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Adaptive synchronization of uncertain underactuated Euler-Lagrange agents

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***Abstract*—This work proposes a framework for adaptive synchronization of uncertain underactuated Euler-Lagrange (EL) agents. The designed distributed controller can handle both state-dependent uncertain system dynamics terms and state-dependent uncertain interconnection terms among neighboring agents. No structural knowledge of such terms is required other than the standard properties of EL systems (positive definite mass matrix, bounded gravity, velocity-dependent friction bound, etc.). The study of stability relies on a suitable analysis of the non-actuated and the actuated synchronization errors, resulting in stable error dynamics perturbed by parametrized state-dependent uncertainty. This uncertainty is tackled via appropriate adaptation laws, giving stability in the uniform ultimate boundedness sense, in line with the available literature on state-dependent uncertain system dynamics and/or state-dependent uncertain interconnections. An example with a network of boom cranes is used to validate the proposed approach.**

***Index Terms*—Underactuated systems, Euler-Lagrange dynamics, adaptive synchronization, distributed control.**

I. Introduction

Underactuated systems have fewer control inputs than degrees of freedom. Lower cost or more tolerance to faults can make underactuated robots [1], underactuated cranes [2], [3], underactuated vehicles [4], [5] etc. preferable to their fully-actuated versions. Although approaches to control underactuated systems span from feedback linearization [6]–[8], passivity [9]–[11], and optimal control [12], [13], the inevitable *system uncertainties* put most of these approaches at stake and call for appropriate adaptive designs [14]. One such class of designs relies on extending sliding mode control [15]–[18] in an adaptive sense, giving adaptive-robust methods originally developed for fully-actuated systems [19]–[21]. These methods only require the knowledge of an uncertainty bound around a nominal mass matrix, while all other system terms (Coriolis, gravity, friction) can be unknown [22], [23].

A. Challenges and related works

Nowadays, adaptive-robust methods constitute a mature and general framework for fully-actuated systems. Unfortunately, such maturity and generality are missing for underactuated systems, where it is common to consider special classes of dynamics [24], specific applications [25], [26] or ad-hoc structural assumptions on the system terms [7]–[9], [17], [18], [27]. The most adopted of these structural assumptions requires the mass matrix to depend on the actuated states only [18], or on the non-actuated states only [9], [27]. This assumption turns out to be falsified in several scenarios, such as biped robots [1], boom cranes [3], surface vessels [5], [26], among others.

When considering interconnected dynamics as in multi-agent systems literature, an additional source of uncertainty arises from the *interconnection terms* among the agents, which may take a *state-dependent form* analogous to the state-dependent system terms. Power systems are a representative example, where power flows across different areas create state-dependent uncertain interconnections that might result in inter-area oscillations or other problems [28]–[31]. When designing distributed approaches to synchronization/consensus, the presence of these uncertain interconnection terms is traditionally overlooked. In most approaches, the interconnection is only the result of the synchronization/consensus protocol, i.e., no intrinsic interconnection is considered before such protocol is designed [32]–[34]. Notable exceptions appear in fully-actuated dynamics controlled in a decentralized fashion, i.e., without communication among the agents: here, state-dependent and possibly uncertain interconnection terms have been considered [35], [36]. For underactuated dynamics, distributed control was considered in [5] without uncertainty on the system dynamics and on the interconnection terms, whereas the adaptive method in [37] and the non-adaptive ones in [9]–[11] are for single agents, i.e., the issue of uncertain interconnections does not arise. These works on underactuated systems still rely on the aforementioned structural assumptions on the system terms.

B. Contributions of this work

The overview above shows that distributed approaches for underactuated dynamics impose crucial structural assumptions e.g., on the mass matrix and on the existence of interconnections. The design of a general underactuated adaptive-robust framework with reduced structural assumptions is still an open problem in the field. This motivates a research on new distributed approaches for underactuated systems with limited knowledge of the system dynamics and of the interconnection terms. The main contributions of this work are:

- A distributed adaptive protocol for synchronization of underactuated Euler-Lagrange (EL) systems is designed in the presence of uncertain system terms.
- No structural assumptions are imposed other than the standard properties of EL systems (positive definite mass matrix, bounded gravity, velocity-dependent friction

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bound, etc.). We do not impose the mass matrix to depend on the actuated states only, or on the non-actuated states only. We just require the mass matrix to satisfy the Strong Inertial Coupling [8] (see Assumption 2), a well-known condition for controllability;
- State-dependent uncertain interconnections among the underactuated agents are considered to exist before the control design, instead of only being a result of coupling caused by the control. Recall that, even for fully-actuated systems, the literature has shown that state-dependent uncertainties prevent asymptotic tracking [29]–[31], [36], [38], seeking stability as uniform ultimate boundedness, which is the approach we also follow.

The proposed approach provides a convenient underactuated extension of adaptive-robust methods, with the following distinguishing contributions that make this extension possible:

a) Suitable dynamics are derived for the actuated and the non-actuated errors so as to fit the adaptive control goal (cf. the parametrized state-dependent perturbations in Sect. IV-B). As state-dependent uncertainties appear in both the actuated and the non-actuated part, a dedicated uncertainty analysis is carried out in Sect. V-A.
b) A novel stability analysis is presented that, in addition to handling state-dependent uncertainties, can handle distributed information and the different state space regions arising from the control law (cf. the proof of Theorem 1 in the appendix).
c) The proposed adaptive laws in Sect. V-B depart from standard adaptive-robust laws, as they are not designed using a standard leakage, but based on an appropriate state-dependent leakage (cf. the discussion in Remark 6).

The rest of the paper is organized as follows. Section II gives the basic notation, followed by the problem formulation in Section III. Section IV gives key steps about the distributed control law and the error dynamics, leading to the uncertainty analysis and stability result in Section V. Using a network of boom cranes as a numerical case study, simulations and comparisons with the state-of-the-art are in Section VI.

II. Notation and Basic graph theory

We adopt a standard notation, with $\mathbb{R}^+$ for the set of positive real numbers, $\otimes$ for the Kronecker product, I_N for the identity matrix of dimension N, 1_N for the N-dimensional vector of ones, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ for the minimum and maximum singular value of a matrix, and $\|\cdot\|$ for the Euclidean norm.

We use graphs to represent a network of nodes (or agents). A directed graph $\mathcal{G}$ is described by the pair $(\mathcal{V}, \mathcal{E})$, comprising the node set $\mathcal{V} \triangleq \{v_1, \dots, v_N\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge is a pair of nodes $(v_j, v_i) \in \mathcal{E}$ representing that agent i has access to the information from agent j, i.e., agent j is a neighbor of agent i (not necessarily vice versa). The neighbor set of agent i is denoted by $\mathcal{N}_i$.

Weighted edges in $\mathcal{E}$ are described by the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ associated with $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij},\ i \neq j$.

The node set $\mathcal{V}$ does not include the leader node v_0: for those agents i that can receive information from v_0, we have an edge with $b_i > 0$; otherwise, $b_i = 0$. The following assumption is standard in multi-agent systems literature [32], [33]:

Assumption 1. *The directed augmented graph comprising $\mathcal{G}$ and edges from the leader node contains a spanning tree with the root being the leader node.*

Remark 1. *Assumption 1 implies that there exists a communication path from the leader to any follower node: it is a condition to make synchronization over a network feasible, cf. Remark 4 and Lemma 1 later on.*

III. Synchronization Problem

Consider the following network of underactuated Euler-Lagrange (EL) agents $(i = 1, \dots, N)$:

$$\begin{aligned} M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) + F_i(\dot{q}_i) \\ + H_i(e_i, \dot{e}_i) + d_i = [0_{(n-m)}^T \ \tau_i^T]^T \end{aligned} \tag{1}$$

where $q_i, \dot{q}_i \in \mathbb{R}^n$ are the generalized coordinates and their derivatives, $d_i \in \mathbb{R}^n$ is an external bounded disturbance with $\|d_i\| \leq \bar{d}_i$ ($\bar{d}_i$ an unknown constant), $\tau_i \in \mathbb{R}^m$ with $n - m \leq m < n$ is the control input. The system dynamics (1) comprises the mass matrix $M_i(q_i) \in \mathbb{R}^{n \times n}$, the Coriolis matrix $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{n \times n}$, the gravity term $G_i(q_i) \in \mathbb{R}^n$, the friction term $F_i(\dot{q}_i) \in \mathbb{R}^n$, and the interconnection term $H_i(e_i, \dot{e}_i) \in \mathbb{R}^n$ which depends on the local synchronization error and its derivative. For convenience of analysis, arrange the generalized coordinates according to non-actuated and actuated dynamics as $q_i = [q_{\mathrm{u}i}^T \ q_{\mathrm{a}i}^T]^T$ with $q_{\mathrm{u}i} \in \mathbb{R}^{n-m}$ and $q_{\mathrm{a}i} \in \mathbb{R}^m$. Accordingly, the local synchronization error e_i can be decomposed as $e_i = [e_{\mathrm{u}i}^T \ e_{\mathrm{a}i}^T]^T$, resulting in

$$e_{\mathrm{u}i}(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \big[q_{\mathrm{u}i}(t) - q_{\mathrm{u}j}(t)\big] + b_i \big[q_{\mathrm{u}i}(t) - q_{\mathrm{u}0}\big] \tag{2a}$$

$$e_{\mathrm{a}i}(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \big[q_{\mathrm{a}i}(t) - q_{\mathrm{a}j}(t)\big] + b_i (q_{\mathrm{a}i}(t) - q_{\mathrm{a}0}) \tag{2b}$$

and analogously for $\dot{e}_i = [\dot{e}_{\mathrm{u}i}^T \ \dot{e}_{\mathrm{a}i}^T]^T$

$$\dot{e}_{\mathrm{u}i}(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \big[\dot{q}_{\mathrm{u}i}(t) - \dot{q}_{\mathrm{u}j}(t))\big] + b_i \dot{q}_{\mathrm{u}i}(t) \tag{3a}$$

$$\dot{e}_{\mathrm{a}i}(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \big[\dot{q}_{\mathrm{a}i}(t) - \dot{q}_{\mathrm{a}j}(t)\big] + b_i \dot{q}_{\mathrm{a}i}(t). \tag{3b}$$

In principle, one could consider time-varying leader trajectories (cf. [22], [23], [36], [38] for fully-actuated and [24] for a class of underactuated EL systems): however, this poses the challenging problem of how to define a feasible trajectory for a general underactuated system. Because this problem goes beyond the scope of this work, we consider a fixed-point leader position $q_0 = [q_{\mathrm{u}0}^T,\ q_{\mathrm{a}0}^T]^T \in \mathbb{R}^n$ in (2a)-(2b), as common in the literature (cf. [7], [10], [11], [15]–[17], [27], [37]).

In line with standard EL literature [19], [39], [40], the following system properties are assumed:

Property 1. *There exist constants $\bar{c}_i$, $\bar{g}_i$, $\bar{f}_i$, $\bar{h}_{1i}$, $\bar{h}_{2i}$, $\bar{h}_{3i}$, $\bar{h}_{4i}$, $\bar{h}_{5i} \in \mathbb{R}^+$ such that the following upper bound structures hold*

$\|C_i(q_i,\dot q_i)\| \leq \bar c_i\|\dot q_i\|,\ \|G_i(q_i)\| \leq \bar g_i,\ \|F_i(\dot q_i)\| \leq \bar f_i\|\dot q_i\|,$
$\|H_i(e_i,\dot e_i)\| \leq \bar h_{1i}+\bar h_{2i}\|e_i\|+\bar h_{3i}\|\dot e_i\|+\bar h_{4i}\|e_i\|^2+\bar h_{5i}\|\dot e_i\|^2.$

Property 2. *The mass matrix $M_i(q_i)$ is symmetric, positive definite and there exist positive constants $\underline{m}$ and $\overline{m}$ such that $0 \leq \underline{m}I_n \leq M_i(q_i) \leq \overline{m}I_n$, $\forall q_i, \forall i$.*

Remark 2. *The interconnection term H_i represents the uncertain interaction between agents, existent before the design of the synchronization protocol. Literature on multi-agent systems typically neglects this term [32]–[34]: its presence requires a novel synchronization protocol.*

The upper bounds of C_i, G_i, F_i, H_i, d_i in Property 1 will be taken to be unknown, i.e., not used in the control design. The upper bound structure of H_i is taken to be quadratic in accordance with the quadratic effect of the term $C_i\dot q_i$ in (1). Note that, although some terms (e.g., d_i and G_i) can reasonably be assumed to be bounded a priori by constants, other terms (e.g., C_i, F_i and H_i) are state-dependent. Indeed, the constants $\bar c_i, \bar f_i, \bar h_{1i}, \bar h_{2i}, \bar h_{3i}, \bar h_{4i}, \bar h_{5i}$ in Property 1 multiply state-dependent terms, so that one cannot assume a priori boundedness of these uncertain terms.

For brevity, let us omit the dependence of the system dynamics terms on the state variables, and let us organize the system dynamics terms as

$$M_i \triangleq \begin{bmatrix} M_{\mathrm{uu}i} & M_{\mathrm{au}i} \\ M_{\mathrm{ua}i} & M_{\mathrm{aa}i} \end{bmatrix}, \tag{4a}$$

$$E_i \triangleq C_i\dot q_i + G_i + F_i + H_i + d_i = [E_{\mathrm{u}i}^T\ E_{\mathrm{a}i}^T]^T \tag{4b}$$

where $M_{\mathrm{uu}i} \in \mathbb{R}^{(n-m)\times(n-m)},\ M_{\mathrm{au}i} \in \mathbb{R}^{(n-m)\times m},\ M_{\mathrm{aa}i} \in \mathbb{R}^{m\times m},\ E_{\mathrm{u}i} \in \mathbb{R}^{n-m},\ E_{\mathrm{a}i} \in \mathbb{R}^{m}$. Therefore, the dynamics (1) for each agent can be represented as

$$\ddot q_{\mathrm{u}i} = -M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\ddot q_{\mathrm{a}i} - R_{\mathrm{u}i} \tag{5a}$$

$$\ddot q_{\mathrm{a}i} = M_{\mathrm{s}i}^{-1}\tau_i + R_{\mathrm{a}i} \tag{5b}$$

with

$$\begin{aligned} R_{\mathrm{u}i} &\triangleq M_{\mathrm{uu}i}^{-1}E_{\mathrm{u}i}, \\ R_{\mathrm{a}i} &\triangleq M_{\mathrm{s}i}^{-1}(M_{\mathrm{ua}i}M_{\mathrm{uu}i}^{-1}E_{\mathrm{u}i} - E_{\mathrm{a}i}), \\ M_{\mathrm{s}i} &\triangleq M_{\mathrm{aa}i} - M_{\mathrm{ua}i}M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}. \end{aligned}$$

As M_i in (4a) is positive definite, $M_{\mathrm{s}i}$ and $M_{\mathrm{uu}i}$ are positive definite (thus invertible). The following assumption, known as Strong Inertial Coupling, ensures controllability of underactuated EL dynamics in line with standard literature [6]–[8].

