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Distributed Adaptive Synchronization in Euler Lagrange Networks with Uncertain Interconnections

Tian Tao, Spandan Roy, Bart De Schutter and Simone Baldi

***Abstract*—In this work we propose a new practical synchronization protocol for multiple Euler Lagrange (EL) systems without structural linear-in-the-parameters (LIP) knowledge of the uncertainty and where the agents can be interconnected before control design by unknown state-dependent interconnection terms. This setting is meant to overcome two standard a priori assumptions in the literature concerning uncertainty with LIP structure and absence of interaction among agents before designing the synchronization protocol. To overcome these assumptions, we propose an adaptive distributed control mechanism having the purpose of estimating the coefficients of the resulting state-dependent uncertainty structure.**

***Index Terms*—Adaptive synchronization, heterogeneous networks, Euler Lagrange dynamics, bounded interconnections.**

I. INTRODUCTION

Euler Lagrange (EL) dynamics can describe the motion of various mechanical systems [1], [2], robotic manipulators [3], [4], aerospace systems [5], and many more. Motivated by the advances in multi-agent systems, the problem of controlling a single EL system to track desired trajectories [6]–[8] has been recently accompanied by the problem of controlling multiple EL systems [9] toward a common behavior. The problem becomes especially challenging in the presence of uncertainty in the EL dynamics. Developments in this field use adaptive control tools and are often referred to as *adaptive synchronization* of multiple uncertain EL systems [10]–[12]. Recent developments consider sinusoidal leader signals or sinusoidal disturbances that guarantee persistence of excitation for proper estimation of uncertainties [13]–[15] (see also [16], [17] for the importance of persistence of excitation in adaptive control and recent efforts to relax this condition).

Crucial aspects worth considering in uncertain EL systems include the a priori assumptions on the uncertainty: a typical assumption is the linear-in-the-parameters (*LIP) structure* [18], [19], which however is rarely met in practical situations. In particular, except for viscous friction, most friction models do not satisfy the LIP structure [20].

Another crucial aspect worth considering in multiple uncertain EL systems includes the assumptions made on the *a priori structure of the interaction*, i.e. how the EL systems interact before the control design. In all aforementioned literature (cf. also [21]–[23]), interconnections between agents are assumed nonexistent before control design. That is, the agents interact with each other only as the result of the synchronization protocol. Before the control design, each agent is assumed to be unaffected by neighboring agents. When a priori interaction is considered, such as in [9], [24], the control strategy is decentralized (i.e. it assumes each agent can access the leader information). These assumptions on the interaction among agents restrict the applicability of synchronization to many practical cases in which agents interact in some state-dependent way. For example, in power systems [25], [26], or in the recently proposed open multi-agent systems [27], interconnections exist before the control design, coming from the state difference between neighboring agents (e.g. power flow between neighboring areas).

Therefore, despite the progress in the field, most approaches rely on two important a priori assumptions concerning uncertainty with LIP structure and absence of interaction among agents before protocol design. These a priori assumptions on structure of the uncertainty and structure of the interaction motivate us towards a novel adaptive distributed design for synchronization of EL networks. First, we consider state-dependent uncertainty (not necessarily LIP). Then, differently from standard literature, we consider that the interaction terms among agents exist before control design, which are also state-dependent. Summarizing, this work addresses and solves the leader-following synchronization for multiple uncertain EL systems with state-dependent uncertainty and without a priori bounded interconnections. As a result of removing the a priori bounded structure [28], we must seek for practical synchronization instead of asymptotic synchronization. To address the presence of state-dependent uncertainty and uncertain state-dependent interconnections, we propose an adaptive distributed control mechanism having the purpose of estimating the coefficients of the resulting uncertainty structure.

The paper is organized as follows: Sect. II introduces basic notation; synchronization problem is formulated in Sect. III. Adaptive synchronization laws are in Sect. VI, with Lyapunov stability analysis in Sect. V. Simulations are in Sect. VI.

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II. BASIC NOTATION

We will adopt standard notation, such as I_N for the identity matrix of dimension N, 1_N for the N-dimensional vector of ones, $\underline{\lambda}(\cdot)$ and $\overline{\lambda}(\cdot)$ for the minimum and maximum singular value of a matrix, $\|\cdot\|$ for the 2-norm.

We use graphs to represent a network of nodes (or agents). A directed graph $\mathcal{G}$ is described by the pair $(\mathcal{V}, \mathcal{E})$, comprising the node set $\mathcal{V} \triangleq \{v_1, \ldots, v_N\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The node set does not include the leader node v_0, due to its special role. An edge is a pair of nodes $(v_j, v_i) \in \mathcal{E}$, which represents that agent i has access to the information from agent j, i.e. agent j is a neighbor of agent i (not necessarily vice versa). The neighbor set of agent i is denoted by $\mathcal{N}_i$.

For those nodes i that can receive information from the leader, we have $b_i > 0$; otherwise, $b_i = 0$. Let $B = \operatorname{diag}(b_1, \ldots, b_N) \in \mathbb{R}^{N \times N}$. The edges in $\mathcal{E}$ are described by the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The following is a standard condition for achieving synchronization in directed graphs [18], [29], [30] (existence of a directed path from the leader to any follower node).

Assumption 1. *The directed augmented graph representing the connections between the graph $\mathcal{G}$ and the leader node v_0 contains a directed spanning tree with the root being v_0.*

III. Synchronization Problem

Let each node $i \in \{1, \ldots, N\}$ be represented by Euler Lagrange (EL) dynamics (in the following, let us remove time dependency for brevity):

$$\begin{aligned} M_i(q_i(t))\ddot{q}_i(t) + C_i(q_i(t), \dot{q}_i(t))\dot{q}_i(t) + G_i(q_i(t)) \\ + F_i(\dot{q}_i(t)) + H_i(e_i(t), \dot{e}_i(t)) + d_i(t) = \tau_i(t) \end{aligned} \quad (1)$$

where $q_i, \dot{q}_i, \ddot{q}_i \in \mathbb{R}^n$ are the generalized coordinates and their derivatives, and $\tau_i \in \mathbb{R}^n$ is the control input. The system dynamics (1) comprise the mass/inertia matrix $M_i(q_i)$, the centripetal term $C_i(q_i, \dot{q}_i)$, the gravity term $G_i(q_i)$, the friction term $F_i(\dot{q}_i)$, and an external bounded disturbance $\|d_i(t)\| \leq \bar{d}_i$ $\forall t$ (with possibly unknown $\bar{d}_i$). In addition, (1) includes an interconnection term $H_i(e_i, \dot{e}_i)$ depending on the local synchronization error $e_i \in \mathbb{R}^n$ and its derivative $\dot{e}_i \in \mathbb{R}^n$:

$$e_i = \sum_{j \in \mathcal{N}_i} a_{ij}(q_i - q_j) + b_i(q_i - q_0) \quad (2a)$$

$$\dot{e}_i = \sum_{j \in \mathcal{N}_i} a_{ij}(\dot{q}_i - \dot{q}_j) + b_i(\dot{q}_i - \dot{q}_0) \quad (2b)$$

where $q_0, \dot{q}_0 \in \mathbb{R}^n$ represent the state of the leader and its derivative. As common in EL literature, we consider $\|\dot{q}_0\| \leq \bar{q}_0$, $\|\ddot{q}_0\| \leq \check{q}_0$ [10], [14]. We take $\bar{q}_0, \check{q}_0$ unknown constants.

Remark 1 (Interconnection before control design)**.** *The dynamics in* (1) *depart from considering a priori disconnected dynamics, i.e. when the dynamics of each agent i are unaffected by neighboring states $q_j, \dot{q}_j$ before control design [10]–[13], [21]–[23]. In fact, the terms $H_i(e_i, \dot{e}_i)$ in* (1) *are active even before control design. These interconnection terms, which cannot be designed nor bounded a priori (cf. Property 4), require a new design not available in the literature.*

The following properties for the dynamic terms in (1) are taken or further extended from standard and recent EL literature [31], [32]:

Property 1. *There exist $\bar{c}_i, \bar{g}_i, \bar{f}_i \in \mathbb{R}^+$ such that $\|C_i(q_i, \dot{q}_i)\| \leq \bar{c}_i\|\dot{q}_i\|$, $\|G_i(q_i)\| \leq \bar{g}_i$, $\|F_i(\dot{q}_i)\| \leq \bar{f}_i\|\dot{q}_i\|$.*

Property 2. *The matrix $M_i(q_i)$ is symmetric and uniformly positive in q_i. There exist positive constants $\underline{m}$ and $\overline{m}$ such that $0 \leq \underline{m}I_n \leq M_i(q_i) \leq \overline{m}I_n$, $\forall q_i, \forall i$.*

Property 3. *The matrix $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric, i.e. for any non-zero vector s, we have $s^T(\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i))s = 0$.*

Property 4. *There exist $\bar{h}_{1i}, \bar{h}_{2i}, \bar{h}_{3i}, \bar{h}_{4i}, \bar{h}_{5i} \in \mathbb{R}^+$ such that $\|H_i(e_i, \dot{e}_i)\| \leq \bar{h}_{1i} + \bar{h}_{2i}\|e_i\| + \bar{h}_{3i}\|\dot{e}_i\| + \bar{h}_{4i}\|e_i\|^2 + \bar{h}_{5i}\|\dot{e}_i\|^2$.*

All the constants in Properties 1, 2, and 4 are possibly unknown for the control design. In Property 4 we take the interconnection term $H_i(e_i, \dot{e}_i)$ with a quadratic upper bound. This is a natural choice in view of the fact that the other forces stemming from centripetal, gravity, or friction terms in Property 1, have linear or quadratic upper bounds.

