input $r(t)$ in (21) can be designed to manipulate the leaderless consensus manifold, as will be illustrated in Example 3 of Sect. IV; however, this requires the knowledge of a common reference among all agents. In case no such knowledge is allowed, it is still possible to set $r(t) \equiv 0$ in (21), in which case the consensus manifold will be specified only by $A_{\rm m}$.

Remark 8: MRASCon (22) encompasses MRACon [11] in the case of known and homogeneous system matrices. Specifically, [11] considers the special case when the agents' dynamics follows a (known) homogeneous pair $(A_{\rm m}, B_{\rm m})$ perturbed by some (unknown) matching gain K_i^*, i.e.,

$$\dot{x}_i = A_{\rm m}x_i + B_{\rm m}(u_i - \Phi_i(t, x_i)K_i^*). \tag{28}$$

Here, Φ_i is a known and bounded continuous function. In this case, the reference models can be designed the same as in (21), considering that $r(t) \equiv 0$ in [11]. Due to the fact that the special dynamics (28) imposes $B_i = B_{\rm m}$, it follows that $L_i^* = I$ and $F_i^* = G$ (with G defined as in (5)), $\forall i$. As these matrices are known, the controller (22) degenerates to MRACon proposed in [11]:

$$\begin{aligned} u_i &= \Phi_i\hat{K}_i(t) + r_i + Ge_i \\ \dot{\hat{K}}_i &= -\gamma_i\Phi_i^T B_{\rm m}^T Pe_i. \end{aligned}$$

The above discussions also imply that MRACon is not directly applicable when the agent dynamics (A_i, B_i) is heterogeneous and unknown, as is the case considered in this paper.

IV. Illustrative Examples

Example 1 (MRASC): For illustration, Table 1 summarizes the classical MRAC, CMRAC, and the proposed MRASC law for a scalar unknown plant. Then we take $a = 1$, $b = 2$ which are assumed to be unknown for the control design.

TABLE I
MRAC, CMRAC and MRASC for a scalar unknown plant.

Plant	$\dot{x} = ax + bu$; only sgn(b) is known, $b \neq 0$ (controllable).
Reference	$\dot{x}_{\rm m} = a_{\rm m}x_{\rm m} + b_{\rm m}r$; $a_{\rm m}, b_{\rm m}, r$ are user-specified, $r(t)$ is bounded.
MRAC (e.g., [3])	$u = \phi^T\theta(t)$, $\dot{\theta} = -\gamma\,\text{sgn}(b)\phi e$. $\phi = (x, r)^T$, $\theta = (\hat{k}, \hat{l})^T$, $e = x - x_{\rm m}$.
CMRAC ([4], [5])	$u = \phi^T\theta(t)$, $\dot{\theta} = -\gamma\,\text{sgn}(b)\phi e^{\rm c}$. $\phi = (x, r)^T$, $\theta = (\hat{k}, \hat{l})^T$, $e^{\rm c} = x - x_{\rm m}^{\rm c}$. $\dot{x}_{\rm m}^{\rm c} = a_{\rm m}x_{\rm m}^{\rm c} + b_{\rm m}r - \rho e^{\rm c}$; $\rho < 0$.
MRASC	$u = \phi^T\theta(t)$, $\dot{\theta} = -\gamma\,\text{sgn}(b)\phi e$. $\phi = (x, r, e)^T$, $\theta = (\hat{k}, \hat{l}, \hat{f})^T$, $e = x - x_{\rm m}$.

We select $b_{\rm m} = 1$, $r = \sin(t)$; let $\rho = -1$ for CMRAC and $\gamma = 1$ for all three methods. Then, we select $a_{\rm m} = -1$ for a stable reference and $a_{\rm m} = 0.1$ for an unstable one, respectively, for comparisons. All initial values are set to zero.

