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Technical report 24-013

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S. Markkassery, T. van den Boom, and B. De Schutter, "Stability of time-invariant max-min-plus-scaling discrete-event systems with diverse states," *Proceedings of the 17th IFAC Workshop on Discrete Event Systems (WODES 2024)*, Rio de Janeiro, Brazil, pp. 60–65, Apr.–May 2024. doi:10.1016/j.ifacol.2024.07.011

Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
phone: +31-15-278.24.73 (secretary)
URL: https://www.dcsc.tudelft.nl

Stability of Time-invariant Max-Min-Plus-Scaling Discrete-Event Systems with Diverse States

Sreeshma Markkassery * **Ton van den Boom** *
Bart De Schutter *

* *Delft Center for Systems and Control (DCSC), TU Delft, The Netherlands (e-mail: {smarkkassery, A.J.J.vandenBoom, B.DeSchutter}@tudelft.nl)*

Abstract: In this paper, we discuss the stability of general time-invariant discrete-event systems modeled as max-min-plus-scaling (MMPS) systems. We analyze MMPS systems with two types of states: time states and quantity states. A set of linear programming problems are proposed to find the growth rates of the time states via a normalization of the MMPS system. Then a framework for stability analysis of the general time-invariant MMPS system is discussed with respect to the normalized system. The approach presented in this paper is an efficient way to study the stability of a general MMPS system.

Keywords: Discrete-event systems, max-min-plus-scaling systems, modeling and stability

1. INTRODUCTION

Max-min-plus-scaling (MMPS) systems are algebraic models that can describe linear and nonlinear discrete-event systems in the max-plus algebra. These systems are extensions of popular max-plus-linear (MPL) systems (Heidergott et al., 2006; Baccelli et al., 1992) and max-min-plus (MMP) systems (Gunawardena, 1994a). MMPS systems model operations such as synchronization, competition, and state-dependent processing times (van den Boom et al., 2023). These operations are used in applications like urban railway networks, flexible manufacturing systems, traffic networks, etc. The scaling operation in MMPS systems provides modeling advantages over MPL and MMP systems when the states are dependent on each other. For example, in an urban railway network, the number of people in the train can be modeled as a fraction of the number of people in the station; in a manufacturing system the processing time can be linearly dependent on previous states or external inputs.

Studies on MMPS systems have appeared in (De Schutter and van den Boom, 2004, 2002; Heemels et al., 2001). Heemels et al. (2001) presents the equivalence of MMPS systems to other hybrid system classes such as piecewise affine systems, mixed-logical dynamical systems, and linear complementarity systems. In (De Schutter and van den Boom, 2004, 2002), the model predictive control problem of the MMPS system has been considered. The MMPS system as a modeling tool for discrete-event systems is only recently discussed in (van den Boom et al., 2023; Markkassery et al., 2024). However, the dynamics of discrete-event MMPS systems have never been studied.

The dynamics of MPL systems and MMP systems have been widely studied in (Bemporad and Morari, 1999; Heidergott et al., 2006; Königsberg, 2001; Gunawardena, 1994a,b). They are focused on finding the additive eigenvalue/growth rate and additive eigenvector/stationary regime/equilibrium point of the system, with all states having the dimension of time (time states). However, discrete-event systems can also have quantity states (e.g. the number of passengers on a train in an urban railway network). Having two different types of states helps in the efficient modeling of discrete-event systems. In (van den Boom et al., 2023), an urban railway network is modeled as an MMPS system with both time and quantity states. In general, MPL systems and MMP systems are monotone and non-expansive and have a unique growth rate, if the growth rate exists (Cochet-Terrasson et al., 1997). The scaling operation in MMPS systems can make the system non-monotone and expansive. As a result, the system can have multiple growth rates and stationary regimes (Markkassery et al., 2024).

The main contributions of this paper are as follows. Given the importance of the MMPS systems in modeling the discrete-event systems, we analyze their dynamics. A novel approach is presented to derive a normalized MMPS system with zero growth rate from a general MMPS system with both time and quantity states. The most important contribution of this paper is the proposed framework for the stability analysis of a general time-invariant discrete-event MMPS system. We establish a connection between the stability analysis of discrete-event systems and linear discrete-time systems via normalization. Finally, we propose an optimization problem to find the invariant regions where the MMPS system is stable.