Assumption 2. *(Strong Inertial Coupling [6]–[8]) The following rank condition holds:*

$$\mathrm{rank}(M_{\mathrm{au}i}(q_i)) = n - m \leq m,\ \forall q_i \in \mathbb{R}^n. \tag{6}$$

Remark 3. *The name 'Strong Inertial Coupling' comes from the coupling created by $M_{\mathrm{au}i}$ between the actuated and the non-actuated states. As the non-actuated states cannot be directly controlled,* (6) *ensures controllability as it allows to design a virtual control for the non-actuated states. This condition appears in most works about underactuated EL systems (also cf. [40]–[42]) and has been mostly used in the framework of backstepping.*

Due to the block structure in (4a), the uncertainty in the mass matrix M_i is addressed in a different way from the other dynamic terms. It is assumed that $M_{\mathrm{s}i} \in \mathbb{R}^{m\times m}$ can be decomposed as $M_{\mathrm{s}i} = \hat M_{\mathrm{s}i} + \Delta M_{\mathrm{s}i}$ where $\hat M_{\mathrm{s}i}$ is the nominal part (used for control design) and $\Delta M_{\mathrm{s}i}$ is the unknown part satisfying the following bound conditions:

Assumption 3. *Define the matrix $T_i = M_{\mathrm{s}i}^{-1}\hat M_{\mathrm{s}i} - I_m$. Then there exists a known scalar $\bar T \in \mathbb{R}^+$ such that*

$$\|T_i\| \leq \bar T < 1. \tag{7}$$

Assumption 3 implies that an upper bound on the uncertainty of $M_{\mathrm{s}i}$ is known. It is often adopted in the literature to describe uncertainty in mass matrix [19]–[21].

Let $B = \mathrm{diag}(b_1,\ldots,b_N) \in \mathbb{R}^{N\times N}$. From (2), we obtain

$$\begin{aligned} e_{\mathrm{u}} &= -(\mathcal{L}+B)\otimes(q_{\mathrm{u}} - \underline{q}_{\mathrm{u}0}) = -(\mathcal{L}+B)\otimes\delta_{\mathrm{u}} \\ e_{\mathrm{a}} &= -(\mathcal{L}+B)\otimes(q_{\mathrm{a}} - \underline{q}_{\mathrm{a}0}) = -(\mathcal{L}+B)\otimes\delta_{a} \end{aligned}$$

where $e_{\mathrm{u}} = [e_{\mathrm{u}1}^T,\ldots,e_{\mathrm{u}N}^T]^T,\ e_{\mathrm{a}} = [e_{\mathrm{a}1}^T,\ldots,e_{\mathrm{a}N}^T]^T,\ q_{\mathrm{u}} = [q_{\mathrm{u}1}^T,\ldots,q_{\mathrm{u}N}^T]^T,\ q_{\mathrm{a}} = [q_{\mathrm{a}1}^T,\ldots,q_{\mathrm{a}N}^T]^T,\ \underline{q}_{\mathrm{a}0} = 1_N\otimes q_{\mathrm{a}0},\ \underline{q}_{\mathrm{u}0} = 1_N\otimes q_{\mathrm{u}0}$. The errors $\delta_{\mathrm{a}} = (q_{\mathrm{a}} - \underline{q}_{\mathrm{a}0}) \in \mathbb{R}^{nN}$, $\delta_{\mathrm{u}} = (q_{\mathrm{u}} - \underline{q}_{\mathrm{u}0}) \in \mathbb{R}^{nN}$ represent the global synchronization error with the leader in actuated and non-actuated states, respectively.

Remark 4. *In a distributed control setting, the leader's state contained in $\delta_{\mathrm{a}},\ \delta_{\mathrm{u}}$ is not directly accessible to all followers. The existence of a directed spanning tree, as in Assumption 1, allows even those agents without direct access to the leader's information to track the leader's state by synchronizing e_{a} and e_{u} with the neighbors with which they can communicate.*

The directed spanning tree property implies the following.

Lemma 1. *[32] Under Assumption 1, the local and global synchronization errors are related by*

$$\|\delta_{\mathrm{u}}\| \leq \frac{\|e_{\mathrm{u}}\|}{\lambda_{\min}(\mathcal{L}+B)} \tag{9a}$$

$$\|\delta_{\mathrm{a}}\| \leq \frac{\|e_{\mathrm{a}}\|}{\lambda_{\min}(\mathcal{L}+B)} \tag{9b}$$

with $\lambda_{\min}(\mathcal{L}+B)$ the minimum singular value of $\mathcal{L}+B$.

Due to the presence of state-dependent uncertainties, it has been shown in the literature that asymptotic tracking is hard to achieve even for fully-actuated system. Therefore, stability is sought as uniform ultimate boundedness, which is in line with the existing literature [29]–[31], [36].

Definition 1. *(Uniform Ultimate Boundedness (UUB) [43]) A signal is uniformly ultimately bounded if there exists a convex and compact set $\mathcal{C}$ such that for $\forall\delta(0) = \delta_0$, there exists a finite time $T(\delta_0)$ such that $\delta \in \mathcal{C}$ for all $t > T(\delta_0)$.*

Problem Formulation. *Let $\delta_i = [\delta_{\mathrm{u}i}^T\ \delta_{\mathrm{a}i}^T]^T$. Design a distributed (i.e., using state information from neighboring agents) adaptive mechanism for the network of underactuated systems* (1) *guaranteeing that the global synchronization error $\delta = [\delta_1^T,\ldots,\ \delta_N^T]^T$ is uniformly ultimately bounded.*

IV. Preliminary design steps

We give the distributed control law (Sect. IV-A) and the dynamics of the synchronization error (Sect. IV-B). These steps are useful to derive the proposed adaptation in Sect. V.

A. Distributed Control Law

Define a tracking error variable:

$$r_i = \Theta_{\mathrm{a}i}\dot{e}_{\mathrm{a}i} + \Xi_{\mathrm{a}i}e_{\mathrm{a}i} + \Theta_{\mathrm{u}i}\dot{e}_{\mathrm{u}i} + \Xi_{\mathrm{u}i}e_{\mathrm{u}i} \tag{10}$$

where $\Theta_{\mathrm{a}i},\ \Xi_{\mathrm{a}i} \in \mathbb{R}^{m\times m}$ are user-defined positive definite matrices, and $\Theta_{\mathrm{u}i},\ \Xi_{\mathrm{u}i} \in \mathbb{R}^{m\times(n-m)}$ are user-defined full rank matrices. The distributed controller is designed as

$$\tau_i = \frac{\hat{M}_{\mathrm{s}i}}{\breve{a}_i}\big(-r_i - \bar{\tau}_i\big), \quad \bar{\tau}_i = \rho_i\,\mathrm{sat}(S_i,\varphi) \tag{11}$$

where $\mathrm{sat}(S_i,\varphi) = \begin{cases} \frac{S_i}{\|S_i\|}, & \|S_i\| \geq \varphi \\ \frac{S_i}{\varphi}, & \|S_i\| < \varphi \end{cases}$ is a saturation term with $S_i = B_1^T P_{\mathrm{a}i}\omega_{\mathrm{a}i}$, $B_1 = [0\ I_m]^T$, $\omega_{\mathrm{a}i} = [e_{\mathrm{a}i}^T\ \dot{e}_{\mathrm{a}i}^T]^T$, $\varphi > 0$ a user-defined scalar, and $P_{\mathrm{a}i} > 0$ the solution to the Lyapunov equation $A_{\mathrm{a}i}^T P_{\mathrm{a}i} + P_{\mathrm{a}i}A_{\mathrm{a}i} = -Q_{\mathrm{a}i}$ where $A_{\mathrm{a}i} = \begin{bmatrix} 0 & I_m \\ -\Xi_{\mathrm{a}i} & -\Theta_{\mathrm{a}i}\end{bmatrix}$ is Hurwitz by design, and $Q_{\mathrm{a}i} > 0$ is user-designed; ρ_i will be defined later in Sect. V-B to deal with the uncertainty in the system dynamics.

B. Synchronization Error Dynamics

Using (3b) and (5b), we obtain the synchronization error dynamics in the actuated dynamics as

$$\ddot{e}_{\mathrm{a}i} = \breve{a}_i\big(M_{\mathrm{s}i}^{-1}\tau_i + R_{\mathrm{a}i}\big) - \sum_{j\in\mathcal{N}_i} a_{ij}\big(M_{\mathrm{s}j}^{-1}\tau_j + R_{\mathrm{a}j}\big) \tag{12}$$

where $\breve{a}_i = b_i + \sum_{j\in\mathcal{N}_i} a_{ij}$. Substituting (11) into (12) gives

$$\begin{aligned}\ddot{e}_{\mathrm{a}i} =& \big(M_{\mathrm{s}i}^{-1}\hat{M}_{\mathrm{s}i} - I_m\big)\big(-r_i - \bar{\tau}_i\big) - \big(r_i + \bar{\tau}_i\big) + \breve{a}_iR_{\mathrm{a}i} \\ &- \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\Big[\big(M_{\mathrm{s}j}^{-1}\hat{M}_{\mathrm{s}j} - I_m\big)\big(-r_j - \bar{\tau}_j\big) \\ &- \big(r_j + \bar{\tau}_j\big)\Big] - a_{ij}R_{\mathrm{a}j} \\ =& -r_i - (I_m + T_i)\bar{\tau}_i + \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}(I_m + T_j)\bar{\tau}_j + \phi_{ij}\end{aligned} \tag{13}$$

where $\bar{a}_{ij} = \frac{a_{ij}}{\breve{a}_j}$ and $\phi_{ij} = -T_ir_i + \breve{a}_iR_{\mathrm{a}i} + \sum_{j\in\mathcal{N}_i}\big[\bar{a}_{ij}(I_m + T_j)r_j - a_{ij}R_{\mathrm{a}j}\big]$. According to (10), (13) can be rewritten as

$$\begin{aligned}\ddot{e}_{\mathrm{a}i} =& -\Theta_{\mathrm{a}i}\dot{e}_{\mathrm{a}i} - \Xi_{\mathrm{a}i}e_{\mathrm{a}i} - (I_m + T_i)\bar{\tau}_i \\ &+ \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}(I_m + T_j)\bar{\tau}_j + \psi_{ij}\end{aligned} \tag{14}$$

with $\psi_{ij} = \phi_{ij} - \big(\Theta_{\mathrm{u}i}\dot{e}_{\mathrm{u}i} + \Xi_{\mathrm{u}i}e_{\mathrm{u}i}\big)$. Using (14), we have

$$\begin{aligned}\dot{\omega}_{\mathrm{a}i} =& A_{\mathrm{a}i}\omega_{\mathrm{a}i} + B_1\Big[-(I_m + T_i)\bar{\tau}_i + \psi_{ij} \\ &+ \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}(I_m + T_j)\bar{\tau}_j\Big].\end{aligned} \tag{15}$$

Similarly, using (3a) and (5a), the synchronization error in the non-actuated dynamics turns out to be

$$\begin{aligned}\ddot{e}_{\mathrm{u}i} =& -\breve{a}_i\big(M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\ddot{q}_{\mathrm{a}i} + R_{\mathrm{u}i}\big) \\ &+ \sum_{j\in\mathcal{N}_i} a_{ij}\big(M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}\ddot{q}_{\mathrm{a}j} + R_{\mathrm{u}j}\big) \\ =& -\breve{a}_i\Big[M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\big(M_{\mathrm{s}i}^{-1}\tau_i + R_{\mathrm{a}i}\big) + R_{\mathrm{u}i}\Big] \\ &+ \sum_{j\in\mathcal{N}_i} a_{ij}\Big[M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}\big(M_{\mathrm{s}j}^{-1}\tau_i + R_{\mathrm{a}j}\big) + R_{\mathrm{u}j}\Big].\end{aligned} \tag{16}$$

Similar to (13), substituting (11) into (16), gives

$$\begin{aligned}\ddot{e}_{\mathrm{u}i} =& -M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\Big[\big(M_{\mathrm{s}i}^{-1}\hat{M}_{\mathrm{s}i} - I_m\big)\big(-r_i - \bar{\tau}_i\big) - \big(r_i + \bar{\tau}_i\big)\Big] \\ &+ \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}\Big[\big(M_{\mathrm{s}j}^{-1}\hat{M}_{\mathrm{s}j} - I_m\big)\big(-r_j - \bar{\tau}_j\big) \\ &- \big(r_j + \bar{\tau}_j\big)\Big] + \breve{a}_i\big(M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}R_{\mathrm{a}i} + R_{\mathrm{u}i}\big) \\ &+ \sum_{j\in\mathcal{N}_i} a_{ij}\big(M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}R_{\mathrm{a}j} + R_{\mathrm{u}j}\big) \\ =& M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\big(I_m + T_i\big)\bar{\tau}_i + \phi'_{ij} \\ &- \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}\big(I_m + T_j\big)\bar{\tau}_j\end{aligned} \tag{17}$$

where

$$\begin{aligned}\phi'_{ij} =& M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\big(I_m + T_i\big)r_i - \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}\big(I_m + T_j\big)r_j \\ &- \breve{a}_i\big(M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}R_{\mathrm{a}i} + R_{\mathrm{u}i}\big) + \sum_{j\in\mathcal{N}_i} a_{ij}\big(M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}R_{\mathrm{a}j} + R_{\mathrm{u}j}\big).\end{aligned}$$

Design a full-rank matrix $\Gamma_i \in \mathbb{R}^{(n-m)\times m}$ such that $\Lambda_{1i} = \Gamma_i\Theta_{\mathrm{u}i} > 0,\ \Lambda_{2i} = \Gamma_i\Xi_{\mathrm{u}i} > 0$. Add and subtract Γ_ir_i to (17),

$$\begin{aligned}\ddot{e}_{\mathrm{u}i} =& M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\big(I_m + T_i\big)\bar{\tau}_i - \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}\big(I_m + T_j\big)\bar{\tau}_j \\ &+ \phi'_{ij} - \Gamma_i\big(\Theta_{\mathrm{a}i}\dot{e}_{\mathrm{a}i} + \Xi_{\mathrm{a}i}e_{\mathrm{a}i} + \Theta_{\mathrm{u}i}\dot{e}_{\mathrm{u}i} + \Xi_{\mathrm{u}i}e_{\mathrm{u}i}\big) + \Gamma_ir_i \\ =& -\Gamma_i\Theta_{\mathrm{u}i}\dot{e}_{\mathrm{u}i} - \Gamma_i\Xi_{\mathrm{u}i}e_{\mathrm{u}i} + M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\big(I_m + T_i\big)\bar{\tau}_i \\ &- \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}\big(I_m + T_j\big)\bar{\tau}_j + \psi'_{ij}\end{aligned} \tag{18}$$

where $\psi'_{ij} = \phi'_{ij} - \big(\Theta_{\mathrm{a}i}\dot{e}_{\mathrm{a}i} + \Xi_{\mathrm{a}i}e_{\mathrm{a}i}\big) + \Gamma_ir_i$. Arrange the non-actuated state error as $\omega_{\mathrm{u}i} = [e_{\mathrm{u}i}^T\ \dot{e}_{\mathrm{u}i}^T]^T$. Using (18), we have

$$\begin{aligned}\dot{\omega}_{\mathrm{u}i} =& A_{\mathrm{u}i}\omega_{\mathrm{u}i} + B_2\Big[M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\big(I_m + T_i\big)\bar{\tau}_i \\ &- \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}\big(I_m + T_j\big)\bar{\tau}_j + \psi'_{ij}\Big]\end{aligned} \tag{19}$$

where we have defined $A_{\mathrm{u}i} = \begin{bmatrix} 0 & I_{(n-m)} \\ -\Lambda_{1i} & -\Lambda_{2i}\end{bmatrix}$, which is Hurwitz by design, and $B_2 = [0\ I_{(n-m)}]^T$.