From (2) and defining $e = [e_1^T, \ldots, e_N^T]^T$, $q = [q_1^T, \ldots, q_N^T]^T$, $\underline{q}_0 = 1_N \otimes q_0$, we can obtain

$$e = -(L + B) \otimes (q - \underline{q}_0) = -(L + B) \otimes \delta \quad (3)$$

where $\otimes$ denotes the Kronecker product and $\delta = (q - \underline{q}_0) \in \mathbb{R}^{nN}$ represents the global synchronization error with the leader.

Note that δ cannot be used for control design as it includes global leader state information (only available to some followers. The following lemma is known from literature:

Lemma 1. *[23] The local and global synchronization error are related by*

$$\|\delta\| \leq \frac{\|e\|}{\underline{\lambda}(L + B)} \quad (4)$$

with $\underline{\lambda}(L + B)$ being the minimum singular value of $(L + B)$, which is positive for a directed graph containing a directed spanning tree with the root being the leader node.

Remark 2 (No structural knowledge)**.** *In Property 1-4, no assumption is made on the LIP structure of the dynamic terms, which marks another difference with standard EL literature, since general friction terms are not in LIP form [20], [32]. The price to be paid as shown in [28], is that a bounded error must be sought in place of asymptotic error.*

Definition 1. *(Uniform Ultimate Bounded (UUB)) The local synchronization error e is uniformly ultimately bounded for any i, if there exists a convex and compact set $\mathcal{C}$ such that $\forall e(0) = e^*$, there exists a finite time $T(e^*)$ such that $e \in \mathcal{C}$ for all $t > T(e^*)$.*

Problem Formulation. *Under Assumption 1 and Properties 1-4, the adaptive synchronization problem is to design a distributed adaptive law for the EL network* (1) *that guarantees the local synchronization error e to be UUB (implying the global synchronization error δ to be UUB from Lemma 1).*

IV. Controller Design

A. Uncertainty Analysis

First, we rewrite (1) as

$$M_i\ddot{q}_i = Q_i(q_i, \dot{q}_i, e_i, \dot{e}_i) + \tau_i \quad (5)$$

where $Q_i(q_i,\dot{q}_i,e_i,\dot{e}_i) = -C_i(q_i,\dot{q}_i)\dot{q}_i - G_i(q_i) - F_i(\dot{q}_i) - H_i(e_i,\dot{e}_i) - d_i$. Using Property 1, we have

$$\|Q_i(q_i,\dot{q}_i,e_i,\dot{e}_i)\| \leq (\bar{g}_i+\bar{d}_i+\bar{h}_{1i}) + \bar{f}_i\|\dot{q}_i\| + \bar{c}_i\|\dot{q}_i\|^2 + \bar{h}_{2i}\|e_i\| + \bar{h}_{3i}\|\dot{e}_i\| + \bar{h}_{4i}\|e_i\|^2 + \bar{h}_{5i}\|\dot{e}_i\|^2 \quad (6)$$

We define a filtered tracking error

$$r_i = \dot{e}_i + P_i e_i \quad (7)$$

with $P_i \in \mathbb{R}^{n\times n}$ a designed positive definite diagonal matrix.

Let us define $\xi_i = [e_i^T, \dot{e}_i^T, q_i^T, \dot{q}_i^T]^T$. The control mechanism using local information is designed as

$$\tau_i = -K_i r_i - \bar{\tau}_i - \bar{K}_i P_i^{-1} e_i \quad (8a)$$

$$\bar{\tau}_i = \omega\rho_i \frac{r_i}{\sqrt{\|r_i\|^2+\varepsilon}} \quad (8b)$$

$$\rho_i = \hat{\theta}_{0i} + \hat{\theta}_{1i}\|\xi_i\| + \hat{\theta}_{2i}\|\xi_i\|^2 + \gamma_i \quad (8c)$$

where $K_i \in \mathbb{R}^{n\times n}$ is a designed positive definite matrix, $\bar{K}_i \in \mathbb{R}^{n\times n}$ is a designed positive definite diagonal matrix, $\omega > 1, \varepsilon$ are user-defined scalars, and $\hat{\theta}_{0i}$, $\hat{\theta}_{1i}$, $\hat{\theta}_{2i}$ are adaptive parameters to be designed later.

The dynamics of $\dot{e}_i$ can be calculated as

$$\ddot{e}_i = \breve{a}_i\ddot{q}_i - \sum_{j\in\mathcal{N}_i} a_{ij}\ddot{q}_j - b_i\ddot{q}_0 \quad (9)$$

where $\breve{a}_i = b_i + \sum_{j\in\mathcal{N}_i} a_{ij} > 0$. We multiply (9) with $\frac{1}{\breve{a}_i}M_i$, and then add and subtract e_i, and use (5) to obtain

$$\frac{1}{\breve{a}_i}M_i\ddot{e}_i = M_i\ddot{q}_i - \sum_{j\in\mathcal{N}_i}\frac{a_{ij}}{\breve{a}_i}(M_iM_j^{-1})M_j\ddot{q}_j - \frac{1}{\breve{a}_i}M_ib_i\ddot{q}_0 = -K_ir_i - \bar{K}_iP_i^{-1}e_i - \bar{\tau}_i + \sum_{j\in\mathcal{N}_i}A_{ij}\bar{\tau}_j + \Delta_{ij} \quad (10)$$

where $A_{ij} = \frac{a_{ij}}{\breve{a}_i}(M_iM_j^{-1})$, and Δ_{ij} is treated as an uncertainty term of agent i and agent j:

$$\Delta_{ij} \triangleq \big[Q_i(q_i,\dot{q}_i,e_i,\dot{e}_i) - \frac{1}{\breve{a}_i}M_ib_i\ddot{q}_0 - \sum_{j\in\mathcal{N}_i}A_{ij}\big[Q_j(q_j,\dot{q}_j,e_j,\dot{e}_j) - K_jr_j\big]. \quad (11)$$

According to (7), we have

$$\frac{1}{\breve{a}_i}M_i\ddot{e}_i = \frac{1}{\breve{a}_i}M_i\dot{r}_i - \frac{1}{\breve{a}_i}M_iP_i\dot{e}_i. \quad (12)$$

Substituting (12) into (10), we get the dynamics of r_i:

$$\frac{1}{\breve{a}_i}M_i\dot{r}_i = -K_ir_i - \bar{K}_iP_i^{-1}e_i - \bar{\tau}_i + \sum_{j\in\mathcal{N}_i}A_{ij}\bar{\tau}_j + \bar{\Delta}_{ij} - \frac{C_ir_i}{\breve{a}_i} \quad (13)$$

where $\bar{\Delta}_{ij} = \Delta_{ij} + \frac{1}{\breve{a}_i}M_iP_i\dot{e}_i + \frac{1}{\breve{a}_i}C_ir_i$. The definition of ξ_i implies that $\|e_i\| \leq \|\xi_i\|$, $\|\dot{e}_i\| \leq \|\xi_i\|$, $\|q_i\| \leq \|\xi_i\|$ and $\|\dot{q}_i\| \leq \|\xi_i\|$. From (7), we can write $\|r_i\| \leq (1+\|P_i\|)\|\xi_i\|$. The following bound on uncertainty $\|\bar{\Delta}_{ij}\|$ can be obtained:

$$\begin{aligned}\|\bar{\Delta}_{ij}\| \leq{}& (\bar{g}_i+\bar{d}_i+\bar{h}_{1i}) + \bar{f}_i\|\dot{q}_i\| + \bar{c}_i\|\dot{q}_i\|^2 \\ &+ \bar{h}_{2i}\|e_i\| + \bar{h}_{3i}\|\dot{e}_i\| + \bar{h}_{4i}\|e_i\|^2 + \bar{h}_{5i}\|\dot{e}_i\|^2 \\ &+ \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\Big[(\bar{g}_j+\bar{d}_j+\bar{h}_{1j}) + \bar{f}_j\|\dot{q}_j\| \\ &+ \bar{c}_j\|\dot{q}_j\|^2 + \bar{h}_{2j}\|e_j\| + \bar{h}_{3j}\|\dot{e}_j\| + \bar{h}_{4j}\|e_j\|^2 \\ &+ \bar{h}_{5j}\|\dot{e}_j\|^2\Big] + \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\|K_j\|(1+\|P_j\|)\|\xi_j\| + \frac{b_i}{\breve{a}_i}\|M_i\|\|\ddot{q}_0\| \\ &+ \frac{1}{\breve{a}_i}\|P_i\|\|M_i\|\|\xi_i\| + \frac{\bar{c}_i}{\breve{a}_i}(1+\|P_i\|)\|\xi_i\| \\ \leq{}& \theta_{0i}^* + \theta_{1i}^*\|\xi_i\| + \theta_{2i}^*\|\xi_i\|^2 + \sum_{j\in\mathcal{N}_i}\varphi_{1j}^*\|\xi_j\| + \sum_{j\in\mathcal{N}_i}\varphi_{2j}^*\|\xi_j\|^2\end{aligned} \quad (14)$$

where $\bar{a}_{ij} = \|A_{ij}\|$, $\theta_{0i}^* = \big(\bar{g}_i+\bar{d}_i+\bar{h}_{1i}\big) + \sum_{j\in\mathcal{N}_i}\big[\bar{a}_{ij}(\bar{g}_j+\bar{d}_j+\bar{h}_{1j}\big] + \frac{b_i}{\breve{a}_i}\overline{m}\breve{q}_0$, $\theta_{1i}^* = \bar{h}_{2i}+\bar{h}_{3i}+\bar{f}_i+\frac{1}{\breve{a}_i}\|P_i\|\|M_i\| + \frac{\bar{c}_i}{\breve{a}_i}(1+\|P_i\|)$, $\theta_{2i}^* = \bar{h}_{4i}+\bar{h}_{5i}+\bar{c}_i$, $\varphi_{1j}^* = \bar{a}_{ij}\big[\bar{h}_{2j}+\bar{h}_{3j}+\bar{f}_j+\|K_j\|(1+\|P_j\|)\big]$, $\varphi_{2j}^* = \bar{a}_{ij}(\bar{h}_{4j}+\bar{h}_{5j}+\bar{c}_j)$. Note that $\|A_{ij}\|$ can be bounded by a constant thanks to the uniform bounds for the mass matrix in Property 2. Also, $\theta_{0i}^*, \theta_{1i}^*, \theta_{2i}^*, \varphi_{1j}^*, \varphi_{2j}^*$ are all unknown constants according to Properties 1 and 4.

B. Adaptive Synchronization Laws

According to the structure of the upper bounds of $\bar{\Delta}_{ij}$ in (14), the adaptive laws for (8c) are designed as:

$$\dot{\hat{\theta}}_{0i} = \|r_i\| - \alpha_0\hat{\theta}_{0i} \quad (15a)$$

$$\dot{\hat{\theta}}_{1i} = \|r_i\|\|\xi_i\| - \alpha_1\hat{\theta}_{1i} \quad (15b)$$

$$\dot{\hat{\theta}}_{2i} = \|r_i\|\|\xi_i\|^2 - \alpha_2\hat{\theta}_{2i} \quad (15c)$$

$$\dot{\gamma}_i = -(\epsilon_0 + \epsilon_1\|\xi_i\|^7 - \epsilon_2\|\xi_i\|^5)\gamma_i + \beta_i \quad (15d)$$

$$\text{where } \hat{\theta}_{0i}(0) > 0, \hat{\theta}_{1i}(0) > 0, \hat{\theta}_{2i}(0) > 0, \gamma_i(0) > 0 \quad (15e)$$

$$\epsilon_0, \epsilon_1, \epsilon_2, \alpha_i, \beta_i \in \mathbb{R}^+ \quad (15f)$$

$$\text{with} \quad \epsilon_0 \geq 1+\epsilon_2, \epsilon_1 \geq \epsilon_2 \quad (15g)$$

V. Stability Analysis

Theorem 1. *Under Properties 1-4 and Assumption 1, the closed-loop trajectories of* (1) *employing control law* (8) *and adaptive law* (15) *are UUB with the following ultimate bound on the local synchronization error e:*

$$U = \sqrt{\frac{2\chi}{\min_{i\in\Omega}\underline{\lambda}(\bar{K}_iP_i^{-1})(\zeta-\kappa)}} \quad (16)$$

where $\chi = \sum_{i=1}^N\Big(\frac{\alpha_0\theta_{0i}^{*\,2}}{2} + \frac{\alpha_1\theta_{1i}^{*\,2}}{2} + \frac{\alpha_2\theta_{2i}^{*\,2}}{2}\Big) + \sum_{i=1}^N\frac{2\zeta\bar{\gamma}_i}{\underline{\gamma}_i}$; *$\kappa$ is a scalar satisfying $0 < \kappa < \zeta$ with* $\zeta = \dfrac{\min\{\min_{i\in\Omega}\underline{\lambda}(K_i), \min_{i\in\Omega}\underline{\lambda}(\bar{K}_i), \alpha_0/2, \alpha_1/2, \alpha_2/2\}}{\max\{\overline{m}/2\min_{i\in\Omega}\{\breve{a}_i\}, \max_{i\in\Omega}\overline{\lambda}(\bar{K}_iP_i^{-1})/2\}}$.

Proof: Construct a Lyapunov function defined by:

$$V(t) = \frac{1}{2}\sum_{i=1}^N\Big(\frac{1}{\breve{a}_i}r_i^T(t)M_i(t)r_i(t) + e_i^T(t)\bar{K}_iP_i^{-1}e_i(t)\Big) + \frac{1}{2}\sum_{i=1}^N\Big\{(\hat{\theta}_{0i}(t)-\theta_{0i}^*)^2 + (\hat{\theta}_{1i}(t)-\theta_{1i}^*)^2$$

$$+(\hat{\theta}_{2i}(t)-\theta^*_{2i})^2+\frac{2\gamma_i(t)}{\underline{\gamma}_i}\bigg\}. \quad (17)$$

Note that (15d) has a stable linear time-varying structure in the variable γ_i thanks to the inequalities (15g), since

a) for $\|\xi\| \geq 1$: According to $\epsilon_1 \geq \epsilon_2$, we have $\epsilon_1\|\xi\|^7 - \epsilon_2\|\xi\|^5 \geq \epsilon_1(\|\xi\|^7 - \|\xi\|^5) \geq 0$. Thus, according to $\epsilon_0 \geq 1+\epsilon_2$ and $\epsilon_2 > 0$, we obtain

$$\epsilon_0 + \epsilon_1\|\xi\|^7 - \epsilon_2\|\xi\|^5 \geq \epsilon_0 \geq 1+\epsilon_2 > 1$$

b) for $\|\xi\| < 1$: According to $\epsilon_0 \geq 1+\epsilon_2$, we have $\epsilon_0 - \epsilon_2\|\xi\|^5 \geq 1+\epsilon_2(1-\|\xi\|^5) > 1$. Thus, according to $\epsilon_1 > 0$, we obtain

$$\epsilon_0 - \epsilon_2\|\xi\|^5 + \epsilon_1\|\xi\|^7 > 1.$$

Then, $\epsilon_0 - \epsilon_2\|\xi\|^5 + \epsilon_1\|\xi\|^7 > 1$ always holds, i.e. the system in (15d) can be seen as a stable linear time-varying system.

Based on the linear time-varying structure of (15d), the positive input β_i and positive initial condition (15e), it can be verified that $\gamma_i(t) \geq \underline{\gamma}_i > 0$ $\forall t \geq t_0$. This condition will be used for subsequent stability analysis.

The proof is organized as follows: first, we calculate the time derivative of the Lyapunov function. Then, based on the structure of (8b), we study the behavior of the Lyapunov function under the three standard [19] possible scenarios:

1) $\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\epsilon}} \geq \|r_i\|$ for all i;
2) $\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\epsilon}} < \|r_i\|$ for all i;
3) $\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\epsilon}} \geq \|r_i\|$ for $i=1,...,k$, and $\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\epsilon}} < \|r_i\|$ for $i=k+1,...,N$.