The paper is organized as follows. Section 2 presents the mathematical preliminaries and definitions. Section 3 presents a method to derive a normalized system from a general MMPS system and provides a set of linear programming problems to calculate all the growth rates and equilibrium points of MMPS systems. The internal stability of MMPS systems is discussed in Section 4 using the normalized system. An optimization problem to find the invariant regions of the stable MMPS system is presented in Section 5. The analysis of an example MMPS system is presented in Section 6. Section 7 presents the conclusions.

2. MATHEMATICAL PRELIMINARIES

In this paper, we consider MMPS systems that are explicit, time-invariant, and autonomous with both time states and quantity states.

Let $\top = \infty$, $\varepsilon = -\infty$, $\mathbb{R}_\top = \mathbb{R} \cup \{\infty\}$, $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$, $\mathbb{R}_c = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ and let $\mathbb{Z}^+$, denote the set of positive integers. We use the set $\mathcal{R}$ to denote one of the following sets: $\mathbb{R}$, $\mathbb{R}_\varepsilon$, $\mathbb{R}_\top$, or $\mathbb{R}_c$. The notations $\mathbf{1}$ and $\mathbf{0}$ are used to denote the vector with all components equal to 1 and the zero vector of the appropriate dimension, respectively. In some cases we use the notation $\mathbf{1}_n$ and $\mathbf{0}_n$ to specify the dimension, n, of vectors $\mathbf{1}$ and $\mathbf{0}$, respectively. An identity matrix of size n is denoted as I_n. Let $\overline{n}$ denote the set of all positive integers up to $n \in \mathbb{Z}^+$. Let $A \in \mathcal{R}^{n\times m}$ be a matrix, then $[A]_i$ denote the i-th row of the matrix. The notation '$\intercal$' is used to denote transpose of a matrix/vector. For $a, b \in \mathcal{R}$, the operations $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ are called max-plus addition and max-plus multiplication. The operations $a \oplus' b = \min(a, b)$ and $a \otimes' b = a + b$ are called min-plus addition and min-plus multiplication. Max-plus addition, max-plus multiplication, min-plus addition and min-plus multiplication (Heidergott et al., 2006) of matrices $A, B \in \mathcal{R}^{m\times n}$ and $C \in \mathcal{R}^{n\times p}$ are defined as:

$$[A\oplus B]_{ij} = \max([A]_{ij}, [B]_{ij}), [A\otimes C]_{ij} = \max_{k\in\overline{n}}([A]_{ik} + [C]_{kj})$$

$$[A\oplus' B]_{ij} = \min([A]_{ij}, [B]_{ij}), [A\otimes' C]_{ij} = \min_{k\in\overline{n}}([A]_{ik} + [C]_{kj})$$

From conventional algebra and matrix theory, we use the following notations and definitions. The matrix-vector product of the matrix, $C \in \mathbb{R}^{n\times p}$ and a vector, $x \in \mathbb{R}^p$ is denoted as $C \cdot x$ and the scalar multiplication of a scalar $\mu \in \mathbb{R}$ and the vector x is denoted as μx. The standard basis vector is denoted as a row vector of appropriate size, $\boldsymbol{e}_j$, with the j-th component equal to 1 and other components equal to 0.

Definition 1 (Kronecker Product). The Kronecker product of a matrix A and vector $\mathbf{1}_n$, $A \boxtimes \mathbf{1}_n$ stacks n copies of every row of the matrix A vertically and $\mathbf{1}_n \boxtimes A$ stacks n copies of the entire A matrix vertically.

Definition 2 (Row-major order of a matrix). The row-major order of a matrix $A \in \mathcal{R}^{n\times m}$ is the order of mapping a matrix to a column vector, $\text{vec}(A)$, in which the rows of matrix A are stacked in one column,

$$\text{vec}(A) = [A_1^\intercal \;\; A_2^\intercal \dots A_n^\intercal]^\intercal$$

where $A_i, i \in \overline{n}$ denote the row i of matrix A.

Definition 3. Given the vector $v \in \mathbb{R}^n$, we define a max-plus diagonal matrix, $\text{d}_\otimes(v)$ and the min-plus diagonal matrix, $\text{d}_{\otimes'}(v)$

$$\text{d}_\otimes(v) = \begin{bmatrix} v_1 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & v_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \varepsilon & \cdots & \cdots & v_n \end{bmatrix}, \text{d}_{\otimes'}(v) = \begin{bmatrix} v_1 & \top & \cdots & \top \\ \top & v_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \top & \cdots & \cdots & v_n \end{bmatrix}$$

The inverse max-plus diagonal matrix is, $[\text{d}_\otimes(v)]^{-1} = \text{d}_\otimes(-v)$ and the inverse min-plus diagonal matrix is $[\text{d}_{\otimes'}(v)]^{-1} = \text{d}_{\otimes'}(-v)$.