The simulation results are shown in Fig. 3 (stable reference model) and Fig. 4 (unstable reference model). When tracking the stable reference model, MRASC improves the transient of classical MRAC thanks to the adaptive stabilization process; meanwhile, MRASC realizes faster tracking as compared with CMRAC since MRASC targets the reference model $x_{\rm m}$ directly instead of the closed-loop reference model $x_{\rm m}^{\rm c}$. More importantly, Fig. 4 shows that MRASC can track the unstable reference asymptotically while MRAC fails and CMRAC realizes bounded tracking with some bias. The estimated

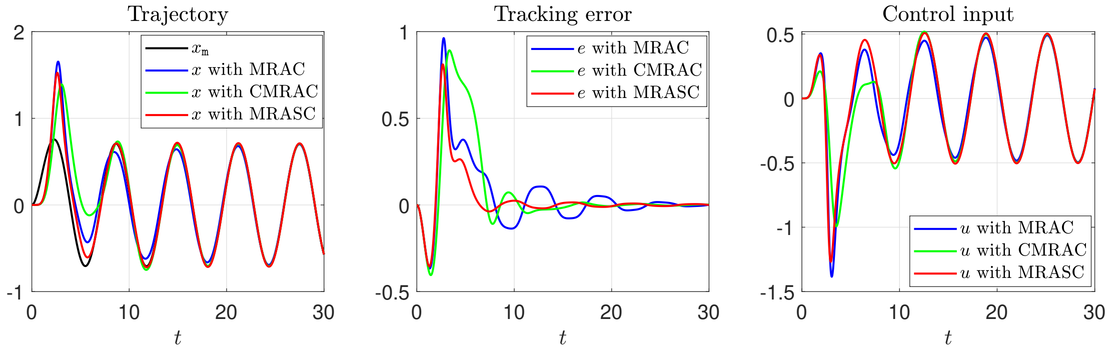

Fig. 3. Example 1: MRAC, CMRAC, and MRASC (6) for tracking a stable reference model ($a_{\mathrm{m}} = -1$).

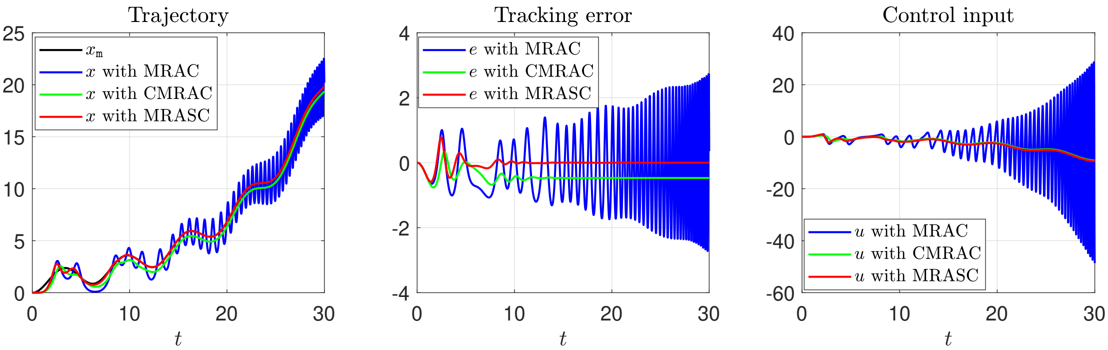

Fig. 4. Example 1: MRAC, CMRAC, and MRASC (6) for tracking an unstable reference model ($a_{\mathrm{m}} = 0.1$).

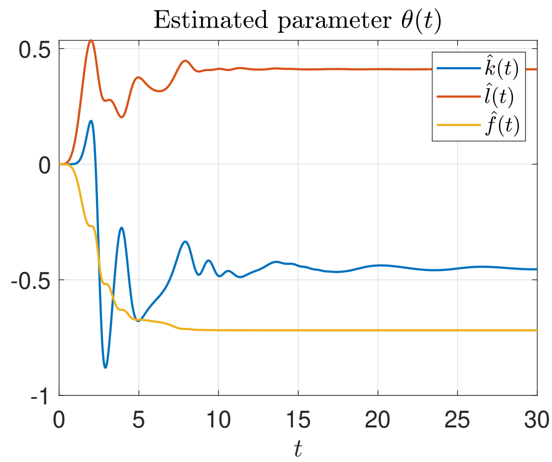

Fig. 5. Example 1: Estimated parameter $\theta(t)$ (with $a_{\mathrm{m}} = 0.1$).

parameters in the unstable case converge to some finite values, as shown in Fig. 5.