Remark 5. *The analysis of the error dynamics has led to* (15) *and* (19)*, which are stable dynamics (due to the Hurwitz state matrices $A_{\mathrm{a}i}$ and $A_{\mathrm{u}i}$) perturbed by state-dependent terms.*

The state-dependent terms in Property 1 make the perturbations in (15) and (19) also state-dependent, so that they cannot be bounded a priori by a constant. In the rest of the analysis, the idea is to find an upper bound for such perturbations and define an appropriate ρ_i for stabilizing the error dynamics.

V. Design of the adaptive protocol

In the following, we provide the uncertainty analysis (Sect. V-A), leading to the adaptive synchronization laws (Sect. V-B).

A. Uncertainty Analysis

Define $\xi_i = [e_i^T \ \dot{e}_i^T \ q_i^T \ \dot{q}_i^T]^T$, $\xi = [\xi_1^T, \dots, \xi_N^T]^T$. Therefore, $\|e_{\text{a}i}\| \le \|\xi_i\|$, $\|e_{\text{u}i}\| \le \|\xi_i\|$, $\|\dot{e}_{\text{a}i}\| \le \|\xi_i\|$, $\|\dot{e}_{\text{u}i}\| \le \|\xi_i\|$. According to (10), we have

$$\|r_i\| \le \vartheta_i \|\xi_i\| \tag{20}$$

with $\vartheta_i = \|\Theta_{\text{a}i}\| + \|\Xi_{\text{a}i}\| + \|\Theta_{\text{u}i}\| + \|\Xi_{\text{u}i}\|$. Using Assumption 3 and (14), the following bound for ψ_{ij} in (15) can be obtained:

$$\begin{aligned}
\|\psi_{ij}\| \le\ & \|T_i r_i\| + \sum_{j\in\mathcal{N}_i} \bar{a}_{ij}\|T_j r_j\| + \check{a}_i\|R_{\text{a}i}\| \\
& + a_{ij}\|R_{\text{a}j}\| + \|\Theta_{\text{u}i}\dot{e}_{\text{u}i}\| + \|\Xi_{\text{u}i}e_{\text{u}i}\| \\
\le\ & \bar{T}\vartheta_i\|\xi_i\| + \bar{T}\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\vartheta_j\|\xi_j\| + \check{a}_i\|M_{\text{s}i}^{-1}\|\big(\|E_{\text{a}i}\| \\
& + \|M_{\text{ua}i}M_{\text{uu}i}^{-1}\|\|E_{\text{u}i}\|\big) + \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\|M_{\text{s}j}^{-1}\|\big(\|E_{\text{a}j}\| \\
& + \|M_{\text{ua}j}M_{\text{uu}j}^{-1}\|\|E_{\text{u}j}\|\big) + \|\Theta_{\text{u}i}\dot{e}_{\text{u}i}\| + \|\Xi_{\text{u}i}e_{\text{u}i}\|.
\end{aligned} \tag{21}$$

According to the definition of ξ_i, $\|q_i\| \le \|\xi_i\|$ can be obtained. Using Property 1, we have

$$\begin{aligned}
\|E_i(q_i, \dot{q}_i, e_i, \dot{e}_i)\| \le\ & \big(\bar{g}_i + d_i + \bar{h}_{1i}\big) + \bar{f}_i\|\dot{q}_i\| + \bar{c}_i\|\dot{q}_i\|^2 \\
& + \bar{h}_{2i}\|e_i\| + \bar{h}_{3i}\|\dot{e}_i\| + \bar{h}_{4i}\|e_i\|^2 + h_{5i}\|\dot{e}_i\|^2 \\
\le\ & (\bar{g}_i + \bar{d}_i + \bar{h}_{1i}) + (\bar{f}_i + \bar{h}_{2i} + \bar{h}_{3i})\|\xi_i\| \\
& + \big(\bar{c}_i + \bar{h}_{4i} + \bar{h}_{5i}\big)\|\xi_i\|^2.
\end{aligned} \tag{22}$$

From (4b), $\|E_{\text{a}i}\| \le \|E_i\|$, $\|E_{\text{u}i}\| \le \|E_i\|$. Then, (21) yields

$$\begin{aligned}
\|P_{\text{a}i}B_1\|\|\psi_{ij}\| \le\ & \|P_{\text{a}i}B_1\|\Big[\bar{T}\vartheta_i\|\xi_i\| + \bar{T}\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\vartheta_j\|\xi_j\| \\
& + \check{a}_i\|M_{\text{s}i}^{-1}\|\big(1 + \|M_{\text{ua}i}M_{\text{uu}i}^{-1}\|\big)\|E_i\| + \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\|M_{\text{s}j}^{-1}\|\big(1 \\
& + \|M_{\text{ua}j}M_{\text{uu}j}^{-1}\|\big)\|E_j\| + \big(\|\Theta_{\text{u}i}\| + \|\Xi_{\text{u}i}\|\big)\|\xi_i\|\Big] \\
\le\ & \theta_{0i} + \theta_{1i}\|\xi_i\| + \theta_{2i}\|\xi_i\|^2 + \sum_{j\in\mathcal{N}_i}\big(\varphi_{1j}\|\xi_j\| + \varphi_{2j}\|\xi_j\|^2\big)
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\theta_{0i} &= \|P_{\text{a}i}B_1\|\Big[\mu_i\big(\bar{g}_i + \bar{d}_i + \bar{h}_{1i}\big) + \sum_{j\in\mathcal{N}_i}\bar{\mu}_{ij}\big(\bar{g}_j + \bar{d}_j + \bar{h}_{1j}\big)\Big] \\
\theta_{1i} &= \|P_{\text{a}i}B_1\|\Big[\mu_i\big(\bar{f}_i + \bar{h}_{2i} + \bar{h}_{3i}\big) + \bar{T}\vartheta_i + \big(\|\Theta_{\text{u}i}\| + \|\Xi_{\text{u}i}\|\big)\Big] \\
\theta_{2i} &= \|P_{\text{a}i}B_1\|\mu_i\big(\bar{c}_i + \bar{h}_{4i} + \bar{h}_{5i}\big) \\
\varphi_{1j} &= \|P_{\text{a}i}B_1\|\Big[\bar{\mu}_{ij}\big(\bar{f}_j + \bar{h}_{2j} + \bar{h}_{3j}\big) + \bar{T}\bar{a}_{ij}\vartheta_j\Big] \\
\varphi_{2j} &= \|P_{\text{a}i}B_1\|\bar{\mu}_{ij}\big(\bar{c}_j + \bar{h}_{4j} + \bar{h}_{5j}\big) \\
\mu_i &= \check{a}_i\|M_{\text{s}i}^{-1}\|\big(1 + \|M_{\text{ua}i}M_{\text{uu}i}^{-1}\|\big) \\
\bar{\mu}_{ij} &= \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\|M_{\text{s}j}^{-1}\|\big(1 + \|M_{\text{ua}j}M_{\text{uu}j}^{-1}\|\big).
\end{aligned}$$

Similar to (21), the upper bound on ψ'_{ij} from (19) is obtained

$$\begin{aligned}
\|\psi'_{ij}\| \le\ & \|\phi'_{ij}\| + \|\Theta_{\text{a}i}\dot{e}_{\text{a}i}\| + \|\Xi_{\text{a}i}e_{\text{a}i}\| + \|\Gamma_i r_i\| \\
\le\ & \Big[\big(1 + \bar{T}\big)\|M_{\text{uu}i}^{-1}M_{\text{au}i}\| + 1\Big]\vartheta_i\|\xi_i\| + \check{a}_i\|M_{\text{uu}i}^{-1}\|\|E_{\text{u}i}\| \\
& + \check{a}_i\|M_{\text{uu}i}^{-1}M_{\text{au}i}\|\|M_{\text{s}i}^{-1}\|\big(\|E_{\text{a}i}\| + \|M_{\text{ua}i}M_{\text{uu}i}^{-1}\|\|E_{\text{u}i}\|\big) \\
& + \big(1 + \bar{T}\big)\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\vartheta_j\|M_{\text{uu}j}^{-1}M_{\text{au}j}\|\|\xi_j\| \\
& + \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\|M_{\text{uu}j}^{-1}M_{\text{au}j}\|\|M_{\text{s}j}^{-1}\|\big(\|E_{\text{a}j}\| \\
& + \|M_{\text{ua}j}M_{\text{uu}j}^{-1}\|\|E_{\text{u}j}\|\big) + \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\|M_{\text{uu}j}^{-1}\|\|E_{\text{u}j}\| \\
& + \|\Theta_{\text{a}i}\dot{e}_{\text{a}i}\| + \|\Xi_{\text{a}i}e_{\text{a}i}\|.
\end{aligned} \tag{24}$$

Let $P_{\text{u}i} > 0$ be the solution to the Lyapunov equation $A_{\text{u}i}^T P_{\text{u}i} + P_{\text{u}i}A_{\text{u}i} = -Q_{\text{u}i}$, with $Q_{\text{u}i}$ a user-designed positive definite matrix. Then, we finally obtain

$$\begin{aligned}
\|P_{\text{u}i}B_2\|\|\psi'_{ij}\| \le\ & \|P_{\text{u}i}B_2\|\bigg\{\Big[\big(1 + \bar{T}\big)\|M_{\text{uu}i}^{-1}M_{\text{au}i}\| \\
& + 1\Big]\vartheta_i\|\xi_i\| + \check{a}_i\|M_{\text{uu}i}^{-1}\|\Big[\|M_{\text{au}i}\|\|M_{\text{s}i}^{-1}\|\big(1 + \|M_{\text{ua}i}M_{\text{uu}i}^{-1}\|\big) \\
& + 1\Big]\|E_i\| + \big(1 + \bar{T}\big)\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\vartheta_j\|M_{\text{uu}j}^{-1}M_{\text{au}j}\|\|\xi_j\| \\
& + \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\|M_{\text{uu}j}^{-1}\|\Big[\|M_{\text{au}j}\|\|M_{\text{s}j}^{-1}\|\big(1 + \|M_{\text{ua}j}M_{\text{uu}j}^{-1}\|\big) \\
& + 1\Big]\|E_j\| + \big(\|\Theta_{\text{a}i}\| + \|\Xi_{\text{a}i}\|\big)\|\xi_i\|\bigg\} \\
\le\ & \theta'_{0i} + \theta'_{1i}\|\xi_i\| + \theta'_{2i}\|\xi_i\|^2 + \sum_{j\in\mathcal{N}_i}\big(\varphi'_{1j}\|\xi_j\| + \varphi'_{2j}\|\xi_j\|^2\big)
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\theta'_{0i} &= \|P_{\text{u}i}B_2\|\Big[\mu'_i\big(\bar{g}_i + \bar{d}_i + \bar{h}_{1i}\big) + \sum_{j\in\mathcal{N}_i}\bar{\mu}'_{ij}\big(\bar{g}_j + \bar{d}_j + \bar{h}_{1j}\big)\Big] \\
\theta'_{1i} &= \|P_{\text{u}i}B_2\|\Big[\mu'_i\big(\bar{f}_i + \bar{h}_{2i} + \bar{h}_{3i}\big) + \big(\|\Theta_{\text{u}i}\| + \|\Xi_{\text{u}i}\|\big)\Big] \\
&\quad + \Big[\big(1 + \bar{T}\big)\|M_{\text{uu}i}^{-1}M_{\text{au}i}\| + 1\Big]\vartheta_i \\
\theta'_{2i} &= \|P_{\text{u}i}B_2\|\mu'_i\big(\bar{c}_i + \bar{h}_{4i} + \bar{h}_{5i}\big) \\
\varphi'_{1j} &= \|P_{\text{u}i}B_2\|\Big[\bar{\mu}'_{ij}\big(\bar{f}_j + \bar{h}_{2j} + \bar{h}_{3j}\big) \\
&\quad + \big(1 + \bar{T}\big)\bar{a}_{ij}\vartheta_j\|M_{\text{uu}j}^{-1}M_{\text{au}j}\|\Big] \\
\varphi'_{2j} &= \|P_{\text{u}i}B_2\|\bar{\mu}'_{ij}\big(\bar{c}_j + \bar{h}_{4j} + \bar{h}_{5j}\big) \\
\mu'_i &= \check{a}_i\|M_{\text{uu}i}^{-1}\|\Big[\|M_{\text{au}i}\|\|M_{\text{s}i}^{-1}\|\big(1 + \|M_{\text{ua}i}M_{\text{uu}i}^{-1}\big) + 1\Big] \\
\bar{\mu}'_{ij} &= \bar{a}_{ij}\|M_{\text{uu}j}^{-1}\|\|\Big[\|M_{\text{au}j}\|\|M_{\text{s}j}^{-1}\|\big(1 + \|M_{\text{ua}j}M_{\text{uu}j}^{-1}\big) + 1\Big].
\end{aligned}$$

Let us stress that the structures of the uncertainty bounds in (23) and (25) are not imposed as per assumption, but a consequence of Property 1. As Property 1 is generally valid for EL dynamics, the upper bounds in (23) and (25) hold generally. These upper bounds put us in the position to design an appropriate ρ_i in (11), as explained later.