Definition 4 (Markkassery et al. (2024)). (ABC canonical form) The autonomous MMPS system $x(k) = f(x(k-1))$ with f being an MMPS function can be formulated as the following canonical form:

$$x(k) = A \otimes (B \otimes' (C \cdot x(k-1))) \tag{1}$$

for some matrices $A \in \mathbb{R}_\varepsilon^{n\times m}$, $B \in \mathbb{R}_\top^{m\times p}$, $C \in \mathbb{R}^{p\times n}$, and $x \in \mathcal{R}^n, k \in \mathbb{Z}^+$.

MMPS systems with time and quantity states gives ease in modeling of complex systems such as an urban railway network and are studied in detail in (van den Boom et al., 2023) along with a case study. In this paper, we define a structure of A, B, and C matrices for MMPS systems with both time and quantity states.

Definition 5. An autonomous MMPS system with both time states and quantity states is defined as

$$\begin{bmatrix} x_t(k) \\ x_q(k) \end{bmatrix} = \underbrace{\begin{bmatrix} A_t & \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} & A_q \end{bmatrix}}_{A} \otimes \left(\underbrace{\begin{bmatrix} B_t & \top \\ \top & B_q \end{bmatrix}}_{B} \otimes' \left(\underbrace{\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}}_{C} \cdot \begin{bmatrix} x_t(k-1) \\ x_q(k-1) \end{bmatrix} \right) \right) \tag{2}$$

where $x_t \in \mathbb{R}^{n_t}$, $x_q \in \mathbb{R}^{n_q}$, $A_t \in \mathbb{R}_\varepsilon^{n_t\times m_t}$, $A_q \in \mathbb{R}_\varepsilon^{n_q\times m_q}$, $B_t \in \mathbb{R}_\top^{m_t\times p_t}$, $B_q \in \mathbb{R}_\top^{m_q\times p_q}$, $C_{11} \in \mathbb{R}^{p_t\times n_t}$, $C_{12} \in \mathbb{R}^{p_t\times n_q}$, $C_{21} \in \mathbb{R}^{p_q\times n_t}$, and $C_{22} \in \mathbb{R}^{p_q\times n_q}$. The notations $\boldsymbol{\varepsilon}$ and $\top$ represent matrices of appropriate sizes with all elements equal to ε and, $\top$ respectively. The subscript 't' is associated with time states, and 'q' is associated with quantity states.

Definition 6 (van den Boom et al. (2023)). (Partial Additive Homogeneity) Consider $x_t \in \mathbb{R}^{n_t}$ and $x_q \in \mathbb{R}^{n_q}$ and MMPS functions $f_t : \mathbb{R}^{n_t\times n_q} \to \mathbb{R}^{n_t}$ and $f_q : \mathbb{R}^{n_t\times n_q} \to \mathbb{R}^{n_q}$. Then the system

$$\begin{bmatrix} x_t(k) \\ x_q(k) \end{bmatrix} = \begin{bmatrix} f_t(x_t(k-1), x_q(k-1)) \\ f_q(x_t(k-1), x_q(k-1)) \end{bmatrix} \tag{3}$$

is partly additive homogeneous if

$$f_t(x_t + h\mathbf{1}, x_q) = f_t(x_t, x_q) + h\mathbf{1}$$
$$f_q(x_t + h\mathbf{1}, x_q) = f_q(x_t, x_q)$$

An MMPS system is time-invariant if it is partly additive homogeneous (van den Boom et al., 2023).

Definition 7. (Additive eigenvalue, additive eigenvector) The time-invariant MMPS system $x(k) = f(x(k-1))$, $x \in \mathcal{R}^n$ and $f : \mathcal{R}^n \to \mathcal{R}^n$ with both time and quantity states is said to have an additive eigenvalue if there exists a real number $\lambda \in \mathcal{R}$ and a vector $v \in \mathbb{R}^n$ such that

$$f(v) = v + \lambda[\mathbf{1}_{n_t}^\intercal \;\; \mathbf{0}_{n_q}^\intercal]^\intercal$$

where n_t and n_q are the number of time states and quantity states, respectively. The scalar λ is then called an additive eigenvalue, and the vector v is called a corresponding additive eigenvector. Further, if v is an eigenvector, $v + h[\mathbf{1}_{n_t}^\intercal \;\; \mathbf{0}_{n_q}^\intercal]^\intercal$ is also an eigenvector for any $h \in \mathbb{R}$.