Finally, combining the results of the three scenarios, we obtain the ultimate bound on the local synchronization error e. Using (7) and (13), the time derivative of (17) satisfies

$$\begin{aligned}
\dot{V} \leq& -\sum_{i=1}^N r_i^T K_i r_i + \sum_{i=1}^N r_i^T \bar{\Delta}_{ij} - \sum_{i=1}^N r_i^T \bar{\tau}_i \\
&+ \sum_{i=1}^N r_i^T \sum_{j\in\mathcal{N}_i} A_{ij}\bar{\tau}_j + \frac{1}{2}\sum_{i=1}^N \frac{1}{\breve{a}_i} r_i{}^T\big(\dot{M}_i - 2C_i\big)r_i \\
&+ \sum_{i=1}^N \left\{\frac{\dot{\gamma}_i}{\underline{\gamma}_i} + \sum_{l=0}^2(\hat{\theta}_{li}-\theta^*_{li})\dot{\hat{\theta}}_{li}\right\} - \sum_{i=1}^N e_i^T\bar{K}_i e_i \\
\leq& -\sum_{i=1}^N r_i^T K_i r_i + \sum_{i=1}^N \|r_i^T\|\|\bar{\Delta}_{ij}\| - \sum_{i=1}^N e_i^T\bar{K}_i e_i \\
&+ \sum_{i=1}^N \left\{\sum_{j\in\mathcal{N}_j}\bar{a}_{ij}\rho_j\omega\frac{\|r_i\|\|r_j\|}{\sqrt{\|r_j\|^2+\varepsilon}} - \rho_i\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\varepsilon}}\right\} \\
&+ \sum_{i=1}^N \left\{\frac{\dot{\gamma}_i}{\underline{\gamma}_i} + \sum_{l=0}^2(\hat{\theta}_{li}-\theta^*_{li})\dot{\hat{\theta}}_{li}\right\}. \quad (18)
\end{aligned}$$

According to (7), we have $\|r_i\| \leq (1+\|P_i\|)\|\xi_i\|$. Combined with (13) and (14), we obtain the uncertainty structure as

$$\begin{aligned}
\sum_{i=1}^N \|r_i\|\|\bar{\Delta}_{ij}\| \leq& \sum_{i=1}^N \|r_i\|\Big\{\theta^*_{0i} + \theta^*_{1i}\|\xi_i\| + \theta^*_{2i}\|\xi_i\|^2\Big\} \\
&+ \sum_{i=1}^N \|r_i\|\Big\{\sum_{j\in\mathcal{N}_i}\varphi^*_{1j}\|\xi_j\| + \sum_{j\in\mathcal{N}_i}\varphi^*_{2j}\|\xi_j\|^2\Big\}. \quad (19)
\end{aligned}$$

Furthermore, the following two bounds hold

$$\sum_{i=1}^N \|r_i\| \sum_{j\in\mathcal{N}_i}\varphi^*_{1j}\|\xi_j\| \leq \sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\varphi^*_{1j}(1+\|P_i\|)\|\xi_i\|\|\xi_j\| \quad (20)$$

$$\sum_{i=1}^N \|r_i\| \sum_{j\in\mathcal{N}_i}\varphi^*_{2j}\|\xi_j\|^2 \leq \sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\varphi^*_{2j}(1+\|P_i\|)\|\xi_i\|\|\xi_j\|^2 \quad (21)$$

The bounded-input-bounded-output property of the stable linear time-varying system (15d) with positive constant input β_i guarantees that $\gamma_i \in \mathcal{L}_\infty$, i.e. there exists $\bar{\gamma}_i \in \mathbb{R}^+$ such that $\gamma_i \leq \bar{\gamma}_i$. From $\frac{\|r_j\|}{\sqrt{\|r_j\|^2+\varepsilon}} \leq 1$, we get

$$\begin{aligned}
&\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\rho_j\omega\frac{\|r_i\|\|r_j\|}{\sqrt{\|r_j\|^2+\varepsilon}} \leq \omega\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\rho_j\|r_i\| \\
&\leq \omega\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\left\{\sum_{k=0}^2\bar{a}_{ij}\hat{\theta}_{kj}\|r_i\|\|\xi_j\|^k + \bar{a}_{ij}\bar{\gamma}_j\|r_i\|\right\}. \quad (22)
\end{aligned}$$

Meanwhile, the fact that the following dynamics $\dot{\hat{\theta}}_{0j} = -\alpha_0\hat{\theta}_{0j}$, $\dot{\hat{\theta}}_{1j} = -\alpha_1\hat{\theta}_{1j}$, $\dot{\hat{\theta}}_{2j} = -\alpha_2\hat{\theta}_{2j}$, in the adaptive laws (15a)-(15c) are first-order stable dynamics gives, the standard input/output stability properties [33, Sect. 3.3] gives

$$\hat{\theta}_{0j} \leq \bar{\theta}_{0j} + \check{\theta}_{0j}\|r_j\| \quad (23a)$$
$$\hat{\theta}_{1j} \leq \bar{\theta}_{1j} + \check{\theta}_{1j}\|r_j\|\|\xi_j\| \quad (23b)$$
$$\hat{\theta}_{2j} \leq \bar{\theta}_{2j} + \check{\theta}_{2j}\|r_j\|\|\xi_j\|^2 \quad (23c)$$

with $\bar{\theta}_{0j}, \check{\theta}_{0j}, \bar{\theta}_{1j}, \check{\theta}_{1j}, \bar{\theta}_{2j}, \check{\theta}_{2j} \in \mathbb{R}^+$. This in turn leads to

$$\begin{aligned}
&\omega\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\hat{\theta}_{2j}\|r_i\|\|\xi_j\|^2 \\
&\leq \omega\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\bar{\theta}_{2j}(1+\|P_i\|)\|\xi_i\|\|\xi_j\| \\
&+ \omega\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\check{\theta}_{2j}(1+\|P_i\|)(1+\|P_j\|)\|\xi_i\|\|\xi_j\|^5. \quad (24)
\end{aligned}$$

Similarly, we obtain the overall terms from the neighboring agents $j \in \mathcal{N}_i$:

$$\begin{aligned}
&\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\Big\{\bar{a}_{ij}\rho_j\omega\frac{\|r_i\|\|r_j\|}{\sqrt{\|r_j\|^2+\varepsilon}} + \|r_i\|\big(\varphi^*_{1j}\|\xi_j\| + \varphi^*_{2j}\|\xi_j\|^2\big)\Big\} \\
\leq& \sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\Big\{\omega\bar{a}_{ij}(1+\|P_i\|)(\bar{\theta}_{0j}+\bar{\gamma}_j)\|\xi_i\| \\
&+ (1+\|P_i\|)\Big[\omega\bar{a}_{ij}(\check{\theta}_{0j}(1+\|P_j\|)+\bar{\theta}_{1j}) + \varphi^*_{1j}\Big]\|\xi_i\|\|\xi_j\| \\
&+ (1+\|P_j\|)\big(\omega\bar{a}_{ij}\bar{\theta}_{2j} + \varphi^*_{2j}\big)\|\xi_i\|\|\xi_j\|^2
\end{aligned}$$

$$+\,\omega\bar{a}_{ij}(1+\|P_i\|)(1+\|P_j\|)\|\xi_i\|\|\xi_j\|^3(\check{\theta}_{1j}+\check{\theta}_{2j}\|\xi_j\|^2)\Bigg\}. \quad (25)$$

Using (15a)-(15c), we have

$$(\hat{\theta}_{li}-\theta^*_{li})\dot{\hat{\theta}}_{li}=(\hat{\theta}_{li}-\theta^*_{li})\|\xi_i\|^l\|r_i\|+\big(\alpha_l\hat{\theta}_{li}\theta^*_{li}-\alpha_l\hat{\theta}^2_{li}\big) \quad (26)$$

for $l=0,1,2$ and $i=1,\cdots,N$. The last term of (26) can be rewritten as

$$\big(\alpha_l\hat{\theta}_{li}\theta^*_{li}-\alpha_l\hat{\theta}^2_{li}\big)=-\frac{\alpha_l(\hat{\theta}_{li}-\theta^*_{li})^2}{2}+\frac{\alpha_l{\theta^*_{li}}^2}{2}. \quad (27)$$

Similarly, with $\gamma_i(t)\geq\underline{\gamma}_i>0$, (15d) leads to

$$\begin{aligned}\frac{\dot{\gamma}_i(t)}{\underline{\gamma}_i}=&\frac{1}{\underline{\gamma}_i}\Big[-(\epsilon_0+\epsilon_1\|\xi_i\|^7-\epsilon_2\|\xi_i\|^5)\gamma_i+\beta_i\Big]\\ \leq&\Big[-\big(\epsilon_0+\epsilon_1|\xi_i\|^7-\epsilon_2\|\xi_i\|^5\big)+(\beta_i/\underline{\gamma}_i)\Big] \quad (28)\end{aligned}$$