B. Adaptive Synchronization Laws

According to the structure of the upper bounds of ψ_{ij} in (23) and ψ'_{ij} in (25), ρ_i is designed as

$$\rho_i = \frac{1}{(1-\bar{T})}\left(\hat{\theta}_{0i} + \hat{\theta}_{1i}\|\xi_i\| + \hat{\theta}_{2i}\|\xi_i\|^2 + \gamma_i\right) \tag{26}$$

with the adaptive laws $(l = 0, 1, 2)$

$$\begin{aligned}\dot{\hat{\theta}}_{li} =& \chi_{li}\big(\|\omega_{\mathrm{a}i}\| + \|\omega_{\mathrm{u}i}\| + \|S_i\|\big)\|\xi_i\|^l \\ &- \alpha_{li}\big(\|\omega_{\mathrm{a}i}\| + \|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l\hat{\theta}_{li} \end{aligned} \tag{27a}$$

$$\begin{aligned}\dot{\gamma}_i =& \epsilon_0\big(\|S_i\| + \|\xi_i\|\big) + \beta_i \\ &- \Big[\epsilon_0 + \epsilon_1(\|\xi_i\|^7 - \|\xi_i\|^5) + \epsilon_2\|\xi_i\|\Big]\gamma_i \end{aligned} \tag{27b}$$

$$\text{where } \hat{\theta}_{0i}(0) > 0, \hat{\theta}_{1i}(0) > 0, \hat{\theta}_{2i}(0) > 0, \gamma_i(0) > 0 \tag{27c}$$

$$\epsilon_0,\ \epsilon_1,\ \epsilon_2,\ \chi_{li},\ \alpha_{li},\ \beta_i \in \mathbb{R}^+ \tag{27d}$$

$$\text{with } \epsilon_2 \geq \epsilon_1. \tag{27e}$$

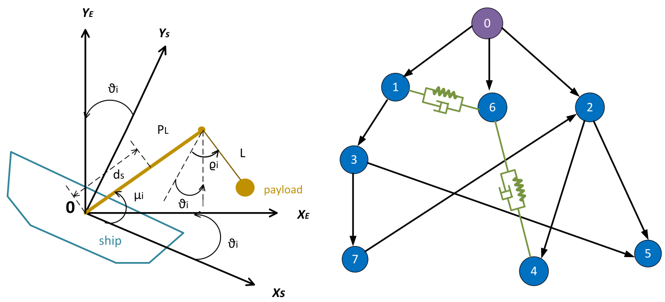
Remark 6. *The proposed adaptive law uses a leakage dependent on the synchronization error. This is useful in the Lyapunov analysis of the derivative of* $(\hat{\theta}_{li} - \bar{\theta}_{li})^2$*, as the common factor* $\big(\|\omega_{\mathrm{a}i}\| + \|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l$ *can be extracted to construct negative square terms of* $\hat{\theta}_{li}$*, cf.* (39)-(40) *in the appendix. The parameters in* (27)*, namely* $\chi_{li}, \alpha_{li}, \epsilon_0, \epsilon_1, \epsilon_2, \beta_i$*, determine the rate of variation of the gains. For example,* α_{li}/χ_{li} *represents the ratio between the decreasing effect of leakage and the increasing effect of the first part of the adaptive law. Compared to standard leakage in the literature (cf. [31], [36], [44, Chapter 8]), the proposed leakage leads to a more concise UUB condition for* $\omega_{\mathrm{a}i}$ *and* $\omega_{\mathrm{u}i}$ *in* (47), (52) *and* (54).

Theorem 1. *Under Properties 1-2 and Assumptions 1-3, the closed-loop trajectories of* (1) *employing the distributed control law* (11) *with adaptive law* (27) *are uniformly ultimately bounded.*

Proof: See the appendix. ■

Remark 7. *Although the proposed framework shares some points with sliding mode control (e.g., the saturation in* (11)*), crucial features make it depart from such methods: sliding mode control of underactuated EL dynamics imposes structural restrictions on the variables the mass matrix depends on (cf. [7, Def. 1] used in most of the literature) and structural restrictions on the uncertainties (cf. [16, Assump. 4-8] imposing invertibility). In the proposed framework, none of these structural restrictions are imposed.*

Remark 8. *The uncertainties considered in this work create another crucial difference with available sliding mode methods for underactuated EL dynamics. There, uncertainties are typically* a priori *bounded, where the constant bound can be unknown a priori in the adaptive sliding mode literature. The adaptive laws involve either monotonically increasing gains, or increasing-decreasing rules [20], [21]. Our method involves neither a priori bounded uncertainties (state-dependent perturbations in* (23) *and* (25) *cannot be bounded a priori), nor increasing gains/increasing-decreasing rules (cf. the discussion about leakage in Remark 6). These aspects require a dedicated stability analysis (cf. Appendix) departing significantly from analyses available in the sliding mode literature.*



(a) Schematic boom crane on ship

(b) Network

Figure 1: System used for simulations.

Remark 9. *The appendix provides estimates for the uniform ultimate bounds, which can be tuned as follows. Larger* β_i *and* ϵ_1 *leads to more negative* $-\underline{\gamma}_i^2\bar{\epsilon}_1$*, which is the fifth degree coefficient of the polynomials* $Z_1(\|\xi\|)$*,* $Z_2(\|\xi\|)$*,* $Z_3(\|\xi\|)$ *in* (46), (51), *and* (53)*. Making this coefficient more negative makes the roots* η_1*,* η_2*,* η_3 *closer to zero, which in turn contributes to reducing the ultimate bound on the error. Larger* χ_{li} *and smaller* ζ*, which can be obtained from larger* $P_{\mathrm{u}i}$*,* $P_{\mathrm{a}i}$*, result in* ι_1*,* ι_2*,* ι_3 *being closer to zero. A larger* ϵ_2 *leads to a smaller* ι_4*. This also contributes to reducing the ultimate bound on the error. Let us mention that a smaller error might require a larger input: this is a standard trade-off, which might be seen from the fact that larger* χ_{li} *and* β_i *leads to larger* ρ_i*.*

VI. Simulation Example

A network of underactuated systems is considered, where each system has boom crane dynamics as in [3]: the network can be thought as an abstraction of a cooperative lifting scenario where the boom cranes are mounted on one or more ships (cf. Fig. 1). In addition to the sensors, actuators and micro-controllers for crane control, a distributed implementation of such a control scenario would require to put in place a communication network among the cranes, e.g. via wireless nodes. Let us consider a graph as in Fig. 1b where the directed spanning tree property in Assumption 1 holds.

For the system in Fig. 1, ϱ_i is the payload swing with respect to Y_s, ϑ_i is the ship roll angle caused by sea waves, μ_i is the luffing angle of the boom, and L denotes the length of the rope. The length, mass, and moment of inertia of the boom are $P_L, m,$ and J. The distance between the barycenter of the boom and the origin is denoted by d_s. The states of the crane system are $q_{1i} = \varrho_i - \vartheta_i$, $q_{2i} = \mu_i - \vartheta_i$, and $q_{3i} = L$ (q_{1i} is the non-actuated state, q_{2i}, q_{3i} are the actuated states), leading to the dynamics as (1) with $n = 3, m = 2$, and

$$M_i = \begin{bmatrix} m_{\mathrm{p}i}q_{3i}^2 & -m_{\mathrm{p}i}P_Lq_{3i}S_{21,i} & 0 \\ -m_{\mathrm{p}i}P_Lq_{3i}S_{21,i} & J_i + m_{\mathrm{p}i}P_L^2 & -m_{\mathrm{p}i}P_LC_{21,i} \\ 0 & -m_{\mathrm{p}i}P_LC_{21,i} & m_{\mathrm{p}i} \end{bmatrix}$$

$$C_i = \begin{bmatrix} m_{\mathrm{p}i}q_{3i}\dot{q}_{3i} & -m_{\mathrm{p}i}P_Lq_{3i}C_{21,i}\dot{q}_{2i} & m_{\mathrm{p}i}q_{3i}\dot{q}_{1i} \\ U_{3i} & 0 & -m_{\mathrm{p}i}P_LS_{21,i}\dot{q}_{1i} \\ -m_{\mathrm{p}i}q_{3i}\dot{q}_{1i} & m_{\mathrm{p}i}P_LS_{21,i}\dot{q}_{2i} & 0 \end{bmatrix}$$

$$U_{3i} = -m_P P_L(S_{21,i}\dot{q}_{3i} - C_{21,i}q_{3i}\dot{q}_{1i}),\ \tau_i = [\tau_{1i} \quad \tau_{2i}]$$

$$G_i = \begin{bmatrix} m_{\mathrm{p}i} g_a q_{2i} \sin(q_{1i}) \\ (m_{\mathrm{p}i}P_L + m_i d_{\mathrm{s}i}) g_a \cos(q_{2i}) \\ -m_{\mathrm{p}i} g_a \cos(q_{1i}) \end{bmatrix},\ q_i = \begin{bmatrix} q_{1i} \\ q_{2i} \\ q_{3i} \end{bmatrix}$$

$$H_i = \sum_{j=0}^{N} s_{ij}(q_i - q_j) + \sum_{j=0}^{N} \delta_{ij}(\dot{q}_i - \dot{q}_j)$$

with $S_{21,i} \triangleq \sin(q_{2i} - q_{1i})$, $C_{21,i} \triangleq \cos(q_{2i} - q_{1i})$ and $F_i = [F_{i1} \ F_{i2} \ F_{i3}]^T$ where $F_{i1}(\dot{q}_{i1}) \triangleq f_{i1}\tanh(f_{i2}\dot{q}_{i1}) - \tanh(f_{i3}\dot{q}_{i1})) + f_{i4}\tanh(f_{i5}\dot{q}_{i1}) + f_{i6}\dot{q}_{i1}$, $F_{i2}(\dot{q}_{2i}) \triangleq f_{i1}\tanh(f_{i2}\dot{q}_{i2}) - \tanh(f_{i3}\dot{q}_{i2})) + f_{i4}\tanh(f_{i5}\dot{q}_{i2}) + f_{i6}\dot{q}_{i2}$, $F_{i3}(\dot{q}_{3i}) \triangleq f_{i1}\tanh(f_{i2}\dot{q}_{i3}) - \tanh(f_{i3}\dot{q}_{i3})) + f_{i4}\tanh(f_{i5}\dot{q}_{i3}) + f_{i6}\dot{q}_{i3}$. The friction term F_i is taken in non-linear-in-the-parameters form according to [45], whereas the interconnection term H_i follows a standard spring-damper model where s_{ij} is the stiffness parameter, and δ_{ij} is the damping factor (this can represent some interconnection among the cranes via the crane wires due to the load).

It is possible to verify, cf. [3], that Properties 1-2 hold for the system dynamics terms reported above. In addition, the Strong Inertial Coupling condition as in Assumption 2 is verified as the term $M_{\mathrm{au}i} = [0 \quad -m_{\mathrm{p}i}P_L C_{21,i}]$ does not lose rank in the operating range of interest (note that $\cos(\mu_i - \varrho_i) = 0$ never occurs for luffing angles and swing angles of interest). The goal is to bring the payload to a desired position defined by

$$q_{01} = 0,\ q_{02} = \arccos(a_L/P_L),\ q_{03} = \sqrt{P_L^2 - a_L^2} - b_L.$$

A. System parameters (uncertain) and design parameters

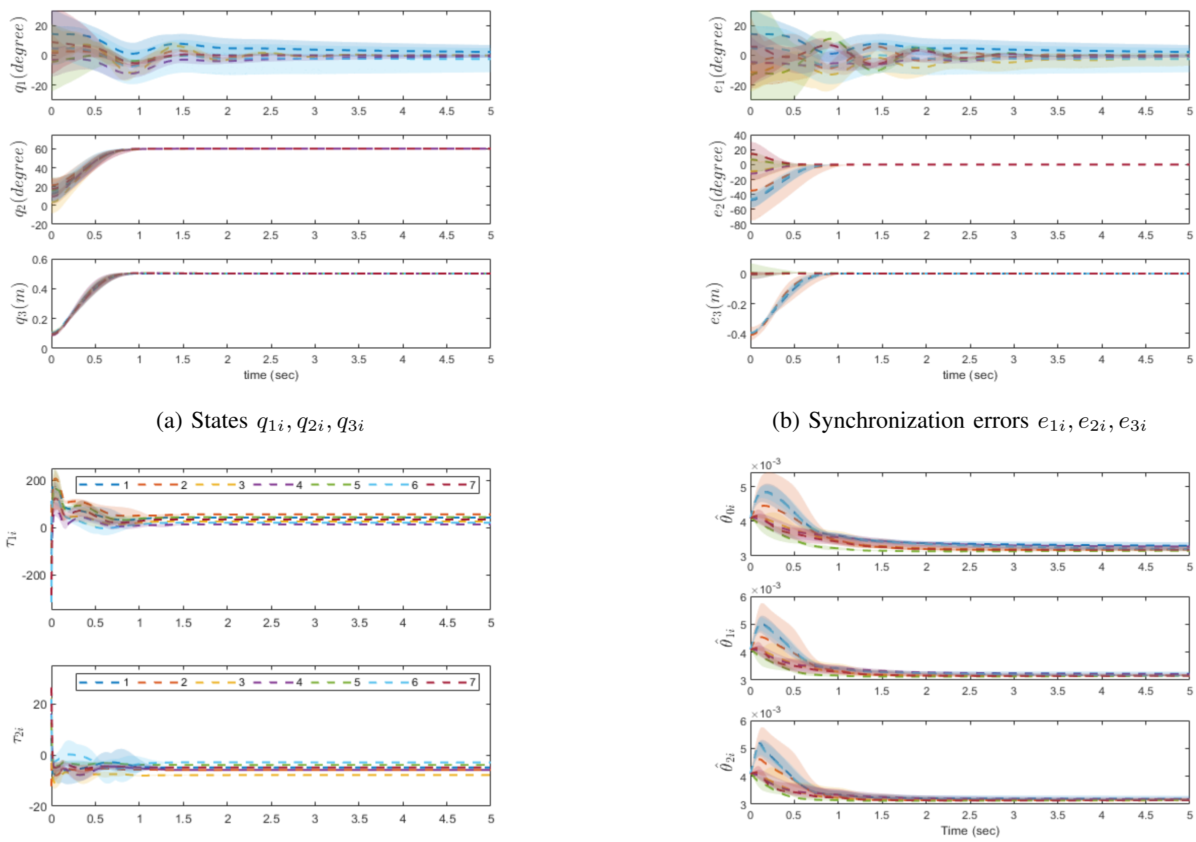
The following system parameters are only used for simulation purpose, but they are unknown for control design. The vector of (unknown) parameters in friction term is compactly represented as $\Theta_i = [f_{i1} \ f_{i2} \ f_{i3} \ f_{i4} \ f_{i5} \ f_{i6}]^T$, where

$$\Theta_1 = [0.5\ 0.8\ 0.9\ 1.2\ 0.5\ 0.4]^T$$
$$\Theta_2 = [0.5\ 0.7\ 0.9\ 1.0\ 0.5\ 0.4]^T$$
$$\Theta_3 = [0.3\ 0.7\ 0.7\ 1.0\ 0.7\ 0.3]^T$$
$$\Theta_4 = [0.5\ 0.9\ 0.7\ 1.2\ 0.4\ 0.5]^T$$
$$\Theta_5 = [0.5\ 0.8\ 0.6\ 1.3\ 0.5\ 0.6]^T$$
$$\Theta_6 = [0.6\ 1.0\ 0.9\ 1.5\ 0.2\ 0.5]^T$$
$$\Theta_7 = [0.4\ 1.0\ 0.8\ 1.2\ 0.4\ 0.8]^T.$$

According to interconnection network in Fig. 1b, the spring-damper parameters are chosen as $s_{16} = s_{61} = 0.37$, $s_{46} = s_{64} = 0.29$ and $\delta_{16} = \delta_{61} = 25$, $\delta_{46} = \delta_{64} = 9$. The disturbance is $d_i(t) = 0.1\sin(0.001it)[1\ 1\ 1]^T$.