Example 2 (MRASCon): Consider a network of $N = 6$ heterogeneous unknown second-order agents. Their true parameters (unknown for control design) are

$$A_i = \begin{pmatrix} 0 & 1 \\ i & i \end{pmatrix}, \; B_i = \begin{pmatrix} 0 \\ (-1)^i i \end{pmatrix}. \tag{29}$$

The communication topology among the agents is a directed ring with edge set $\mathcal{E} = \{\mathcal{E}_{12}, \mathcal{E}_{23}, \mathcal{E}_{34}, \mathcal{E}_{45}, \mathcal{E}_{56}, \mathcal{E}_{61}\}$ along with unitary weights: in this case, $\Re(\lambda_2(\mathcal{L})) = 0.5$, leading to the lower bound of the coupling gain $c \geq 1$. Let us select $c = 1$.

Consider a stabilizable (but unstable) pair

$$A_{\mathrm{m}} = \begin{pmatrix} 0 & 1 \\ -1 & 0.2 \end{pmatrix}, \; B_{\mathrm{m}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{30}$$

The control direction $\mathrm{sgn}(L_i^*)$ is known as $\mathrm{sgn}(L_i^*) = -1$ for $i = 1, 3, 5$, and $\mathrm{sgn}(L_i^*) = 1$ for $i = 2, 4, 6$. It can be easily verified that Assumptions 2-4 hold. Assume $r(t) \equiv 0$ in (21).

Solving the algebraic Riccati equation (4) with the `are` command in Matlab gives

$$P = \begin{pmatrix} 1.4765 & 0.3090 \\ 0.3090 & 0.6880 \end{pmatrix},$$

resulting in $G = (-0.6180, -1.3759)$.

In order to highlight the advantages of MRASCon as defined in (22), let us first consider two methods, namely MRACon*[2] and CMRACon. MRACon* can be obtained by simply removing the adaptive feedback term $F_i(t)e_i$ from the MRASCon law (22). CMRACon is the leaderless consensus version of CMRAC [4], [5], and is expressed as follows

$$\begin{aligned} u_i &= \hat{K}_i(t)x_i + \hat{L}_i(t)r_i \\ \dot{\hat{K}}_i &= -\gamma_i \, \mathrm{sgn}(L_i^*) B_{\mathrm{m}}^T P e_i^{\mathrm{cl}} x_i^T \\ \dot{\hat{L}}_i &= -\gamma_i \, \mathrm{sgn}(L_i^*) B_{\mathrm{m}}^T P e_i^{\mathrm{cl}} r_i^T \\ \dot{x}_{\mathrm{m},i}^{\mathrm{cl}} &= A_{\mathrm{m}} x_{\mathrm{m},i}^{\mathrm{cl}} + B_{\mathrm{m}} r_i - \rho e_i^{\mathrm{cl}} \end{aligned} \tag{31}$$

[2]To avoid confusion, notice that MRACon* is not the same as MRACon in [11]. In fact, MRACon in [11] is not directly applicable to heterogeneous and unknown agents (cf. Remark 8). Therefore, MRACon* is a reformulation of the method in [11] to handle heterogeneous and unknown agents.

where $e_i^{\text{cl}} = x_i - x_{\text{m},i}^{\text{cl}}$, $r_i = cG\sum_{j=1}^{N} a_{ij}(x_i - x_j)$ and $\rho < 0$ is an observer gain. Clearly, CMRACon converges to MRACon* as ρ converges to zero.

Without the adaptive stabilization term, both MRACon* and CMRACon give a closed-loop dynamics of x_i as

$$\dot{x}_i = A_{\text{m}} x_i + B_{\text{m}} r_i + B_i(\tilde{K}_i x_i + \tilde{L}_i r_i).$$

This indicates that a singular case may happen near the origin $\mathbf{x} = 0$: the states of all agents decay to zero, while all local (open-loop) reference models diverge to infinity. This singular case is extremely undesired since the agents are expected to track the corresponding local reference models. Fig. 6 shows an example of this singular case occurring in MRACon*, where, although consensus (to zero) is achieved, the tracking task is completely neglected. In the simulation of Fig. 6, we select $\gamma_i = 1$; the initial adaptive gains are set to zero, and the initial states of the agents and the reference models are randomly chosen according to a Gaussian distribution with standard derivation 5.