According to (19)-(28), from (18) we have

$$\begin{aligned}\dot{V}\leq&-\min_{i\in\Omega}\underline{\lambda}(K_i)\sum_{i=1}^N\|r_i\|^2-\min_{i\in\Omega}\underline{\lambda}(\bar{K}_i)\sum_{i=1}^N\|e_i\|^2\\ &+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\Bigg\{\omega\bar{a}_{ij}(1+\|P_i\|)(\bar{\theta}_{0j}+\bar{\gamma}_j)\|\xi_i\|\\ &+(1+\|P_i\|)\Big[\omega\bar{a}_{ij}(\check{\theta}_{0j}(1+\|P_j\|)+\bar{\theta}_{1j})+\varphi^*_{1j}\Big]\|\xi_i\|\|\xi_j\|\\ &+(1+\|P_j\|)\big(\omega\bar{a}_{ij}\bar{\theta}_{2j}+\varphi^*_{2j}\big)\|\xi_i\|\|\xi_j\|^2\\ &+\omega\bar{a}_{ij}(1+\|P_i\|)(1+\|P_j\|)\|\xi_i\|\|\xi_j\|^3(\check{\theta}_{1j}+\check{\theta}_{2j}\|\xi_j\|^2)\Bigg\}\\ &-\sum_{i=1}^N\rho_i\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\varepsilon}}+\sum_{i=1}^N\sum_{l=0}^2\theta^*_{li}\|\xi_i\|^l\|r_i\|\\ &+\sum_{i=1}^N\sum_{l=0}^2\Bigg\{(\hat{\theta}_{li}-\theta^*_{li})\|\xi_i\|^l\|r_i\|-\Big[\frac{\alpha_l(\hat{\theta}_{li}-\theta^*_{li})^2}{2}-\frac{\alpha_l{\theta^*_{li}}^2}{2}\Big]\Bigg\}\\ &-\sum_{i=1}^N\Bigg[\big(\epsilon_0+\epsilon_1|\xi_i\|^7-\epsilon_2\|\xi_i\|^5\big)+(\beta_i/\underline{\gamma}_i)\Bigg]. \quad (29)\end{aligned}$$

We study the behavior of the Lyapunov function for the three aforementioned scenarios:

Scenario 1: We have $\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\varepsilon}}\geq\|r_i\|$ for all $i=1,\ldots,N$. Then, according to (8c), we obtain

$$\begin{aligned}-\sum_{i=1}^N\rho_i\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\varepsilon}}\leq&-\sum_{i=1}^N\rho_i\|r_i\|\\ \leq&-\sum_{i=1}^N\sum_{l=0}^2\Big[\hat{\theta}_{li}\|\xi_i\|^l+\gamma_i\Big]\|r_i\|. \quad (30)\end{aligned}$$

Substituting (30) into (29), yields

$$\begin{aligned}\dot{V}\leq&-\min_{i\in\Omega}\underline{\lambda}(K_i)\sum_{i=1}^N\|r_i\|^2-\min_{i\in\Omega}\underline{\lambda}(\bar{K}_i)\sum_{i=1}^N\|e_i\|^2\\ &-\sum_{i=1}^N\sum_{l=0}^2\Bigg\{\frac{\alpha_l(\hat{\theta}_{li}-\theta^*_{li})^2}{2}-\frac{\alpha_l{\theta^*_{li}}^2}{2}\Bigg\}+Z_1(\|\xi\|) \quad (31)\end{aligned}$$

where $\Omega=\{1,\ldots,N\}$ and $\xi=[\xi_1^T,\ldots,\xi_N^T]^T$ with

$$\begin{aligned}Z_1(\|\xi\|)\triangleq&-\epsilon_1\sum_{i=1}^N\|\xi_i\|^7+\epsilon_2\sum_{i=1}^N\|\xi_i\|^5+\sum_{i=1}^N\Big(-\epsilon_0+\frac{\beta_i}{\underline{\gamma}_i}\Big)\\ &+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\Bigg\{\omega\bar{a}_{ij}(1+\|P_i\|)(\bar{\theta}_{0j}+\bar{\gamma}_j)\|\xi_i\|\\ &+(1+\|P_i\|)\Big[\omega\bar{a}_{ij}(\check{\theta}_{0j}(1+\|P_j\|)+\bar{\theta}_{1j})+\varphi^*_{1j}\Big]\|\xi_i\|\|\xi_j\|\\ &+(1+\|P_j\|)\big(\omega\bar{a}_{ij}\bar{\theta}_{2j}+\varphi^*_{2j}\big)\|\xi_i\|\|\xi_j\|^2\\ &+\omega\bar{a}_{ij}(1+\|P_i\|)(1+\|P_j\|)\|\xi_i\|\|\xi_j\|^3(\check{\theta}_{1j}+\check{\theta}_{2j}\|\xi_j\|^2)\Bigg\}.\end{aligned}$$

Using Descartes' rules of sign change and Bolzano's Theorem [34], the polynomial Z_1 has a unique positive real root $\eta_1\in\mathbb{R}^+$. The coefficient of the highest degree of Z_1 is negative: $-\epsilon_1$. Therefore, $Z_1(\|\xi\|)\leq 0$ when $\|\xi\|\geq\eta_1$.

Since the first-order differential equations as in (15a)-(15c) with positive initial conditions give $\hat{\theta}_{0i}(t)>0,\hat{\theta}_{1i}(t)>0,\hat{\theta}_{2i}(t)>0$, $\forall t\geq 0$, the Lyapunov function (17) satisfies

$$\begin{aligned}V\leq&\frac{\overline{m}}{2\hat{a}}\sum_{i=1}^N\|r_i\|^2+\frac{\max_{i\in\Omega}\overline{\lambda}(\bar{K}_iP_i^{-1})}{2}\sum_{i=1}^N||e_i||^2\\ &+\frac{1}{2}\sum_{i=1}^N\Bigg\{\sum_{l=0}^2(\hat{\theta}_{li}-\theta^*_{li})^2+\frac{2\gamma_i}{\underline{\gamma}_i}\Bigg\}. \quad (32)\end{aligned}$$

Substituting (32) into (31) yields

$$\dot{V}\leq-\zeta V+\sum_{i=1}^N\Bigg\{\sum_{l=0}^2\frac{\alpha_l{\theta^*_{li}}^2}{2}+\frac{2\zeta\bar{\gamma}_i}{\underline{\gamma}_i}\Bigg\}+Z_1(\|\xi\|). \quad (33)$$

Defining a scalar $0<\kappa<\zeta$, (33) is further simplified to

$$\dot{V}\leq-\kappa V-(\zeta-\kappa)V+\chi \quad (34)$$

where $Z_1(\|\xi\|)$ is defined as in (16).

Scenario 2: In this case, we have $0\leq\frac{\omega\|r_i\|^2}{\sqrt{\|r_i\|^2+\epsilon}}\leq\|r_i\|$ for all $i=1,\ldots,N$. Then,

$$-\sum_{i=1}^N\rho_i\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\varepsilon}}\leq 0 \quad (35)$$

Substituting (35) into (29), the time derivative of V satisfies

$$\begin{aligned}\dot{V}\leq&-\min_{i\in\Omega}\underline{\lambda}(K_i)\sum_{i=1}^N\|r_i\|^2-\min_{i\in\Omega}\underline{\lambda}(\bar{K}_i)\sum_{i=1}^N\|e_i\|^2\\ &+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\Bigg\{\omega\bar{a}_{ij}(1+\|P_i\|)(\bar{\theta}_{0j}+\bar{\gamma}_j)\|\xi_i\|\\ &+(1+\|P_i\|)\Big[\omega\bar{a}_{ij}(\check{\theta}_{0j}(1+\|P_j\|)+\bar{\theta}_{1j})+\varphi^*_{1j}\Big]\|\xi_i\|\|\xi_j\|\\ &+(1+\|P_j\|)\big(\omega\bar{a}_{ij}\bar{\theta}_{2j}+\varphi^*_{2j}\big)\|\xi_i\|\|\xi_j\|^2\\ &+\omega\bar{a}_{ij}(1+\|P_i\|)(1+\|P_j\|)\|\xi_i\|\|\xi_j\|^3(\check{\theta}_{1j}+\check{\theta}_{2j}\|\xi_i\|^2)\Bigg\}\end{aligned}$$

$$+\sum_{i=1}^{N}\sum_{l=0}^{2}\theta_{li}^{*}\|\xi_i\|^{l}\|r_i\|+\sum_{i=1}^{N}\sum_{l=0}^{2}(\hat{\theta}_{li}-\theta_{li}^{*})\|\xi_i\|^{l}\|r_i\|$$
$$-\sum_{i=1}^{N}\sum_{l=0}^{2}\left\{\frac{\alpha_l(\hat{\theta}_{li}-\theta_{li}^{*})^2}{2}-\frac{\alpha_l\theta_{li}^{*2}}{2}\right\}$$
$$-\sum_{i=1}^{N}\left[\left(\epsilon_0+\epsilon_1|\xi_i\|^{7}-\epsilon_2\|\xi_i\|^{5}\right)+(\beta_i/\underline{\gamma}_i)\right]. \tag{36}$$