To test the effect of heterogeneity in the agents, the physical parameters are chosen to be different for different agents:

$$[m_1\ m_2\ m_3\ m_4\ m_5\ m_6\ m_7] = [20\ \ 18\ \ 15\ \ 22\ \ 17\ \ 19\ \ 16]$$
$$[m_{\mathrm{p}1}\ m_{\mathrm{p}2}\ m_{\mathrm{p}3}\ m_{\mathrm{p}4}\ m_{\mathrm{p}5}\ m_{\mathrm{p}6}\ m_{\mathrm{p}7}] = [0.5\ 0.6\ 0.8\ 0.6$$



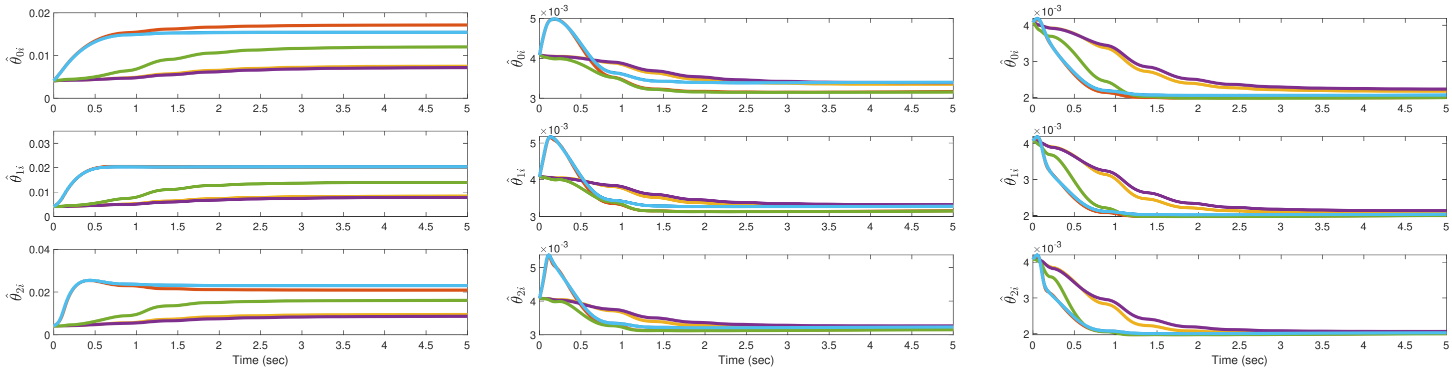
(a) States q_{1i}, q_{2i}, q_{3i}

(b) Synchronization errors e_{1i}, e_{2i}, e_{3i}

(c) Control inputs τ_{1i} and τ_{2i}

(d) Adaptive gains $\hat{\theta}_{li},\ l = 0, 1, 2$

Figure 2: Performance with proposed adaptive method.



(a) $\alpha_{li}/\chi_{li} = 50$ (b) $\alpha_{li}/\chi_{li} = 300$ (c) $\alpha_{li}/\chi_{li} = 500$

Figure 3: Evolution of adaptive gains for different ratios of α_{li}/χ_{li}.

Table I: Tracking error L_2 norm and input L_2 norm for different sinusoidal disturbances

Disturbance	Norm of error	Norm of input
$d_i(t) = 0.1\sin(0.001it)$	82.73	2.56 $\times 10^4$
$d_i(t) = 0.1\sin(it)$	82.60	2.56 $\times 10^4$
$d_i(t) = 0.1\sin(100it)$	82.72	2.56 $\times 10^4$

$0.4\ 0.3\ 0.5]$

$$[J_1\ J_2\ J_3\ J_4\ J_5\ J_6\ J_7] = [6.5\ 7.8\ 5.3\ 6.2\ 7.2\ 6.8\ 6.6]$$

$$[d_{s1}\ d_{s2}\ d_{s3}\ d_{s4}\ d_{s5}\ d_{s6}\ d_{s7}] = [0.4\ 0.6\ 0.3\ 0.1\ 0.5\ 0.2\ 0.4].$$

The parameters $a_L = 0.4m,\ b_L = 0.2m,\ P_L = 0.8m$ give the desired position $q_{01} = 0,\ q_{02} = 1.05,\ q_{03} = 0.5$.

The nominal parameters (used for control design of $\hat{M}_s$) are selected as $[\hat{m}_{p1}\ \hat{m}_{p2}\ \hat{m}_{p3}\ \hat{m}_{p4}\ \hat{m}_{p5}\ \hat{m}_{p6}\ \hat{m}_{p7}] = [0.45\ 0.55\ 0.75\ 0.55\ 0.35\ 0.25\ 0.45]$, $[\hat{J}_1\ \hat{J}_2\ \hat{J}_3\ \hat{J}_4\ \hat{J}_5\ \hat{J}_6\ \hat{J}_7] = [6\ 7\ 5\ 6\ 7\ 6\ 8]$. It can be verified these nominal parameters make Assumption 3 hold with $\bar{T} = 0.5$.

The control design parameters are $\Theta_{ai} = 335I_2$, $\Xi_{ai} = 0.003I_2$, $\Theta_{ui} = 0.03[1\ 1]^T$, $\Xi_{ui} = 0.01[1\ 1]^T$, $\Gamma_i = 0.01[1\ 1]^T$, $Q_{ai} = 0.015I_2$, $Q_{ui} = 560I_2$, $\alpha_{li} = 3.15$, $\chi_{li} = 0.01, \epsilon_0 = 0.001, \epsilon_1 = 0.003,\ \epsilon_2 = 0.015,\ \beta_i = 3150$.

B. Simulation Results and Sensitivity Analysis

To validate the robustness of the proposed method in a statistical way, Fig. 2 provides the evolution of the closed-loop signals, averaged over 20 runs with random initial states $q_{1i} \sim \mathcal{N}(0.1, 0.175^2), q_{2i} \sim \mathcal{N}(0.2, 0.2^2), q_{3i} \sim \mathcal{N}(0.1, 0.01^2)$. The states (with the angles reported in degrees for better understanding) and the corresponding synchronization errors are shown in Fig. 2a and Fig. 2b, from which it can be seen that the errors converge to a neighborhood of zero for all runs. The control inputs are in Fig. 2c, with the adaptive gains $\hat{\theta}_{li},\ l = 0, 1, 2$ in Fig. 2d. The control inputs converge to different values for different systems, due to the heterogeneity of the systems in terms of mass, inertia, friction, etc.

To validate the effect of choosing different parameters as in Remark 6, let us choose different ratios for α_{li}/χ_{li} in the adaptive law. Fig. 3 shows that the gains θ_{li} tend to grow larger as the ratio α_{li}/χ_{li} decreases (i.e. as the leakage decreases).

We then perform a sensitivity analysis with different disturbances d_i, namely, sinusoidal disturbances with different frequencies and Gaussian disturbances with different variances (with 100 realizations of the disturbances). For sinusoidal disturbances, Table I shows minor differences in terms of error and state norm. For Gaussian disturbances, Table II shows a slight increase in the error norm as the variance increases. This validates the robustness of the proposed approach.

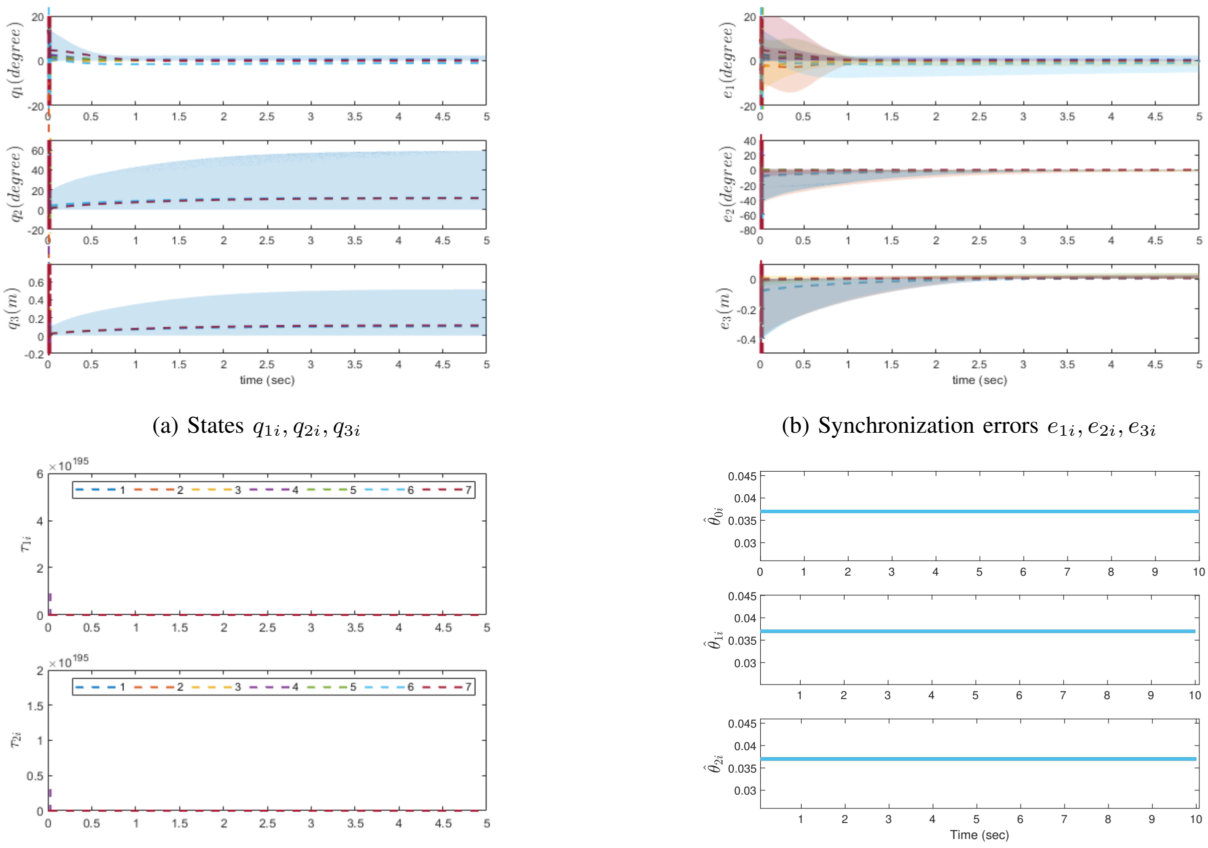
C. Comparisons with State-of-the-Art Methods

To validate the discussions in Remarks 7 and 8, we provide comparisons with state-of-the-art sliding mode control and adaptive sliding mode control. The former is obtained by fixing the gains $\hat{\theta}_{li}$ to be constant, whereas the latter is obtained by removing the leakage from the adaptive law, so as to have monotonically increasing gains. Similar to the simulation results of proposed method, Fig. 4 and Fig. 5 provide the evolution of the closed-loop signals, averaged over 20 runs with random initial states $q_{1i} \sim \mathcal{N}(0.1, 0.175^2), q_{2i} \sim \mathcal{N}(0.2, 0.2^2), q_{3i} \sim \mathcal{N}(0.1, 0.01^2)$. Fig. 4c shows that sliding mode control requires unrealistically large control effort (10^{195}), while still getting large synchronization errors in Fig. 4b (except for the visible shading error in this plot, there are some agents whose error is too large to be seen due to out of scale in the plot). That is, sliding mode control fails to achieve stability with reasonable control input.

Meanwhile, differently from the proposed method that is stable for every run, the state-of-the-art adaptive sliding mode control results in many unstable runs, so that we report only the stable runs. Thus, the errors reported in Fig. 5b are selected from the stable runs, showing that the performance of these selected runs is still not better than the proposed method. Most importantly, state-of-the-art adaptive sliding mode control requires large inputs, cf. Fig. 5c and the increasing adaptive gains $\hat{\theta}_{li}$ in Fig. 5d. Thus, we conclude that the proposed method overcomes state-of-the-art sliding mode and adaptive sliding mode methods in dealing with the problem at hand.

VII. Conclusions

This work has proposed for the first time an adaptive distributed protocol for synchronization of uncertain underac-



(a) States q_{1i}, q_{2i}, q_{3i}

(b) Synchronization errors e_{1i}, e_{2i}, e_{3i}

(c) Control inputs τ_{1i} and τ_{2i}

(d) Fixed gains $\hat{\theta}_{li},\ l = 0, 1, 2$

Figure 4: Performance with state-of-the-art sliding mode method.

Table II: Tracking error L_2 norm and input L_2 norm for different Gaussian disturbances

Disturbance	Norm of error	Norm of input
$d_i(t) \sim \mathcal{N}(0, 0.03)$	82.73	2.56 $\times 10^4$
$d_i(t) \sim \mathcal{N}(0, 0.9)$	82.82	2.56 $\times 10^4$
$d_i(t) \sim \mathcal{N}(0, 10)$	83.22	2.56 $\times 10^4$

tuated EL systems. The protocol can tackle not only unknown system terms, but also uncertain state-dependent interconnections among agents. Handling all these aspects required a new stability analysis and a new design approach departing from state-of-the-art results not considering unknown parameters, unstructured uncertainty in the mass matrix, in the system terms and in the interconnections. In future work, it is of interest to further improve the method by considering non-holonomic constraints or by tackling more uncertainties in the mass matrix and in the network topology.

Appendix

Proof: Construct a Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}\left\{\omega_{\mathrm{a}i}^T(t)P_{\mathrm{a}i}\omega_{\mathrm{a}i}(t) + \omega_{\mathrm{u}i}^T(t)P_{\mathrm{u}i}\omega_{\mathrm{u}i}(t)\right\} + \frac{1}{2}\sum_{i=1}^{N}\left\{\sum_{l=0}^{2}\frac{1}{\chi_{li}}(\hat{\theta}_{li}(t) - \bar{\theta}_{li})^2 + \frac{\gamma_i^2(t)}{\epsilon_0}\right\} \tag{28}$$

where $\bar{\theta}_{li} = \max\{\theta_{li}, \theta'_{li}\},\ l = 0, 1, 2$.

The proof is organized in three steps as follows:

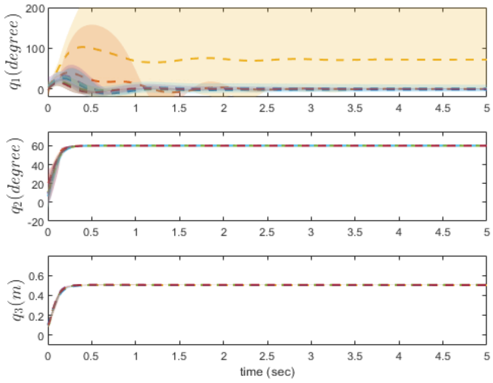
a) the bound of uncertainty for the overall network is calculated;
b) based on such overall uncertainty bound, we calculate the time derivative of the Lyapunov function;
c) based on different regions of saturation function $\mathrm{sat}(S_i, \varphi)$, we study the behavior of the Lyapunov function for three possible scenarios.