In the rest of the paper, we call the additive eigenvalue the growth rate, and the additive eigenvector the equilibrium point.

Definition 8 (Heidergott et al. (2006)). The Hilbert's projective norm of a vector $x \in \mathbb{R}^n$ in max-plus algebra is defined as

$$||x||_{\mathbb{P}} = \max_{i\in\overline{n}} x_i - \min_{i\in\overline{n}} x_i$$

3. GROWTH RATES OF TIME-INVARIANT MMPS SYSTEMS

In (Markkassery et al., 2024), we proposed a method to find the growth rates and equilibrium points of a general time-invariant MMPS system with only time states. In this section, we extend that result to MMPS systems with both time and quantity states.

Proposition 1. The MMPS system (2) is time-invariant if and only if:

$$\sum_{i\in\overline{n_{\rm t}}}[C_{11}]_{\ell i}=1,\ \forall \ell\in\overline{p_{\rm t}},\quad \sum_{i\in\overline{n_{\rm t}}}[C_{21}]_{ti}=0,\ \forall t\in\overline{p_{\rm q}}$$

Proof. It is proved in (Markkassery et al., 2024) that for the time-invariant MMPS system (1), we have

$$\begin{aligned}A\otimes\Big(B\otimes'\big(C\cdot(x(k-1)+h\mathbf{1})\big)\Big)&=\\ A\otimes\Big(B\otimes'\big(C\cdot x(k-1)\big)\Big)&+h\mathbf{1}\end{aligned}\tag{4}$$

for $h\in\mathbb{R}$. From Definition 6, the MMPS system is time-invariant if it is partly additive homogeneous. That is, for some $h\in\mathbb{R}$

$$A_{\rm t}\otimes\Big(B_{\rm t}\otimes'\big(C_{11}\cdot(x_{\rm t}(k-1)+h\mathbf{1})+C_{12}\cdot x_{\rm q}(k-1)\big)\Big)$$
$$=A_{\rm t}\otimes\Big(B_{\rm t}\otimes'\big(C_{11}\cdot x_{\rm t}(k-1)+C_{12}\cdot x_{\rm q}(k-1)\big)\Big)+h\mathbf{1},$$
$$A_{\rm q}\otimes\Big(B_{\rm q}\otimes'\big(C_{21}\cdot(x_{\rm t}(k-1)+h\mathbf{1})+C_{22}\cdot x_{\rm q}(k-1)\big)\Big)$$
$$=A_{\rm q}\otimes\Big(B_{\rm q}\otimes'\big(C_{21}\cdot x_{\rm t}(k-1)+C_{22}\cdot x_{\rm q}(k-1)\big)\Big)$$

By using (4), this leads to

$$A_{\rm t}\otimes\Big(B_{\rm t}\otimes'\big(C_{11}\cdot x_{\rm t}(k-1)+C_{12}\cdot x_{\rm q}(k-1)+C_{11}\cdot h\mathbf{1}\big)\Big)$$
$$=A_{\rm t}\otimes\Big(B_{\rm t}\otimes'\big(C_{11}\cdot x_{\rm t}(k-1)+C_{12}\cdot x_{\rm q}(k-1)\big)\Big)+C_{11}\cdot h\mathbf{1},$$
$$A_{\rm q}\otimes\Big(B_{\rm q}\otimes'\big(C_{21}\cdot x_{\rm t}(k-1)+C_{22}\cdot x_{\rm q}(k-1)+C_{21}\cdot h\mathbf{1}\big)\Big)$$
$$=A_{\rm q}\otimes\Big(B_{\rm q}\otimes'\big(C_{21}\cdot x_{\rm t}(k-1)+C_{22}\cdot x_{\rm q}(k-1)\big)\Big)+C_{21}\cdot h\mathbf{1}$$

Therefore it is required that $C_{11}\cdot h\mathbf{1}=h\mathbf{1}$ and $C_{21}\cdot h\mathbf{1}=\mathbf{0}$. So, $\sum_i [C_{11}]_{li}=1\ \forall l$ and $\sum_r [C_{21}]_{tr}=0\ \forall t$. □

A stable MMPS system cannot have growing quantity variables over events. Hence, the growth rate of the quantity states should always be zero to maintain stability. But, time states should have a constant growth rate. Let λ be the temporal growth rate and