Then, following a similar reasoning as in Scenario 1, we have

$$\dot{V}\leq-\min_{i\in\Omega}\underline{\lambda}(K_i)\sum_{i=1}^{N}\|r_i\|^2-\min_{i\in\Omega}\underline{\lambda}(\bar{K}_i)\sum_{i=1}^{N}\|e_i\|^2+Z_1(\|\xi\|)$$
$$+\sum_{i=1}^{N}\sum_{l=0}^{2}\left\{\hat{\theta}_{li}\|\xi_i\|^{l}\|r_i\|-\left[\frac{\alpha_l(\hat{\theta}_{li}-\theta_{li}^{*})^2}{2}-\frac{\alpha_l\theta_{li}^{*2}}{2}\right]\right\}. \tag{37}$$

According to (23), with $\|r_i\|\leq(1+\|P_i\|)\|\xi_i\|$, if follows that

$$\sum_{i=1}^{N}\sum_{l=0}^{2}\hat{\theta}_{li}\|\xi_i\|^{l}\|r_i\|\leq\sum_{i=1}^{N}\left(\bar{\theta}_{li}+\check{\theta}_{li}\|r_i\|\|\xi_i\|^{l}\right)\|\xi_i\|^{l}\|r_i\|$$
$$\leq\sum_{i=1}^{N}\bar{\theta}_{li}(1+\|P_i\|)\|\xi_i\|^{l+1}+\check{\theta}_{li}(1+\|P_i\|)^2\|\xi_i\|^{2(l+1)}. \tag{38}$$

Substituting (38) into (37), yields

$$\dot{V}\leq-\min_{i\in\Omega}\underline{\lambda}(K_i)\sum_{i=1}^{N}\|r_i\|^2-\min_{i\in\Omega}\underline{\lambda}(\bar{K}_i)\sum_{i=1}^{N}\|e_i\|^2$$
$$-\sum_{i=1}^{N}\sum_{l=0}^{2}\left\{\frac{\alpha_l(\hat{\theta}_{li}-\theta_{li}^{*})^2}{2}-\frac{\alpha_l\theta_{li}^{*2}}{2}\right\}+Z_2(\|\xi\|) \tag{39}$$

where $Z_2(\|\xi\|)=Z_1(\|\xi\|)+\sum_{i=1}^{N}\bar{\theta}_{0i}(1+\|P_i\|)\|\xi_i\|+\sum_{i=1}^{N}\bar{\theta}_{li}(1+\|P_i\|)\|\xi_i\|^{l+1}+\check{\theta}_{li}(1+\|P_i\|)^2\|\xi_i\|^{2(l+1)}$. Similarly, there exists a unique positive real root $\eta_2\in\mathbb{R}^{+}$ so that $Z_2(\|\xi\|)\leq 0$ when $\|\xi\|\geq\eta_2$. The coefficient of Z_2 with the highest degree is still $-\epsilon_1$. Finally, we get

$$\dot{V}\leq-\kappa V-(\zeta-\kappa)V+\chi. \tag{40}$$

Scenario 3: $\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\varepsilon}}\geq\|r_i\|$ for $i=1,...,k$, and $\omega\frac{\|r_i\|^2}{\sqrt{\|r_i\|^2+\varepsilon}}<\|r_i\|$ for $i=k+1,...,N$. Then, following the steps as in Scenario 1 and Scenario 2, we derive

$$\dot{V}\leq-\min_{i\in\Omega}\underline{\lambda}(K_i)\sum_{i=1}^{N}\|r_i\|^2-\min_{i\in\Omega}\underline{\lambda}(\bar{K}_i)\sum_{i=1}^{N}\|e_i\|^2+Z_1(\|\xi\|)$$
$$+\sum_{i=k+1}^{N}\sum_{l=0}^{2}\hat{\theta}_{li}\|\xi_i\|^{l}\|r_i\|-\sum_{i=1}^{N}\sum_{l=0}^{2}\left\{\frac{\alpha_l(\hat{\theta}_{li}-\theta_{li}^{*})^2}{2}-\frac{\alpha_l\theta_{li}^{*2}}{2}\right\}.$$
$$\leq-\min_{i\in\Omega}\underline{\lambda}(K_i)\sum_{i=1}^{N}\|r_i\|^2-\min_{i\in\Omega}\underline{\lambda}(\bar{K}_i)\sum_{i=1}^{N}\|e_i\|^2+Z_3(\|\xi\|)$$
$$-\sum_{i=1}^{N}\sum_{l=0}^{2}\left\{\frac{\alpha_l(\hat{\theta}_{li}-\theta_{li}^{*})^2}{2}-\frac{\alpha_l\theta_{li}^{*2}}{2}\right\} \tag{41}$$

where $Z_3(\|\xi\|)=Z_1(\|\xi\|)+\sum_{i=k+1}^{N}\sum_{i=1}^{N}\bar{\theta}_{li}(1+\|P_i\|)\|\xi_i\|^{l+1}+\check{\theta}_{li}(1+\|P_i\|)^2\|\xi_i\|^{2(l+1)}$. There will exist a unique root η_3 such that $Z_3(\|\xi\|)\leq 0$ when $\|\xi\|\geq\eta_3$. Similarly, it is obtained

$$\dot{V}\leq-\kappa V-(\zeta-\kappa)V+\chi. \tag{42}$$

Combining (34), (40) and (42) from Scenarios 1, 2 and 3 respectively, it can be concluded that $\dot{V}\leq-\kappa V$ when $V\geq Y$ and $\|\xi\|\geq\max\{\eta_1,\eta_2,\eta_3\}$ where

$$Y=\frac{\chi}{(\zeta-\kappa)} \tag{43}$$

and thus, the closed-loop system remains UUB with the bound

$$V(t)\leq\max\{V(0),Y\},\ \forall t\geq 0 \tag{44}$$

The definition of the Lyapunov function (17) satisfies

$$V(t)\geq\frac{\min_{i\in\Omega}\underline{\lambda}(\bar{K}_iP_i^{-1})}{2}\|e\|^2 \tag{45}$$

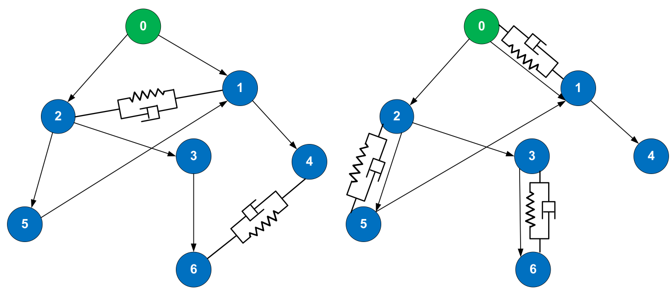
where $e=[e_i^T,\ldots,e_N^T]^T$. Using (44) and (45), it can be obtained that $\|e\|^2\leq\frac{2}{\min_{i\in\Omega}\underline{\lambda}(\bar{K}_iP_i^{-1})}\max\{V(0),Y\},\ \forall t\geq 0$, giving the uniform ultimate bound U in (16). ■

Remark 3 (Ultimate bound and gain tuning)**.** *Owing to the user-defined diagonal matrices $\bar{K}_i$ and P_i, one can notice that the ultimate bound U in (16) reduces by tuning $\bar{K}_i$ and P_i (i.e. with large values of $\bar{K}_iP_i^{-1}$). However, the fact that $\bar{m}$, θ_{li}^{*} are completely unknown prevents reduction of the bound to user-defined levels: this is consistent with robust adaptive control literature with leakage terms α_i as in (15a)-(15c) [33]. In addition, it can be noticed from (31), (37) and (41) that higher values of K_i, ϵ_1, ϵ_0 and lower values of ϵ_2 lead to faster convergence of the Lyapunov function, which may in turn cause a larger control effort. Therefore, tuning choices have to be made according to application requirements.*