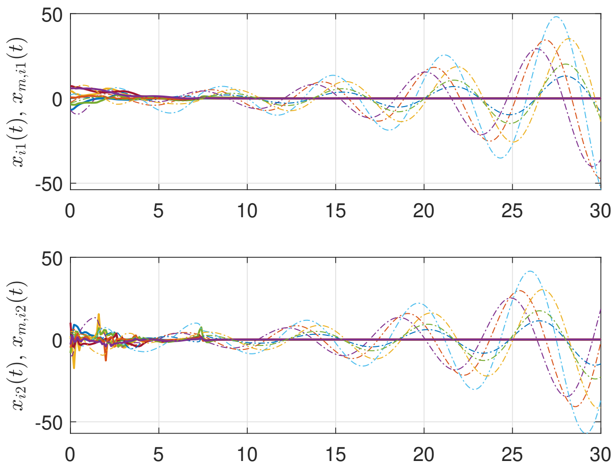

Fig. 6. Example 2: States of the agents under MRACon* law (obtained by removing $F_i(t)e_i$ in MRASCon (22)). The agents (solid lines) reach a trivial consensus at zero, but fail to track the corresponding local reference models (dashed lines): that is, a trivial consensus is reached without tracking.

CMRACon law (31) can also make the agents trapped into the origin $\mathbf{x} = 0$, especially when ρ is small in magnitude: in this case, the closed-loop reference model acts more as a reference model than as an observer. For example, by selecting $\rho = -1$, CMRACon leads to similar results as in Fig. 6 (omitted due to the space limits), i.e., a trivial consensus is reached without tracking. In addition, we show in Fig. 7 that, even with a larger (in magnitude) $\rho = -100$, CMRACon fails to drive the agents to consensus. The main reason for this phenomenon is that CMRAC cannot guarantee asymptotic tracking performance when the reference model is unstable.

The scenarios mentioned in Fig. 6-7 will not occur in MRASCon since all tracking errors e_i are guaranteed to converge to zero asymptotically, i.e., consensus is achieved along with tracking of the reference models. Fig. 8 shows the results of the proposed MRASCon approach (22) with the same initial conditions as in Fig. 6-7. The comparisons

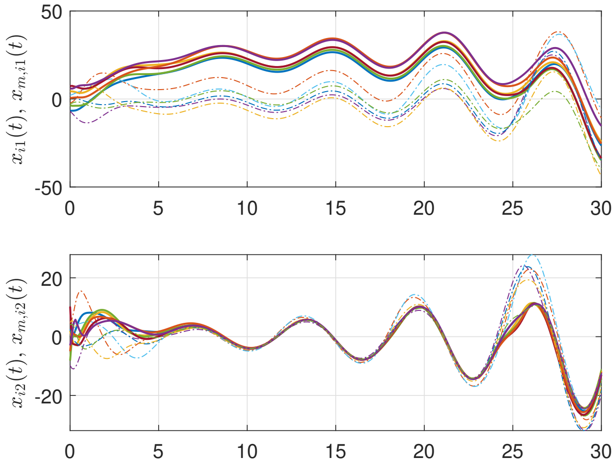

Fig. 7. Example 2: States of the agents under CMRASCon law (31) with $\rho = -100$. The agents (solid lines) fail to accomplish the consensus and tracking tasks.

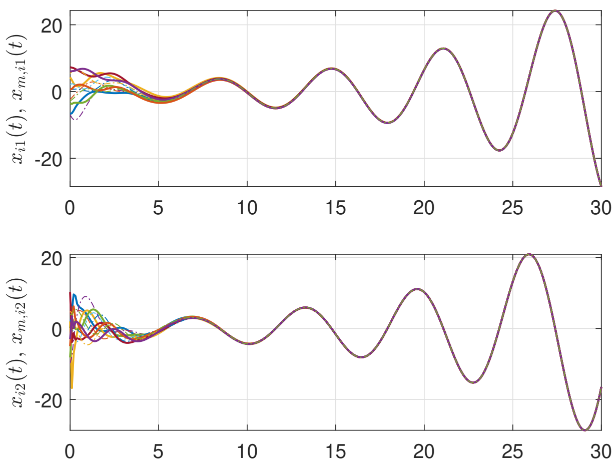

Fig. 8. Example 2: States of the agents under the proposed MRASCon law (22) with the same initial conditions as in Fig. 6-7. Consensus and tracking are both achieved.

highlight the effectiveness of MRASC and MRASCon in tracking unstable reference dynamics.