Combining all the results, we finally obtain a uniform ultimate bound on the actuated error $\omega_{\mathrm{a}i}$ and on the non-actuated error $\omega_{\mathrm{u}i}$.

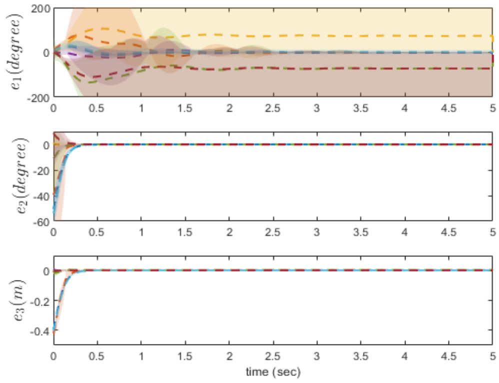
a) The overall uncertainty term

According to (15), we obtain

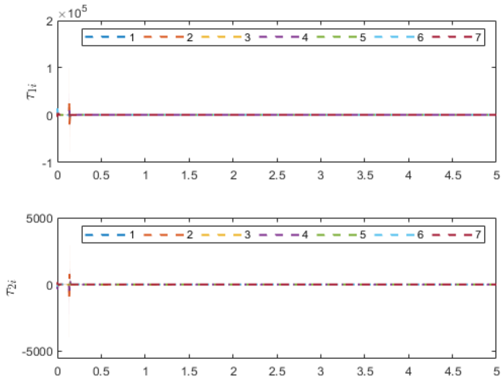
$$\begin{aligned}\omega_{\mathrm{a}i}^T P_{\mathrm{a}i}\dot{\omega}_{\mathrm{a}i} &= \omega_{\mathrm{a}i}^T P_{\mathrm{a}i}\Big\{A_{\mathrm{a}i}\omega_{\mathrm{a}i} + B_1\big[-(I_m + T_i)\bar{\tau}_i \\ &\quad + \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}(I_m + T_j)\bar{\tau}_j + \psi_{ij}\big]\Big\} \\ &\le -\frac{1}{2}\omega_{\mathrm{a}i}^T Q_{\mathrm{a}i}\omega_{\mathrm{a}i} + \|\omega_{\mathrm{a}i}^T\|\|P_{\mathrm{a}i}B_1\|\|\psi_{ij}\| \\ &\quad - \omega_{\mathrm{a}i}^T P_{\mathrm{a}i}B_1(I_m + T_i)\rho_i\,\mathrm{sat}(S_i, \varphi) \\ &\quad + \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\omega_{\mathrm{a}i}^T P_{\mathrm{a}i}B_1(I_m + T_j)\rho_j\,\mathrm{sat}(S_j, \varphi).\end{aligned} \tag{29}$$



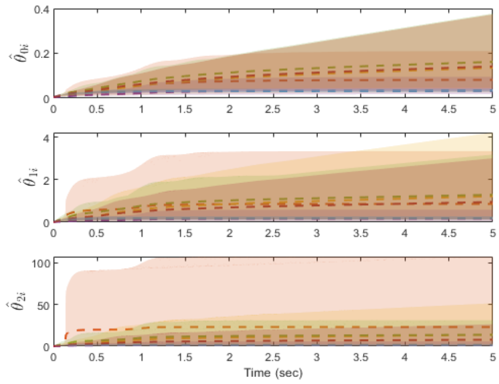
(a) States q_{1i}, q_{2i}, q_{3i}



(b) Synchronization errors e_{1i}, e_{2i}, e_{3i}



(c) Control inputs τ_{1i} and τ_{2i}



(d) Adaptive gains $\hat{\theta}_{li},\ l = 0, 1, 2$

Figure 5: Performance with state-of-the-art adaptive sliding mode method.

Analogously, according to (19), we obtain

$$\begin{aligned}
&\omega_{\mathrm{ui}}^T P_{\mathrm{ui}}\dot{\omega}_{\mathrm{ui}} = \omega_{\mathrm{ui}}^T P_{\mathrm{ui}}\Big\{A_{\mathrm{ui}}\omega_{\mathrm{ui}} + B_2\big[M_{\mathrm{uui}}^{-1}M_{\mathrm{aui}}(I_m \\
&+ T_i)\bar{\tau}_i - \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}M_{\mathrm{uuj}}^{-1}M_{\mathrm{auj}}(I_m+T_j)\bar{\tau}_j + \psi'_{ij}\big]\Big\} \\
&\le -\frac{1}{2}\omega_{\mathrm{ui}}^T Q_{\mathrm{ui}}\omega_{\mathrm{ui}} + \|\omega_{\mathrm{ui}}^T\|\|P_{\mathrm{ui}}B_2\|\|\psi'_{ij}\| \\
&+ \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\omega_{\mathrm{ui}}^T P_{\mathrm{ui}}B_2M_{\mathrm{uuj}}^{-1}M_{\mathrm{auj}}(I_m+T_j)\rho_j\operatorname{sat}(S_j,\varphi) \\
&+ \omega_{\mathrm{ui}}^T P_{\mathrm{ui}}B_2M_{\mathrm{uui}}^{-1}M_{\mathrm{aui}}(I_m+T_i)\rho_i\operatorname{sat}(S_i,\varphi).
\end{aligned} \tag{30}$$

Adding (29) and (30), combined with (23)-(25), we obtain

$$\begin{aligned}
&\omega_{\mathrm{ai}}^T P_{\mathrm{ai}}\dot{\omega}_{\mathrm{ai}} + \omega_{\mathrm{ui}}^T P_{\mathrm{ui}}\dot{\omega}_{\mathrm{ui}} \\
\le& -\lambda_{\mathrm{m}i}\Big[\|\omega_{\mathrm{ai}}\|^2 + \|\omega_{\mathrm{ui}}\|^2\Big] + \sum_{l=0}^{2}\bar{\theta}_{li}\|\xi_i\|^l\big(\|\omega_{\mathrm{ai}}\| + \|\omega_{\mathrm{ui}}\|\big) \\
&+ \sum_{j\in\mathcal{N}_i}\Big[\bar{\varphi}_{1j}\|\xi_j\| + \bar{\varphi}_{2j}\|\xi_j\|^2\Big]\big(\|\omega_{\mathrm{ai}}\| + \|\omega_{\mathrm{ui}}\|\big) \\
&- \omega_{\mathrm{ai}}^T P_{\mathrm{ai}}B_1(I_m+T_i)\rho_i\operatorname{sat}(S_i,\varphi) \\
&+ \omega_{\mathrm{ui}}^T P_{\mathrm{ui}}B_2M_{\mathrm{uui}}^{-1}M_{\mathrm{aui}}(I_m+T_i)\rho_i\operatorname{sat}(S_i,\varphi) \\
&+ \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\omega_{\mathrm{ui}}^T P_{\mathrm{ui}}B_2M_{\mathrm{uuj}}^{-1}M_{\mathrm{auj}}(I_m+T_j)\rho_j\operatorname{sat}(S_j,\varphi) \\
&+ \sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\omega_{\mathrm{ai}}^T P_{\mathrm{ai}}B_1(I_m+T_j)\rho_j\operatorname{sat}(S_j,\varphi)
\end{aligned} \tag{31}$$

where $\lambda_{\mathrm{m}i} = \min\Big\{\lambda_{\min}(Q_{\mathrm{ai}})/2, \lambda_{\min}(Q_{\mathrm{ui}})/2\Big\}$, $\bar{\varphi}_{1j} = \max\big\{\varphi_{1j},\ \varphi'_{1j}\big\}$, $\bar{\varphi}_{2j} = \max\big\{\varphi_{2j},\ \varphi'_{2j}\big\}$, $j\in\mathcal{N}_i$.

Next, we will analyze the last three terms in (31) by using the inequality $\|\operatorname{sat}(S_i,\varphi)\| \le 1$. From the input-output property of the adaptive law in (27), it can be verified that

$$\hat{\theta}_{li} \le \mathring{\theta}_{li} + \check{\theta}_{li}\big(\|\omega_{\mathrm{ai}}\| + \|\omega_{\mathrm{ui}}\| + \|S_i\|\big)\|\xi_i\|^l \tag{32a}$$

$$\gamma \le \mathring{\gamma}_i + \check{\gamma}_i\big(\|S_i\| + \|\xi_i\|\big) \tag{32b}$$

with $\mathring{\theta}_{li},\ \check{\theta}_{li},\ \mathring{\gamma}_i,\ \check{\gamma}_i \in \mathbb{R}^+,\ l = 0, 1, 2.$

Using (32), together with $\|\omega_{\mathrm{ai}}\| \le \|\xi_i\|$ and $\|\omega_{\mathrm{ui}}\| \le \|\xi_i\|$, the following can be obtained:

$$\begin{aligned}
&\omega_{\mathrm{ui}}^T P_{\mathrm{ui}}B_2M_{\mathrm{uui}}^{-1}M_{\mathrm{aui}}(I_m+T_i)\rho_i\operatorname{sat}(S_i,\varphi) \\
\le&\bar{\bar{T}}_{1i}\Big[\sum_{l=0}^{2}\hat{\theta}_{li}\|\xi_i\|^l + \gamma_i\Big]\|\omega_{\mathrm{ui}}^T\| \\
\le&\bar{\bar{T}}_{1i}\Bigg\{\sum_{l=0}^{2}\mathring{\theta}_{li}\|\xi_i\|^{l+1} + \check{\theta}_{li}\big(2+\|B_1^TP_{ai}\|\big)\|\xi_i\|^{2l+2} \\
&+ \mathring{\gamma}_i\|\xi_i\| + \check{\gamma}_i\big(1+\|B_1^TP_{ai}\|\big)\|\xi_i\|^2\Bigg\}
\end{aligned} \tag{33}$$

where $\bar{\bar{T}}_{1i} = \frac{\left(1+\bar{T}\right)\|P_{\mathrm{u}i}B_2\|\|M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}\|}{\left(1-\bar{T}\right)}$.

In an analogous way, the following can be obtained

$$
\begin{aligned}
&\sum_{j\in\mathcal{N}_i} \bar{a}_{ij}\omega_{\mathrm{u}i}^T P_{\mathrm{u}i}B_2 M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}(I_m+T_j)\rho_j \,\mathrm{sat}(S_j,\varphi)\\
&\le \sum_{j\in\mathcal{N}_i} \bar{\bar{T}}_{2j}\Big[\sum_{l=0}^{2}\hat{\theta}_{lj}\|\xi_j\|^l+\gamma_j\Big]\|\omega_{\mathrm{u}i}^T\|\\
&\le \sum_{j\in\mathcal{N}_i} \bar{\bar{T}}_{2j}\Bigg\{\sum_{l=0}^{2}\mathring{\theta}_{lj}\|\xi_i\|\|\xi_j\|^l+\check{\gamma}_j\big(1+\|B_1^TP_{ai}\|\big)\|\xi_i\|^2\\
&\quad+\mathring{\gamma}_j\|\xi_i\|+\check{\theta}_{lj}\big(2+\|B_1^TP_{aj}\|\big)\|\xi_i\|\|\xi_j\|^{2l+1}\Bigg\}
\end{aligned}
\tag{34}
$$

where $\bar{\bar{T}}_{2j} = \frac{\bar{a}_{ij}\left(1+\bar{T}\right)\|P_{\mathrm{u}i}B_2\|\|M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}\|}{\left(1-\bar{T}\right)}$. In addition,

$$
\begin{aligned}
&\sum_{j\in\mathcal{N}_i} \bar{a}_{ij}\omega_{\mathrm{a}i}^T P_{\mathrm{a}i}B_1(I_m+T_j)\rho_j \,\mathrm{sat}(S_j,\varphi)\\
&\le \sum_{j\in\mathcal{N}_i} \bar{\bar{T}}_{3j}\Big[\sum_{l=0}^{2}\hat{\theta}_{lj}\|\xi_j\|^l+\gamma_j\Big]\|\omega_{\mathrm{a}i}^T\|\\
&\le \sum_{j\in\mathcal{N}_i} \bar{\bar{T}}_{3j}\Bigg\{\sum_{l=0}^{2}\mathring{\theta}_{lj}\|\xi_i\|\|\xi_j\|^l+\check{\gamma}_j\big(1+\|B_1^TP_{ai}\|\big)\|\xi_i\|^2\\
&\quad+\mathring{\gamma}_j\|\xi_i\|+\check{\theta}_{lj}\big(2+\|B_1^TP_{aj}\|\big)\|\xi_i\|\|\xi_j\|^{2l+1}
\end{aligned}
\tag{35}
$$

where $\bar{\bar{T}}_{3j} = \frac{\left(1+\bar{T}\right)\|P_{ai}B_1\|}{\left(1-\bar{T}\right)}$. Using (33)-(35), the following aggregate term Ψ_{ij} can be defined from (31)