VI. Numerical validation

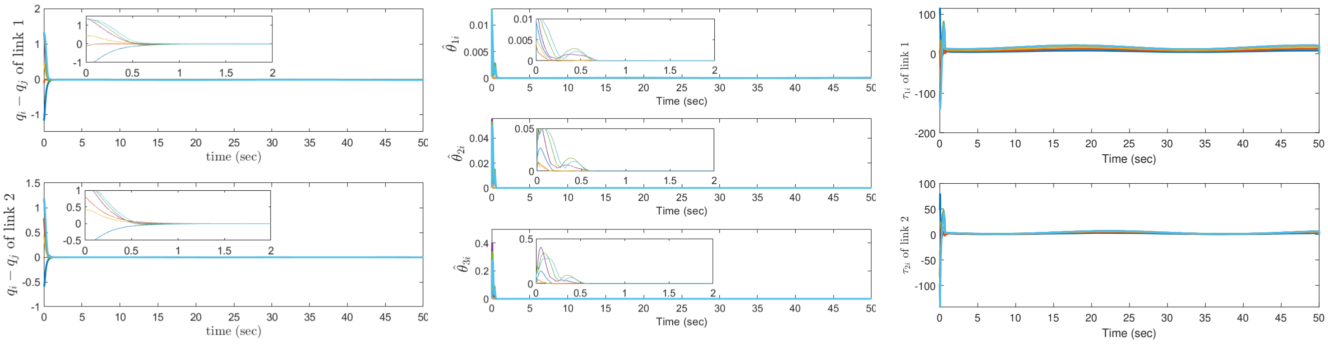
We will consider six EL systems (cf. Fig. 1), representing two-link robot arms with equations of motion as [35]:

$$\begin{bmatrix}M_i^{11} & M_i^{12}\\ M_i^{12} & M_i^{22}\end{bmatrix}\begin{bmatrix}\ddot{q}_{i1}\\ \ddot{q}_{i2}\end{bmatrix}+\begin{bmatrix}c_i\dot{q}_{i2} & c_i(\dot{q}_{i1}+\dot{q}_{i2})\\ -c_i\dot{q}_{i1} & 0\end{bmatrix}\begin{bmatrix}\dot{q}_{i1}\\ \dot{q}_{i2}\end{bmatrix}+d_i$$



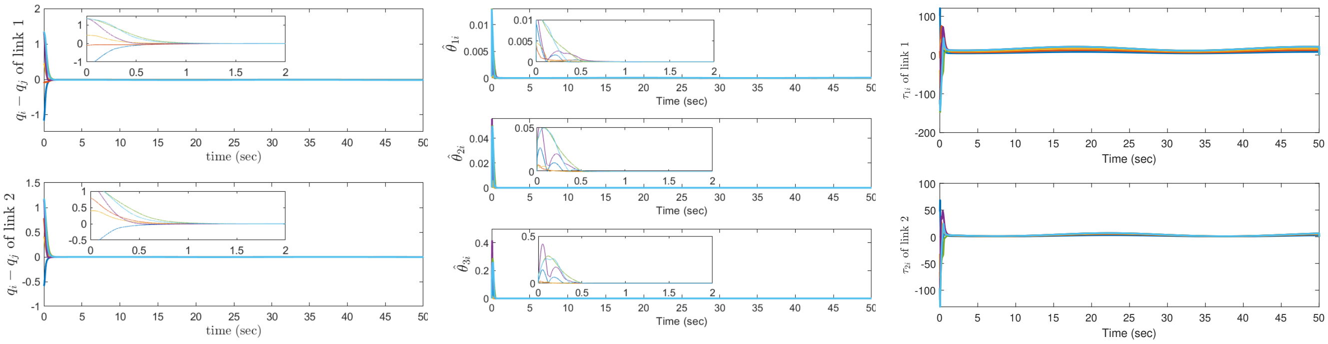
(a) Interconnection 1 (b) Interconnection 2

Figure 1: Networks used for simulations.



(a) Local position errors (b) Adaptive parameters $\hat{\theta}_{li}, l = 0, 1, 2$ (c) Control inputs τ_i

Figure 2: Adaptive synchronization behavior for interconnection 1.



(a) Local position errors (b) Adaptive parameters $\hat{\theta}_{li}, l = 0, 1, 2$ (c) Control inputs τ_i

Figure 3: Adaptive synchronization behavior for interconnection 2.

$$+ \begin{bmatrix} m_{i4} g \cos(q_{i1}) + g_i \\ g_i \end{bmatrix} + \begin{bmatrix} F_{i1}(\dot{q}_i) \\ F_{i2}(\dot{q}_i) \end{bmatrix} + H_i(e_i, \dot{e}_i) = \begin{bmatrix} \tau_{i1} \\ \tau_{i2} \end{bmatrix} \tag{46}$$

where $c_i = -m_{i3} \sin(q_{i2})$ and

$$M_i^{11} = m_{i1} + m_{i2} + 2m_{i3} \cos(q_{i2}),$$
$$M_i^{12} = m_{i2} + m_{i3} \cos(q_{i2}),$$
$$M_i^{22} = m_{i2}, \;\; g_i = m_{i5} g \cos(q_{i1} + q_{i2}).$$

The friction term is taken in non-LIP form as [20]: $F_{i1}(\dot{q}_{i1}) = f_{i1}(\tanh(f_{i2}\dot{q}_{i1}) - \tanh(f_{i3}\dot{q}_{i1})) + f_{i4} \tanh(f_{i5}\dot{q}_{i1}) + f_{i6}\dot{q}_{i1}$, $F_{i2}(\dot{q}_i) = f_{i1}(\tanh(f_{i2}\dot{q}_{i2}) - \tanh(f_{i3}\dot{q}_{i2})) + f_{i4} \tanh(f_{i5}\dot{q}_{i2}) + f_{i6}\dot{q}_{i2}$. The parameters are compactly represented as $\Theta_i = [m_{i1}\ m_{i2}\ m_{i3}\ m_{i4}\ m_{i5}\ f_{i1}\ f_{i2}\ f_{i3}\ f_{i4}\ f_{i5}\ f_{i6}]^T$ with

$$\Theta_1 = \text{col}(0.6, 1.1, 0.1, 0.6, 0.3, 0.5, 0.8, 0.9, 1.2, 0.5, 0.4),$$
$$\Theta_2 = \text{col}(0.8, 1.2, 0.1, 0.9, 0.5, 0.5, 0.8, 0.9, 1.2, 0.5, 0.4),$$
$$\Theta_3 = \text{col}(0.9, 1.3, 0.2, 1.3, 0.6, 0.5, 0.8, 0.9, 1.2, 0.5, 0.4),$$
$$\Theta_4 = \text{col}(1.1, 1.4, 0.3, 1.7, 0.7, 0.5, 0.8, 0.9, 1.2, 0.5, 0.4),$$
$$\Theta_5 = \text{col}(1.1, 1.4, 0.3, 1.7, 0.7, 0.5, 0.8, 0.9, 1.2, 0.5, 0.4),$$
$$\Theta_6 = \text{col}(1.1, 1.4, 0.3, 1.7, 0.7, 0.5, 0.8, 0.9, 1.2, 0.5, 0.4)$$

(all these values, inspired by [13], are used for simulation but are unknown for control design). We select $d_i(t) = 0.1 \sin(0.001it)[1\ 1]^T$.

Inspired by the viscoelasticity model in [36], [37], the interconnections among some agents in the form of springs-dampers

$$H_i = \sum_{j=0}^{N} s_{ij}(q_i - q_j) + \sum_{j=0}^{N} \delta_{ij}(\dot{q}_i - \dot{q}_j) \tag{47}$$

where s_{ij} is the stiffness parameter, δ_{ij} is the damping factor (which are $s_{10} = s_{01} = 0.48, s_{12} = s_{21} = 1.21, s_{25} = s_{52} = 0.085, s_{36} = s_{63} = 0.37, s_{46} = s_{64} = 0.29$ and $\delta_{01} = \delta_{10} = 40, \delta_{12} = \delta_{21} = 20, \delta_{25} = \delta_{52} = 25, \delta_{36} = \delta_{63} = 19, \delta_{46} = \delta_{64} = 9$ (all these values, inspired by [36], [37], are used for simulation and are unknown for control design).

To test the robustness, we consider two different interconnected structures as shown in Fig. 1. Let us remark that each local controller is only aware of which agents are its neighbors: it does not know neither the dynamics of the neighbors, nor whether there are spring-damper interconnections.

The controller is as in (8) with $K_i = 7.5I_2,\ \bar{K}_i = I_2,\ \omega = 2, \varepsilon = 0.1,\ P_i = 33I_2$. The parameters in the adaptive law (15) are $\epsilon_0 = 1, \epsilon_1 = 3 \cdot 10^{-4}, \epsilon_2 = 7.5 \cdot 10^{-5}, \alpha_{0i} = \alpha_{1i} = \alpha_{2i} = 3000, \beta_i = 10$.