Example 3 (MRASCon with switching reference models): To support Remarks 6-7, this example illustrates how the final consensus value can be easily designed by MRASCon. As a matter of fact, MRASCon is able to specify a priori the consensus dynamics via A_{m} and $r(t)$, thanks to the guarantee of asymptotic tracking and consensus. Note that a priori knowledge of the consensus manifold is absent in [17] since the unstable consensus manifold depends on the unknown system dynamics and thus cannot be specified a priori.

Let us consider the following specifications for consensus of agents (29): diverging for 50 seconds; then oscillating for 30 seconds; finally converging to a stable equilibrium point $(200; 0)$. This can be accomplished by choosing A_{m} to be the same as in (30) with $r(t) = 0$ for $t \leq 50$; $A_\text{m} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $r(t) = 0$ for $50 < t \leq 80$; and $A_\text{m} = \begin{pmatrix} 0 & 1 \\ -1 & -0.5 \end{pmatrix}$ with $r(t) = 100$ thereafter. The states and control inputs of the agents are shown in Fig. 9. The consensus behavior meets the prior specifications, thus resulting in a predictable behavior of the leaderless, heterogeneous, and uncertain agents.

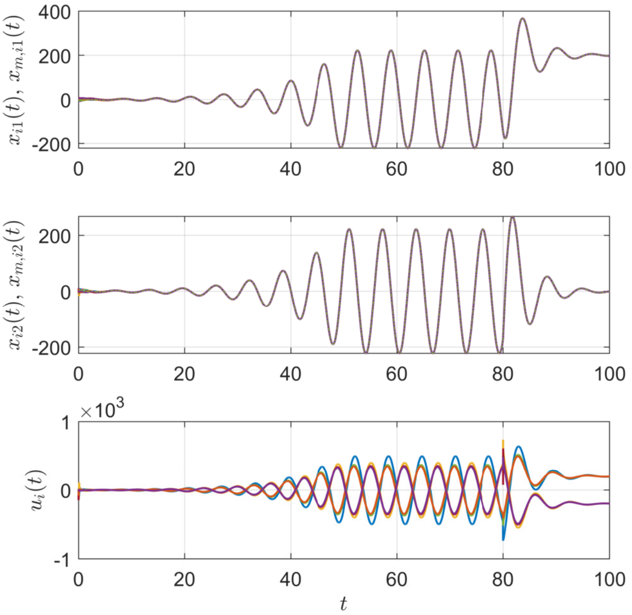

Fig. 9. Example 3: States and control inputs of the agents under the proposed MRASCon law (22) with switching reference models (unstable for $t \leq 50$; marginally stable for $50 < t \leq 80$; stable for $t > 80$).

V. Conclusions

We have proposed a natural extension of the classical MRAC approach, where the main novelty arises from an extra adaptive stabilization loop. For this reason, this extension is called model reference adaptive stabilizing control (MRASC). It has been rigorously proved that with MRASC the tracking error is driven to zero even when the reference model is unstable. This feature, absent in state-of-the-art MRAC schemes, has allowed us to extend MRASC into a model reference adaptive stabilizing consensus (MRASCon) law for a leaderless consensus problem.

We can identify two future lines of research. The first is to study MRASC with switched dynamics rigorously: by handling unstable reference dynamics, MRASC can potentially address switched reference dynamics where one or more subsystems are unstable. The second is to further increase the allowed uncertainty in the multiagent MRASCon scenario, e.g., to get rid of the knowledge of the Laplacian eigenvalues, possibly with the methods of [13]–[16], [23].

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References

[1] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*. Wiley New York, 1995.

[2] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. Courier Corporation, 2012.

[3] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Courier Corporation, 2012.

[4] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, "On adaptive control with closed-loop reference models: transients, oscillations, and peaking," *IEEE Access*, vol. 1, pp. 703–717, 2013.