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Psi_{ij} = \sum_{j\in\mathcal{N}_i}\Big[\bar{\varphi}_{1j}\|\xi_j\|+\bar{\varphi}_{2j}\|\xi_j\|^2\Big]\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\\
&+\omega_{\mathrm{u}i}^TP_{\mathrm{u}i}B_2M_{\mathrm{uu}i}^{-1}M_{\mathrm{au}i}(I_m+T_i)\rho_i\,\mathrm{sat}(S_i,\varphi)\\
&+\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\omega_{\mathrm{u}i}^TP_{\mathrm{u}i}B_2M_{\mathrm{uu}j}^{-1}M_{\mathrm{au}j}(I_m+T_j)\rho_j\,\mathrm{sat}(S_j,\varphi)\\
&+\sum_{j\in\mathcal{N}_i}\bar{a}_{ij}\omega_{\mathrm{a}i}^TP_{\mathrm{a}i}B_1(I_m+T_j)\rho_j\,\mathrm{sat}(S_j,\varphi)\\
\le&\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\check{\theta}_{2j}\big(\bar{\bar{T}}_{2j}+\bar{\bar{T}}_{3j}\big)\big(2+\|B_1^TP_{aj}\|\big)\|\xi_i\|\|\xi_j\|^5\\
&+\sum_{i=1}^{N}\bar{\bar{T}}_{1i}\check{\theta}_{2i}\big(2+\|B_1^TP_{ai}\|\big)\|\xi_i\|^6\\
&+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\check{\theta}_{1j}\big(\bar{\bar{T}}_{2j}+\bar{\bar{T}}_{3j}\big)\big(2+\|B_1^TP_{aj}\|\big)\|\xi_i\|\|\xi_j\|^3\\
&+\sum_{i=1}^{N}\bar{\bar{T}}_{1i}\check{\theta}_{1i}\big(2+\|B_1^TP_{ai}\|\big)\|\xi_i\|^4+\bar{\bar{T}}_{1i}\mathring{\theta}_{2i}\|\xi_i\|^3\\
&+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Big[2\bar{\varphi}_{2j}+\big(\bar{\bar{T}}_{2j}+\bar{\bar{T}}_{3j}\big)\mathring{\theta}_{2j}\Big]\|\xi_i\|\|\xi_j\|^2\\
&+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Bigg\{2\bar{\varphi}_{1j}+\big(\bar{\bar{T}}_{2j}+\bar{\bar{T}}_{3j}\big)\Big[\check{\theta}_{0j}\big(2+\|B_1^TP_{aj}\|\big)\\
&+\mathring{\theta}_{1j}\Big]\Bigg\}\|\xi_i\|\|\xi_j\|+\Bigg\{\bar{\bar{T}}_{1i}\Big[\check{\theta}_{0i}\big(2+\|B_1^TP_{ai}\|\big)+\mathring{\theta}_{1i}\\
&+\check{\gamma}_i\big(1+\|B_1^TP_{ai}\|\Big]+\check{\gamma}_j\big(\bar{\bar{T}}_{2j}+\bar{\bar{T}}_{3j}\big)\big(1+\|B_1^TP_{ai}\|\big)\Bigg\}\|\xi_i\|^2\\
&+\Big[\bar{\bar{T}}_{1i}\big(\mathring{\theta}_{0i}+\mathring{\gamma}_i\big)+\big(\bar{\bar{T}}_{2j}+\bar{\bar{T}}_{3j}\big)\big(\mathring{\gamma}_j+\mathring{\theta}_{0j}\big)\Big]\|\xi_i\|.
\end{aligned}
\tag{36}
$$

b) Time derivative of the Lyapunov function based on uncertainty bound

Up to now, we have calculated the time derivative of the first line in (28). We will proceed with the time derivative of the other terms. Using the adaptive laws (27a)-(27c), we have

$$
\begin{aligned}
&\sum_{l=0}^{2}\frac{1}{\chi_{li}}(\hat{\theta}_{li}-\bar{\theta}_{li})\dot{\hat{\theta}}_{li} = \sum_{l=0}^{2}\frac{1}{\chi_{li}}(\hat{\theta}_{li}-\bar{\theta}_{li})\Big[\chi_{li}\big(\|\omega_{\mathrm{a}i}\|\\
&+\|\omega_{\mathrm{u}i}\|+\|S_i\|\big)\|\xi_i\|^l-\alpha_{li}\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l\hat{\theta}_{li}\Big]\\
=&\sum_{l=0}^{2}\hat{\theta}_{li}\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|+\|S_i\|\big)\|\xi_i\|^l\\
&-\sum_{l=0}^{2}\bar{\theta}_{li}\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}+\|S_i\|\|\big)\|\xi_i\|^l\\
&-\sum_{l=0}^{2}\bar{\alpha}_{li}\hat{\theta}_{li}^2\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l\\
&+\sum_{l=0}^{2}\bar{\alpha}_{li}\hat{\theta}_{li}\bar{\theta}_{li}\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l
\end{aligned}
\tag{37}
$$

where $\bar{\alpha}_{li}=\alpha_{li}/\chi_{li}$. In addition,

$$
\begin{aligned}
\frac{\gamma_i\dot{\gamma}_i}{\epsilon_0} =&\frac{\gamma_i}{\epsilon_0}\Bigg\{-\Big[\epsilon_0+\epsilon_1(\|\xi_i\|^7-\|\xi_i\|^5)+\epsilon_2\|\xi_i\|\Big]\gamma_i\\
&+\epsilon_0\big(\|S_i\|+\|\xi_i\|\big)+\beta_i\Bigg\}\\
=&-\gamma_i^2\Big[1+\bar{\epsilon}_1(\|\xi_i\|^7-\|\xi_i\|^5)+\bar{\epsilon}_2\|\xi_i\|\Big]\\
&+\gamma_i\big(\|S_i\|+\|\xi_i\|\big)+\gamma_i\bar{\beta}_i
\end{aligned}
\tag{38}
$$

where $\bar{\epsilon}_1=\frac{\epsilon_1}{\epsilon_0},\bar{\epsilon}_2=\frac{\epsilon_2}{\epsilon_0},\bar{\beta}_i=\frac{\beta_i}{\epsilon_0}$. Using (31), (36) and (37)-(38), the time derivative of V satisfies

$$
\begin{aligned}
\dot{V}\le&-\sum_{i=1}^{N}\lambda_{\mathrm{m}i}\Big[\|\omega_{\mathrm{a}i}\|^2+\|\omega_{\mathrm{u}i}\|^2\Big]+\sum_{l=0}^{2}\bar{\theta}_{li}\|\xi_i\|^l\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\\
&-\omega_{\mathrm{a}i}^TP_{\mathrm{a}i}B_1(I_m+T_i)\rho_i\,\mathrm{sat}(S_i,\varphi)+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Psi_{ij}\\
&+\sum_{i=1}^{N}\sum_{l=0}^{2}\hat{\theta}_{li}\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|+\|S_i\|\big)\|\xi_i\|^l\\
&-\sum_{i=1}^{N}\sum_{l=0}^{2}\bar{\theta}_{li}\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}+\|S_i\|\|\big)\|\xi_i\|^l\\
&-\sum_{i=1}^{N}\sum_{l=0}^{2}\bar{\alpha}_{li}\hat{\theta}_{li}^2\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l+\sum_{i=1}^{N}\gamma_i\big(\|S_i\|+\|\xi_i\|\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\sum_{l=0}^{2}\bar{\alpha}_{li}\hat{\theta}_{li}\bar{\theta}_{li}\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l+\gamma_i\bar{\beta}_i\\
&-\sum_{i=1}^{N}\gamma_i^2\Big[1+\bar{\epsilon}_1(\|\xi_i\|^7-\|\xi_i\|^5)+\bar{\epsilon}_2\|\xi_i\|\Big]\\
\leq&-\sum_{i=1}^{N}\lambda_{\mathrm{m}i}\Big[\|\omega_{\mathrm{a}i}\|^2+\|\omega_{\mathrm{u}i}\|^2\Big]-\gamma_i^2\bar{\epsilon}_1(\|\xi_i\|^7-\|\xi_i\|^5)\\
&-\omega_{\mathrm{a}i}^TP_{\mathrm{a}i}B_1(I_m+T_i)\rho_i\,\mathrm{sat}(S_i,\varphi)+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Psi_{ij}\\
&+\sum_{i=1}^{N}\sum_{l=0}^{2}\Big[\hat{\theta}_{li}-\bar{\alpha}_{li}\hat{\theta}_{li}^2+\bar{\alpha}_{li}\hat{\theta}_{li}\bar{\theta}_{li}\Big]\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l\\
&-\sum_{i=1}^{N}\sum_{l=0}^{2}\bar{\theta}_{li}\|S_i\|\|\xi_i\|^l+\sum_{i=1}^{N}\Big[\sum_{l=0}^{2}\hat{\theta}_{li}\|\xi_i\|^l+\gamma_i\Big]\|S_i\|\\
&+\sum_{i=1}^{N}\Big[\gamma_i\bar{\beta}_i-\gamma_i^2\big(1+\bar{\epsilon}_2\|\xi_i\|\big)\Big]+\sum_{i=1}^{N}\gamma_i\|\xi_i\|.
\end{aligned}\tag{39}
$$

The following inequality holds:

$$
\begin{aligned}
&\sum_{l=0}^{2}\Big[\hat{\theta}_{li}-\bar{\alpha}_{li}\hat{\theta}_{li}^2+\bar{\alpha}_{li}\hat{\theta}_{li}\bar{\theta}_{li}\Big]\\
\leq&\sum_{l=0}^{2}-\frac{\bar{\alpha}_{li}}{3}\hat{\theta}_{li}^2-\Big[\frac{\bar{\alpha}_{li}}{3}\big(\hat{\theta}_{li}+\frac{3}{2\bar{\alpha}_{li}}\big)^2-\frac{3}{4\bar{\alpha}_l^2}\Big]\\
&-\Big[\frac{\bar{\alpha}_{li}}{3}\big(\hat{\theta}_{li}-\frac{3}{2\bar{\alpha}_{li}}\bar{\theta}_{li}\big)^2-\frac{3\bar{\theta}_{li}^2}{4\bar{\alpha}_l^2}\Big]\\
\leq&\sum_{l=0}^{2}\Big[-\frac{\bar{\alpha}_{li}}{3}\hat{\theta}_{li}^2+\frac{3}{4\bar{\alpha}_{li}}+\frac{3\bar{\theta}_{li}^2}{4\bar{\alpha}_{li}}\Big].
\end{aligned}\tag{40}
$$

In addition,

$$
\begin{aligned}
&\sum_{i=1}^{N}-\gamma_i^2\big(1+\bar{\epsilon}_2\|\xi_i\|\big)+\gamma_i\bar{\beta}_i+\gamma_i\|\xi_i\|\\
\leq&\sum_{i=1}^{N}-\frac{\bar{\epsilon}_2}{2}\|\xi_i\|\gamma_i^2-\left\{\Big[\gamma_i-\frac{1}{2}\bar{\beta}_i\Big]^2-\frac{1}{4}\bar{\beta}_i^2\right\}\\
&-\frac{\bar{\epsilon}_2}{2}\|\xi_i\|\left\{\Big[\gamma_i-\frac{1}{\bar{\epsilon}_2}\Big]^2-\frac{1}{2\bar{\epsilon}_2^2}\right\}\\
\leq&\sum_{i=1}^{N}\Big[-\frac{\bar{\epsilon}_2}{2}\|\xi_i\|\gamma_i^2+\frac{1}{4}\bar{\beta}_i^2+\frac{1}{4\bar{\epsilon}_2}\|\xi_i\|\Big].
\end{aligned}\tag{41}
$$

According to the adaptive law (27b), there exist $\underline{\gamma}_i\in\mathbb{R}^+$ such that $\gamma_i>\underline{\gamma}_i$. Substituting (40)-(41) into (39), yields

$$
\begin{aligned}
\dot{V}\leq&-\sum_{i=1}^{N}\lambda_{\mathrm{m}i}\Big[\|\omega_{\mathrm{a}i}\|^2+\|\omega_{\mathrm{u}i}\|^2\Big]-\sum_{i=1}^{N}\underline{\gamma}_i^2\bar{\epsilon}_1(\|\xi_i\|^7-\|\xi_i\|^5)\\
&+\sum_{i=1}^{N}\sum_{l=0}^{2}\Big[-\frac{\bar{\alpha}_{li}}{3}\hat{\theta}_{li}^2+\frac{3}{4\bar{\alpha}_{li}}+\frac{3\bar{\theta}_{li}^2}{4\bar{\alpha}_{li}}\Big]\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l\\
&+\sum_{i=1}^{N}\Big[-\frac{\bar{\epsilon}_2}{2}\|\xi_i\|\gamma_i^2+\frac{1}{4}\bar{\beta}_i^2+\frac{1}{4\bar{\epsilon}_2}\|\xi_i\|\Big]+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Psi_{ij}\\
&-\omega_{\mathrm{a}i}^TP_{\mathrm{a}i}B_1(I_m+T_i)\rho_i\,\mathrm{sat}(S_i,\varphi)\\
&+\sum_{i=1}^{N}\Big[\sum_{l=0}^{2}\hat{\theta}_{li}\|\xi_i\|^l+\gamma_i\Big]\|S_i\|.
\end{aligned}\tag{42}
$$

c) Behavior of the Lyapunov function on saturation regions

Based on the saturation $\mathrm{sat}(S_i,\varphi)$, we study the behavior of the Lyapunov function along three scenarios similar to [33]:

- **Scenario 1:** $\|S_i\|\geq\varphi$, $i=1,\ldots,N$.
  In this scenario, we have $\mathrm{sat}(S_i,\varphi)=\frac{S_i}{\|S_i\|}$. According to (26), we obtain the following as $S_i^T=\omega_{\mathrm{a}i}^TP_{\mathrm{a}i}B_1$:

$$
\begin{aligned}
-S_i^T(I_m+T_i)\rho_i\,\mathrm{sat}(S_i,\varphi)\leq&-\sum_{i=1}^{N}(1-\bar{T})\frac{S_i^TS_i}{\|S_i\|}\rho_i\\
\leq&-\sum_{i=1}^{N}\Big[\sum_{l=0}^{2}\hat{\theta}_{li}\|\xi_i\|^l+\gamma_i\Big]\|S_i\|.
\end{aligned}\tag{43}
$$

Using (43), the time derivative (42) is simplified to

$$
\begin{aligned}
\dot{V}\leq&-\sum_{i=1}^{N}\lambda_{\mathrm{m}i}\Big[\|\omega_{\mathrm{a}i}\|^2+\|\omega_{\mathrm{u}i}\|^2\Big]-\sum_{i=1}^{N}\underline{\gamma}_i^2\bar{\epsilon}_1(\|\xi_i\|^7-\|\xi_i\|^5)\\
&+\sum_{i=1}^{N}\sum_{l=0}^{2}\Big[-\frac{\bar{\alpha}_{li}}{3}\hat{\theta}_{li}^2+\frac{3}{4\bar{\alpha}_{li}}+\frac{3\bar{\theta}_{li}^2}{4\bar{\alpha}_{li}}\Big]\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l\\
&+\sum_{i=1}^{N}\Big[-\frac{\bar{\epsilon}_2}{2}\|\xi_i\|\gamma_i^2+\frac{1}{4}\bar{\beta}_i^2+\frac{1}{4\bar{\epsilon}_2}\|\xi_i\|\Big]+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Psi_{ij}.
\end{aligned}\tag{44}
$$