Figs. 2a and 3a show that the synchronization error converges close to zero for both interconnection structures and, consequently, the adaptive gains in Figs. 2b and 3b also converge close to zero. The inputs are in Figs. 2c and 3c, where it can be noticed that input oscillations are in a bounded range caused by the sinusoidal disturbance d.

VII. Conclusions

A new adaptive synchronization protocol for Euler Lagrange networks has been proposed addressing problems usually neglected in related literature. The main feature of the protocol is to cope with reduced structural knowledge, i.e. not requiring linear-in-the-parameter structure of the uncertainty and allowing the agents to be interconnected before control design by unknown state-dependent terms with no a priori bound.

References

[1] A. M. Bloch, D. E. Chang, N. E. Leonard, and J. E. Marsden, "Controlled Lagrangians and the stabilization of mechanical systems. ii. potential shaping," *IEEE Transactions on Automatic Control*, vol. 46, no. 10, pp. 1556–1571, 2001.

[2] S. Roy and S. Baldi, "Towards structure-independent stabilization for uncertain underactuated Euler-Lagrange systems," *Automatica*, vol. 113, p. 108775, 2020.

[3] Z. Meng, W. Ren, and Z. You, "Distributed finite-time attitude containment control for multiple rigid bodies," *Automatica*, vol. 46, no. 12, pp. 2092–2099, 2010.

[4] J. Lee, C. Lee, N. Tsagarakis, and S. Oh, "Residual-based external torque estimation in series elastic actuators over a wide stiffness range: Frequency domain approach," *IEEE Robotics and Automation Letters*, vol. 3, no. 3, pp. 1442–1449, 2018.

[5] S.-J. Chung, S. Bandyopadhyay, I. Chang, and F. Y. Hadaegh, "Phase synchronization control of complex networks of Lagrangian systems on adaptive digraphs," *Automatica*, vol. 49, no. 5, pp. 1148–1161, 2013.

[6] N. Sharma, S. Bhasin, Q. Wang, and W. E. Dixon, "Predictor-based control for an uncertain Euler-Lagrange system with input delay," *Automatica*, vol. 47, no. 11, pp. 2332–2342, 2011.

[7] R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar, "Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems," *Automatica*, vol. 38, no. 4, pp. 585–596, 2002.

[8] S. Roy and S. Baldi, "The role of uncertainty in adaptive control of switched Euler-Lagrange systems," in *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE, 2019, pp. 72–77.

[9] Y. Tang, M. Tomizuka, G. Guerrero, and G. Montemayor, "Decentralized robust control of mechanical systems," *IEEE Transactions on Automatic control*, vol. 45, no. 4, pp. 771–776, 2000.

[10] J. Mei, W. Ren, and G. Ma, "Distributed coordinated tracking with a dynamic leader for multiple Euler-Lagrange systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 1415–1421, 2011.

[11] A. Abdessameud, A. Tayebi, and I. G. Polushin, "Leader-follower synchronization of Euler-Lagrange systems with time-varying leader trajectory and constrained discrete-time communication," *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2539–2545, 2016.

[12] H. Cai and J. Huang, "The leader-following consensus for multiple uncertain Euler-Lagrange systems with an adaptive distributed observer," *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 3152–3157, 2015.

[13] M. Lu and L. Liu, "Leader–following consensus of multiple uncertain Euler-Lagrange systems with unknown dynamic leader," *IEEE Transactions on Automatic Control*, vol. 64, no. 10, pp. 4167–4173, 2019.

[14] M. Lu, L. Liu, and G. Feng, "Adaptive tracking control of uncertain Euler-Lagrange systems subject to external disturbances," *Automatica*, vol. 104, pp. 207–219, 2019.

[15] S. Wang, H. Zhang, S. Baldi, and R. Zhong, "Leaderless consensus of heterogeneous multiple euler-lagrange systems with unknown disturbance," *IEEE Transactions on Automatic Control*, pp. 1–1, 2022.

[16] N. Cho, H. Shin, Y. Kim, and A. Tsourdos, "Composite model reference adaptive control with parameter convergence under finite excitation," *IEEE Transactions on Automatic Control*, vol. 63, no. 3, pp. 811–818, 2018.

[17] S. B. Roy, S. Bhasin, and I. N. Kar, "Combined MRAC for unknown MIMO LTI systems with parameter convergence," *IEEE Transactions on Automatic Control*, vol. 63, no. 1, pp. 283–290, 2018.

[18] S. El-Ferik, A. Qureshi, and F. L. Lewis, "Neuro-adaptive cooperative tracking control of unknown higher-order affine nonlinear systems," *Automatica*, vol. 50, no. 3, pp. 798–808, 2014.

[19] Z. Li, Z. Duan, and F. L. Lewis, "Distributed robust consensus control of multi-agent systems with heterogeneous matching uncertainties," *Automatica*, vol. 50, no. 3, pp. 883–889, 2014.

[20] C. Makkar, G. Hu, W. G. Sawyer, and W. E. Dixon, "Lyapunov-based tracking control in the presence of uncertain nonlinear parameterizable friction," *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 1988–1994, 2007.

[21] J. Huang, W. Wang, C. Wen, J. Zhou, and G. Li, "Distributed adaptive leader–follower and leaderless consensus control of a class of strict-feedback nonlinear systems: a unified approach," *Automatica*, vol. 118, p. 109021, 2020.

[22] L. Zhu, Z. Chen, and R. H. Middleton, "A general framework for robust output synchronization of heterogeneous nonlinear networked systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 8, pp. 2092–2107, 2015.

[23] C. P. Bechlioulis and G. A. Rovithakis, "Decentralized robust synchronization of unknown high order nonlinear multi-agent systems with prescribed transient and steady state performance," *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 123–134, 2016.

[24] B. C. Gruenwald, E. Arabi, T. Yucelen, A. Chakravarthy, and D. McNeely, "Decentralised adaptive architectures for control of large-scale active–passive modular systems with stability and performance guarantees," *International Journal of Control*, vol. 93, no. 3, pp. 490–504, 2020.

[25] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Synchronization and power sharing for droop-controlled inverters in islanded microgrids," *Automatica*, vol. 49, no. 9, pp. 2603–2611, 2013.

[26] T. Tao, S. Roy, and S. Baldi, "Stable adaptation in multi-area load frequency control under dynamically-changing topologies," *IEEE Transactions on Power Systems*, vol. 36, no. 4, pp. 2946–2956, 2021.

[27] M. Franceschelli and P. Frasca, "Stability of open multi-agent systems and applications to dynamic consensus," *IEEE Transactions on Automatic Control*, pp. 1–1, 2020.

[28] S. Roy, S. Baldi, and L. M. Fridman, "On adaptive sliding mode control without a priori bounded uncertainty," *Automatica*, vol. 111, p. 108650, 2020.

[29] H. Modares, F. L. Lewis, W. Kang, and A. Davoudi, "Optimal synchronization of heterogeneous nonlinear systems with unknown dynamics," *IEEE Transactions on Automatic Control*, vol. 63, no. 1, pp. 117–131, 2017.

[30] C. Chen, K. Xie, F. L. Lewis, S. Xie, and A. Davoudi, "Fully distributed resilience for adaptive exponential synchronization of heterogeneous multi-agent systems against actuator faults," *IEEE Transactions on Automatic Control*, vol. 64, no. 8, pp. 3347–3354, 2018.

[31] M. W. Spong, S. Hutchinson, and M. Vidyasagar, *Robot modeling and control*. John Wiley & Sons, 2020.

[32] J.-X. Zhang and G.-H. Yang, "Fault-tolerant output-constrained control of unknown Euler-Lagrange systems with prescribed tracking accuracy," *Automatica*, vol. 111, p. 108606, 2020.

[33] P. Ioannou and J. Sun, *Robust Adaptive Control*. Dover Books on Electrical Engineering, 2012.

[34] S. Russ, "A translation of Bolzano's paper on the intermediate value theorem," *Historia Mathematica*, vol. 7, no. 2, pp. 156–185, 1980.

[35] F. L. Lewis, D. Dawson, and C. T. Abdallah, *Control of Robot Manipulators*. Prentice Hall PTR, 1993.

[36] S. Arimoto and M. Sekimoto, "Human-like movements of robotic arms with redundant DOFs: virtual spring-damper hypothesis to tackle the Bernstein problem," in *Proceedings 2006 IEEE International Conference on Robotics and Automation (ICRA 2006)*, 2006, pp. 1860–1866.

[37] H. Kimura, Y. Fukuoka, and A. H. Cohen, "Biologically inspired adaptive walking of a quadruped robot," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 365, no. 1850, pp. 153–170, 2007.