$v=(x_{\rm te},x_{\rm qe},y_{\rm te},y_{\rm qe},z_{\rm te},z_{\rm qe})$ be the equilibrium point of the system (2). Then, from the Definition 7, we can get the following.

$$\begin{aligned}z_{\rm te}&=C_{11}\cdot(x_{\rm te}-\lambda\mathbf{1})+C_{12}\cdot x_{\rm qe}\\ z_{\rm qe}&=C_{21}\cdot(x_{\rm te}-\lambda\mathbf{1})+C_{22}\cdot x_{\rm qe}\\ y_{\rm te}&=B_{\rm t}\otimes' z_{\rm te},\quad y_{\rm qe}=B_{\rm q}\otimes' z_{\rm qe}\\ x_{\rm te}&=A_{\rm t}\otimes y_{\rm te},\quad x_{\rm qe}=A_{\rm q}\otimes y_{\rm qe}\end{aligned}\tag{5}$$

Let $A_{t,\lambda}=[A_{\rm t}]_{ij}-\lambda\ \forall i,j$ and $x_{{\rm te},\lambda}=x_{\rm te}-\lambda\mathbf{1}$, then

$$\begin{aligned}z_{\rm te}&=C_{11}\cdot x_{{\rm te},\lambda}+C_{12}\cdot x_{\rm qe}\\ z_{\rm qe}&=C_{21}\cdot x_{{\rm te},\lambda}+C_{22}\cdot x_{\rm qe}\\ y_{\rm te}&=B_{\rm t}\otimes' z_{\rm te},\quad y_{\rm qe}=B_{\rm q}\otimes' z_{\rm qe}\\ x_{{\rm te},\lambda}&=A_{{\rm t},\lambda}\otimes y_{\rm te},\quad x_{\rm qe}=A_{\rm q}\otimes y_{\rm qe}\end{aligned}\tag{6}$$

Now define the transformation matrices $X_{\rm t}={\rm d}_\otimes(x_{{\rm te},\lambda})$, $X_{\rm q}={\rm d}_\otimes(x_{\rm qe})$, $Y_{\rm t}={\rm d}_\otimes(y_{\rm te})$, $Y_{\rm q}={\rm d}_\otimes(y_{\rm qe})$, $Y'_{\rm t}={\rm d}_{\otimes'}(y_{\rm te})$, $Y'_{\rm q}={\rm d}_{\otimes'}(y_{\rm qe})$, $Z'_{\rm t}={\rm d}_{\otimes'}(z_{\rm te})$ and $Z'_{\rm q}={\rm d}_{\otimes'}(z_{\rm qe})$. From Definition 3 inverses of these matrices are obtained by replacing the diagonal entries with their corresponding negative values. Then we have

$$\begin{aligned}X_{\rm t}^{-1}\otimes x_{{\rm te},\lambda}&=\mathbf{0}, & X_{\rm q}^{-1}\otimes x_{\rm qe}&=\mathbf{0}\\ Y_{\rm t}^{-1}\otimes y_{\rm te}&=\mathbf{0}, & Y_{\rm q}^{-1}\otimes y_{\rm qe}&=\mathbf{0}\\ (Y'_{\rm t})^{-1}\otimes' y_{\rm te}&=\mathbf{0}, & (Y'_{\rm q})^{-1}\otimes' y_{\rm qe}&=\mathbf{0}\\ (Z'_{\rm t})^{-1}\otimes' z_{\rm te}&=\mathbf{0}, & (Z'_{\rm q})^{-1}\otimes' z_{\rm qe}&=\mathbf{0}.\end{aligned}$$