[5] T. E. Gibson, Z. Qu, A. M. Annaswamy, and E. Lavretsky, "Adaptive output feedback based on closed-loop reference models," *IEEE Trans. Autom. Control*, vol. 60, no. 10, pp. 2728–2733, 2015.

[6] S. Baldi, S. Yuan, and P. Frasca, "Output synchronization of unknown heterogeneous agents via distributed model reference adaptation," *IEEE Trans. Control Netw. Syst.*, vol. 6, no. 2, pp. 515–525, 2018.

[7] I. A. Azzollini, W. Yu, S. Yuan, and S. Baldi, "Adaptive leader–follower synchronization over heterogeneous and uncertain networks of linear systems without distributed observer," *IEEE Trans. Autom. Control*, vol. 66, no. 4, pp. 1925–1931, 2020.

[8] R. Goel and S. B. Roy, "Closed-loop reference model based distributed model reference adaptive control for multi-agent systems," in *Proc. Amer. Control Conf.* IEEE, 2021, pp. 1082–1087.

[9] R. Goel, T. Garg, and S. B. Roy, "Closed-loop reference model based distributed MRAC using cooperative initial excitation and distributed reference input estimation," *IEEE Trans. Control Netw. Syst.*, vol. 9, no. 1, pp. 37–49, 2022.

[10] S. Baldi and P. Frasca, "Leaderless synchronization of heterogeneous oscillators by adaptively learning the group model," *IEEE Trans. Autom. Control*, vol. 65, no. 1, pp. 412–418, 2019.

[11] J. Mei, W. Ren, and Y. Song, "A unified framework for adaptive leaderless consensus of uncertain multi-agent systems under directed graphs," *IEEE Trans. Autom. Control*, vol. 66, no. 12, pp. 6179–6186, 2021.

[12] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Trans. Circuits Syst. I, Reg. Papers,*, vol. 57, no. 1, pp. 213–224, 2010.

[13] W. Yu, J. Lü, X. Yu, and G. Chen, "Distributed adaptive control for synchronization in directed complex networks," *SIAM J. Control Optim.*, vol. 53, no. 5, pp. 2980–3005, 2015.

[14] Z. Yu, D. Huang, H. Jiang, C. Hu, and W. Yu, "Distributed consensus for multiagent systems via directed spanning tree based adaptive control," *SIAM J. Control Optim.*, vol. 56, no. 3, pp. 2189–2217, 2018.

[15] D. Yue, S. Baldi, J. Cao, Q. Li, and B. De Schutter, "A directed spanning tree adaptive control solution to time-varying formations," *IEEE Trans. Control Netw. Syst.*, vol. 8, no. 2, pp. 690–701, 2021.

[16] D. Yue, S. Baldi, J. Cao, and B. De Schutter, "Distributed adaptive optimization with weight-balancing," *IEEE Trans. Autom. Control*, vol. 67, no. 4, pp. 2068–2075, 2022.

[17] K. S. Narendra and P. Harshangi, "Unstable systems stabilizing each other through adaptation," in *Proc. Amer. Control Conf.* IEEE, 2014, pp. 7–12.

[18] V. Kucera, "A contribution to matrix quadratic equations," *IEEE Trans. Autom. Control*, vol. 17, no. 3, pp. 344–347, 1972.

[19] H. K. Khalil, *Nonlinear Systems*. Patience Hall, 2002, vol. 115.

[20] A. Franci, G. Drion, and R. Sepulchre, "Robust and tunable bursting requires slow positive feedback," *J. Neurosci.*, vol. 119, no. 3, pp. 1222–1234, 2018.

[21] N. Cho, H.-S. Shin, Y. Kim, and A. Tsourdos, "Composite model reference adaptive control with parameter convergence under finite excitation," *IEEE Trans. Autom. Control*, vol. 63, no. 3, pp. 811–818, 2017.

[22] S. B. Roy, S. Bhasin, and I. N. Kar, "Combined MRAC for unknown MIMO LTI systems with parameter convergence," *IEEE Trans. Autom. Control*, vol. 63, no. 1, pp. 283–290, 2017.

[23] Z. Li, Z. Ding, J. Sun, and Z. Li, "Distributed adaptive convex optimization on directed graphs via continuous-time algorithms," *IEEE Trans. Autom. Control*, vol. 63, no. 5, pp. 1434–1441, 2017.