The definition of Lyapunov function (28) satisfies

$$
\begin{aligned}
V\leq&\sum_{i=1}^{N}\lambda_{\mathrm{M}i}\big(\|\omega_{\mathrm{a}i}\|^2+\|\omega_{\mathrm{u}i}\|^2\big)\\
&+\sum_{i=1}^{N}\Big[\sum_{l=0}^{2}\frac{1}{\chi_{li}}\Big(\hat{\theta}_{li}^2+\bar{\theta}_{li}^2\Big)+\frac{\gamma_i^2}{\epsilon_0}\Big]
\end{aligned}\tag{45}
$$

where $\lambda_{\mathrm{M}i}=\max\Big\{\lambda_{\max}(P_{\mathrm{a}i})/2,\lambda_{\max}(P_{\mathrm{u}i})/2\Big\}$. Define $\zeta=\frac{\min_i\{\lambda_{\mathrm{m}i}\}}{\max_i\{\lambda_{\mathrm{M}i}\}}$. Substituting (45) into (44) yields

$$
\begin{aligned}
\dot{V}\leq&-\zeta V+\sum_{i=1}^{N}\Big[\sum_{l=0}^{2}\frac{\zeta}{\chi_{li}}\Big(\hat{\theta}_{li}^2+\bar{\theta}_{li}^2\Big)+\frac{\zeta\gamma_i^2}{\epsilon_0}\Big]\\
&+\sum_{i=1}^{N}\sum_{l=0}^{2}\Big\{-\frac{\bar{\alpha}_{li}}{3}\hat{\theta}_{li}^2\big(\|\omega_{\mathrm{a}i}\|^{l+1}+\|\omega_{\mathrm{u}i}\|^{l+1}\big)\\
&+\big(\frac{3}{4\bar{\alpha}_{li}}+\frac{3\bar{\theta}_{li}^2}{4\bar{\alpha}_{li}}\big)\big(\|\omega_{\mathrm{a}i}\|+\|\omega_{\mathrm{u}i}\|\big)\|\xi_i\|^l\Big\}\\
&+\sum_{i=1}^{N}\Big[-\frac{\bar{\epsilon}_2}{2}\|\xi_i\|\gamma_i^2+\frac{1}{4}\bar{\beta}_i^2+\frac{1}{4\bar{\epsilon}_2}\|\xi_i\|\Big]\\
&-\sum_{i=1}^{N}\underline{\gamma}_i^2\bar{\epsilon}_1(\|\xi_i\|^7-\|\xi_i\|^5)+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Psi_{ij}\\
\leq&-\zeta V-\sum_{i=1}^{N}\sum_{l=0}^{2}\hat{\theta}_{li}^2\Big[\frac{\bar{\alpha}_{li}}{3}\big(\|\omega_{\mathrm{a}i}\|^{l+1}+\|\omega_{\mathrm{u}i}\|^{l+1}\big)\\
&-\frac{\zeta}{\chi_{li}}\Big]-\sum_{i=1}^{N}\gamma_i^2\Big[\frac{\bar{\epsilon}_2}{2}\|\xi_i\|-\frac{\zeta}{\epsilon_0}\Big]+Z_1(\|\xi\|).
\end{aligned}\tag{46}
$$

According to (36),

$$Z_1(\|\xi\|) \triangleq \sum_{i=1}^{N} -\bar{\epsilon}_1\underline{\gamma}_i^2\|\xi_i\|^7 + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{11}\|\xi_i\|\|\xi_j\|^5 + \sum_{i=1}^{N} c_6 + \sum_{i=1}^{N} c_{12}\|\xi_i\|^6 + \sum_{i=1}^{N}\bar{\epsilon}_1\underline{\gamma}_i^2\|\xi_i\|^5 + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{21}\|\xi_i\|\|\xi_j\|^3 + \sum_{i=1}^{N} c_{22}\|\xi_i\|^4 + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{31}\|\xi_i\|\|\xi_j\|^2 + \sum_{i=1}^{N} c_{32}\|\xi_i\|^3 + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{41}\|\xi_i\|\|\xi_j\| + \sum_{i=1}^{N} c_{42}\|\xi_i\|^2 + \sum_{i=1}^{N} c_5\|\xi_i\|$$

with

$$\begin{aligned}
c_{11} =& \check{\theta}_{2j}\big(\bar{\bar{T}}_{2j} + \bar{\bar{T}}_{3j}\big)\big(2 + \|B_1^T P_{aj}\|\big)\\
c_{12} =& \bar{\bar{T}}_{1i}\check{\theta}_{2i}\big(2 + \|B_1^T P_{ai}\|\big),\ c_6 = \frac{1}{4}\bar{\beta}_i^2 + \zeta\bar{\theta}_{li}^2\\
c_{21} =& \check{\theta}_{1j}\big(\bar{\bar{T}}_{2j} + \bar{\bar{T}}_{3j}\big)\big(2 + \|B_1^T P_{aj}\|\big)\\
c_{22} =& \bar{\bar{T}}_{1i}\check{\theta}_{1i}\big(2 + \|B_1^T P_{ai}\|\big)\\
c_{31} =& 2\bar{\varphi}_{2j} + \big(\bar{\bar{T}}_{2j} + \bar{\bar{T}}_{3j}\big)\bar{\theta}_{2j}\\
c_{32} =& 2\Big(\frac{3}{4\alpha_{2i}} + \frac{3\bar{\theta}_{2i}^2}{4\alpha_{2i}}\Big) + \bar{\bar{T}}_{1i}\mathring{\theta}_{2i}\\
c_{41} =& 2\bar{\varphi}_{1j} + \big(\bar{\bar{T}}_{2j} + \bar{\bar{T}}_{3j}\big)\Big[\check{\theta}_{0j}\big(2 + \|B_1^T P_{aj}\|\big) + \mathring{\theta}_{1j}\Big]\\
c_{42} =& 2\Big(\frac{3}{4\alpha_{1i}} + \frac{3\bar{\theta}_{1i}^2}{4\alpha_{1i}}\Big) + \bar{\bar{T}}_{1i}\Big[\check{\theta}_{0i}\big(2+\|B_1^T P_{ai}\|\big) + \mathring{\theta}_{1i}\\
&+ \check{\gamma}_i\big(1+\|B_1^T P_{ai}\|\Big] + \check{\gamma}_j\big(\bar{\bar{T}}_{2j}+\bar{\bar{T}}_{3j}\big)\big(1+\|B_1^T P_{ai}\|\big)\\
c_5 =& \bar{\bar{T}}_{1i}\big(\mathring{\theta}_{0i} + \mathring{\gamma}_i\big) + \big(\bar{\bar{T}}_{2j} + \bar{\bar{T}}_{3j}\big)\big(\mathring{\gamma}_j + \mathring{\theta}_{0j}\big)\\
&+ 2\Big(\frac{3}{4\alpha_{0i}} + \frac{3\bar{\theta}_{0i}^2}{4\alpha_{0i}}\Big) + \frac{1}{4\bar{\varphi}_2}.
\end{aligned}$$

Using Descartes' rules of sign change and Bolzano's Theorem [46], the polynomial Z_1 has exactly one positive real root $\eta_1 \in \mathbb{R}^+$. The coefficient of the highest degree of Z_1 is negative as $-\underline{\gamma}_i^2\bar{\epsilon}_1$. Therefore, $Z_1(\|\xi\|) \le 0$ when $\|\xi\| \ge \eta_1$, where $\xi = [\xi_1, \ldots, \xi_N]^T$. Define $\iota_1 = \frac{3\zeta}{2\chi_{0i}\bar{\alpha}_{0i}}$, $\iota_2 = \sqrt{\frac{3\zeta}{2\chi_{1i}\bar{\alpha}_{1i}}}$, $\iota_3 = \left(\frac{3\zeta}{2\chi_{2i}\bar{\alpha}_{2i}}\right)^{1/3}$, $\iota_4 = \frac{2\zeta}{\bar{\epsilon}_2}$. According to (46), $\dot{V} \le -\zeta V$ when

$$\begin{aligned}
&\min\{\|\omega_{\mathrm{a}i}\|, \|\omega_{\mathrm{u}i}\|, \|\xi_i\|\} \ge \max\{\eta_1, \iota_1, \iota_2, \iota_3, \iota_4\}\\
&\Rightarrow \min\{\|\omega_{\mathrm{a}i}\|, \|\omega_{\mathrm{u}i}\|\} \ge \max\{\eta_1, \iota_1, \iota_2, \iota_3, \iota_4\}
\end{aligned} \tag{47}$$

- **Scenario 2:** $\|S_i\| < \varphi$, $i = 1, \ldots, N$.
  In this scenario $\mathrm{sat}(S_i, \varphi) = \frac{S_i}{\varphi}$. Using (26), we have

$$-\sum_{i=1}^{N} S_i^T(I_m + T_i)\rho_i\,\mathrm{sat}(S_i, \varphi) \le 0. \tag{48}$$

Substituting (48) into (42) gives

$$\dot{V} \le -\zeta V - \sum_{i=1}^{N}\sum_{l=0}^{2}\hat{\theta}_{li}^2\left[\frac{\bar{\alpha}_{li}}{3}\big(\|\omega_{\mathrm{a}i}\|^{l+1} + \|\omega_{\mathrm{u}i}\|^{l+1}\big) - \frac{\zeta}{\chi_{li}}\right] - \sum_{i=1}^{N}\gamma_i^2\left[\frac{\bar{\epsilon}_2}{2}\|\xi_i\| - \frac{\zeta}{\epsilon_0}\right] + Z_1(\|\xi\|) + \sum_{i=1}^{N}\left[\sum_{l=0}^{2}\hat{\theta}_{li}\|\xi_i\|^l + \gamma_i\right]\|S_i\|. \tag{49}$$

According to (32), together with $\|S_i\| < \varphi$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\left[\sum_{l=0}^{2}\hat{\theta}_{li}\|\xi_i\|^l + \gamma_i\right]\|S_i\|\\
\le& \sum_{i=1}^{N}\big(\mathring{\theta}_{0i} + \varphi\check{\theta}_{0i} + \varphi\mathring{\gamma}_i\big) + \check{\gamma}_i\big(1 + \|B_1^T P_{ai}\|\big)\varphi\\
&+ \big(2\check{\theta}_{0i} + \mathring{\theta}_{1i}\big)\|\xi_i\| + \sum_{i=1}^{N}\Big[\varphi\check{\theta}_{1i} + \mathring{\theta}_{1i}\Big]\|\xi_i\|^2\\
&+ 2\check{\theta}_{1i}\|\xi_i\|^3 + \varphi\check{\theta}_{2i}\|\xi_i\|^4 + \sum_{i=1}^{N} 2\check{\theta}_{2i}\|\xi_i\|^5.
\end{aligned} \tag{50}$$

Substituting (50) into (49) gives

$$\dot{V} \le -\zeta V - \sum_{i=1}^{N}\sum_{l=0}^{2}\hat{\theta}_{li}^2\left[\frac{\bar{\alpha}_{li}}{3}\big(\|\omega_{\mathrm{a}i}\|^{l+1} + \|\omega_{\mathrm{u}i}\|^{l+1}\big) - \frac{\zeta}{\chi_{li}}\right] - \sum_{i=1}^{N}\gamma_i^2\left[\frac{\bar{\epsilon}_2}{2}\|\xi_i\| - \frac{\zeta}{\epsilon_0}\right] + Z_2(\|\xi\|) \tag{51}$$

with $Z_2(\|\xi\|) = Z_1(\|\xi\|) + \sum_{i=1}^{N}\big(\mathring{\theta}_{0i} + \varphi\check{\theta}_{0i} + \varphi\mathring{\gamma}_i\big) + \check{\gamma}_i\big(1 + \|B_1^T P_{ai}\|\big)\varphi + \big(2\check{\theta}_{0i} + \mathring{\theta}_{1i}\big)\|\xi_i\| + \sum_{i=1}^{N}\Big[\varphi\check{\theta}_{1i} + \mathring{\theta}_{1i}\Big]\|\xi_i\|^2 + 2\check{\theta}_{1i}\|\xi_i\|^3 + \varphi\check{\theta}_{2i}\|\xi_i\|^4 + \sum_{i=1}^{N} 2\check{\theta}_{2i}\|\xi_i\|^5$.
Analogously to Scenario 1, $\dot{V} \le -\zeta V$ when

$$\begin{aligned}
&\min\{\|\omega_{\mathrm{a}i}\|, \|\omega_{\mathrm{u}i}\|, \|\xi_i\|\} \ge \max\{\eta_2, \iota_1, \iota_2, \iota_3, \iota_4\}\\
&\Rightarrow \min\{\|\omega_{\mathrm{a}i}\|, \|\omega_{\mathrm{u}i}\|\} \ge \max\{\eta_2, \iota_1, \iota_2, \iota_3, \iota_4\}
\end{aligned} \tag{52}$$

where η_2 is the positive real root of Z_2 such that $Z_2(\|\xi\|) \le 0$ when $\|\xi\| \ge \eta_2$.

- **Scenario 3:** Without loss of generality, consider $\|S_i\| \ge \varphi$ for $i = 1, \ldots, k$, and $\|S_i\| < \varphi$ for $i = k+1, \ldots, N$ where $1 \le k \le N-1$. For $i = 1, \ldots, k$, we have $\mathrm{sat}(S_i, \varphi) = \frac{S_i}{\|S_i\|}$; For $i = k+1, \ldots, N$, $\mathrm{sat}(S_i, \varphi) = \frac{S_i}{\varphi}$. Similarly to Scenario 1 and Scenario 2, we get

$$\dot{V} \le -\zeta V - \sum_{i=1}^{N}\sum_{l=0}^{2}\hat{\theta}_{li}^2\left[\frac{\bar{\alpha}_{li}}{3}\big(\|\omega_{\mathrm{a}i}\|^{l+1} + \|\omega_{\mathrm{u}i}\|^{l+1}\big) - \frac{\zeta}{\chi_{li}}\right] - \sum_{i=1}^{N}\gamma_i^2\left[\frac{\bar{\epsilon}_2}{2}\|\xi_i\| - \frac{\zeta}{\epsilon_0}\right] + Z_3(\|\xi\|) \tag{53}$$

with $Z_3(\|\xi\|) = Z_1(\|\xi\|) + \sum_{i=k+1}^{N}\big(\mathring{\theta}_{0i} + \varphi\check{\theta}_{0i} + \varphi\mathring{\gamma}_i\big) + \check{\gamma}_i\big(1 + \|B_1^T P_{ai}\|\big)\varphi + \big(2\check{\theta}_{0i} + \mathring{\theta}_{1i}\big)\|\xi_i\| + \sum_{i=k+1}^{N}\Big[\varphi\check{\theta}_{1i} + \mathring{\theta}_{1i}\Big]\|\xi_i\|^2 + 2\check{\theta}_{1i}\|\xi_i\|^3 + \varphi\check{\theta}_{2i}\|\xi_i\|^4 + \sum_{i=k+1}^{N} 2\check{\theta}_{2i}\|\xi_i\|^5$.
Analogously to Scenario 1, $\dot{V} \le -\zeta V$ when

$$\begin{aligned}
&\min\{\|\omega_{\mathrm{a}i}\|, \|\omega_{\mathrm{u}i}\|, \|\xi_i\|\} \ge \max\{\eta_3, \iota_1, \iota_2, \iota_3, \iota_4\}\\
&\Rightarrow \min\{\|\omega_{\mathrm{a}i}\|, \|\omega_{\mathrm{u}i}\|\} \ge \max\{\eta_3, \iota_1, \iota_2, \iota_3, \iota_4\}
\end{aligned} \tag{54}$$

where η_3 is the positive real root of Z_3 such that $Z_3(\|\xi\|) \leq 0$ when $\|\xi\| \geq \eta_3$.

Finally, combining (47) in Scenario 1 with (52) in Scenario 2 and (54) in Scenario 3, we obtain $\omega_{ui}, \omega_{ai} \in \mathcal{L}_\infty$ when $\|\xi\| \geq \max\{\eta_1, \eta_2, \eta_3, \iota_1, \iota_2, \iota_3, \iota_4\}$, which leads to $e_u, e_a \in \mathcal{L}_\infty$. Both the local actuated synchronization error and local non-actuated synchronization error are thus proved to reach UUB. According to (9) in Lemma 1, the global synchronization errors $\delta_u,\ \delta_a$ are also uniformly ultimately bounded. ∎

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