By applying the transformation matrices to (6), we get

$$\begin{aligned}(Y'_{\rm t})^{-1}\otimes' y_{\rm te}&=(Y'_{\rm t})^{-1}\otimes' B_{\rm t}\otimes' Z'_{\rm t}\otimes'(Z'_{\rm t})^{-1}\otimes' z_{\rm te}\\ &=\underbrace{(Y'_{\rm t})^{-1}\otimes' B_{\rm t}\otimes' Z'_{\rm t}}_{\tilde{B}_{\rm t}}\otimes'\mathbf{0}\\ (Y'_{\rm q})^{-1}\otimes' y_{\rm qe}&=(Y'_{\rm q})^{-1}\otimes' B_{\rm q}\otimes' Z'_{\rm q}\otimes'(Z'_{\rm q})^{-1}\otimes' z_{\rm qe}\\ &=\underbrace{(Y'_{\rm q})^{-1}\otimes' B_{\rm q}\otimes' Z'_{\rm q}}_{\tilde{B}_{\rm q}}\otimes'\mathbf{0}\\ X_{\rm t}^{-1}\otimes x_{{\rm te},\lambda}&=X_{\rm t}^{-1}\otimes A_{{\rm t},\lambda}\otimes Y_{\rm t}\otimes Y_{\rm t}^{-1}\otimes y_{\rm te}\\ &=\underbrace{X_{\rm t}^{-1}\otimes A_{{\rm t},\lambda}\otimes Y_{\rm t}}_{\tilde{A}_{\rm t}}\otimes\mathbf{0}\\ X_{\rm q}^{-1}\otimes x_{\rm qe}&=X_{\rm q}^{-1}\otimes A_{\rm q}\otimes Y_{\rm q}\otimes Y_{\rm q}^{-1}\otimes y_{\rm qe}\\ &=\underbrace{X_{\rm q}^{-1}\otimes A_{\rm q}\otimes Y_{\rm q}}_{\tilde{A}_{\rm q}}\otimes\mathbf{0}\end{aligned}\tag{7}$$

Therefore,

$$\mathbf{0}=\begin{bmatrix}\tilde{B}_{\rm t}&\top\\ \top&\tilde{B}_{\rm q}\end{bmatrix}\otimes'\ \mathbf{0},\quad \mathbf{0}=\begin{bmatrix}\tilde{A}_{\rm t}&\varepsilon\\ \varepsilon&\tilde{A}_{\rm q}\end{bmatrix}\otimes\ \mathbf{0}.\tag{8}$$

Consider the normalized MMPS system,

$$\begin{bmatrix}\tilde{x}_{\rm t}(k)\\ \tilde{x}_{\rm q}(k)\end{bmatrix}=\overbrace{\begin{bmatrix}\tilde{A}_{\rm t}&\boldsymbol{\varepsilon}\\ \boldsymbol{\varepsilon}&\tilde{A}_{\rm q}\end{bmatrix}}^{\tilde{A}}\otimes\left(\overbrace{\begin{bmatrix}\tilde{B}_{\rm t}&\top\\ \top&\tilde{B}_{\rm q}\end{bmatrix}}^{\tilde{B}}\otimes'\left(\overbrace{\begin{bmatrix}C_{11}&C_{12}\\ C_{21}&C_{22}\end{bmatrix}}^{C}\cdot\begin{bmatrix}\tilde{x}_{\rm t}(k-1)\\ \tilde{x}_{\rm q}(k-1)\end{bmatrix}\right)\right)\tag{9}$$

This system has a growth rate $\tilde{\lambda}=0$ and equilibrium point $(\tilde{x}_{\rm te},\tilde{x}_{\rm qe},\tilde{y}_{\rm te},\tilde{y}_{\rm qe},\tilde{z}_{\rm te},\tilde{z}_{\rm qe})=(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$. Furthermore, there holds:

$$\begin{aligned}x_{\rm t}(k)&=\tilde{x}_{\rm t}(k)+(k\lambda)\mathbf{1}+x_{\rm te}, & x_{\rm q}(k)&=\tilde{x}_{\rm q}(k)+x_{\rm qe}\\ y_{\rm t}(k)&=\tilde{y}_{\rm t}(k)+(k\lambda)\mathbf{1}+y_{\rm te}, & y_{\rm q}(k)&=\tilde{y}_{\rm q}(k)+y_{\rm qe}\\ z_{\rm t}(k)&=\tilde{z}_{\rm t}(k)+(k\lambda)\mathbf{1}+z_{\rm te}, & z_{\rm q}(k)&=\tilde{z}_{\rm q}(k)+z_{\rm qe}\end{aligned}\tag{10}$$

From (8), it can be deduced that

$$\begin{aligned}&\max_{j\in\overline{m}_t}\ [\tilde{A}_{\rm t}]_{ij}=0\ \forall i\in\overline{n}_t,\ \max_{s\in\overline{m}_q}\ [\tilde{A}_{\rm q}]_{rs}=0\ \forall r\in\overline{n}_q\\ &\min_{l\in\overline{p}_t}\ [\tilde{B}_{\rm t}]_{jl}=0\ \forall j\in\overline{m}_t,\ \min_{t\in\overline{p}_q}\ [\tilde{B}_{\rm q}]_{st}=0\ \forall s\in\overline{m}_q\end{aligned}\tag{11}$$

From (9) and (11), we can infer that

$$[\tilde{A}]_{ij}\le 0\quad\forall i\in\overline{n}\quad\text{and}\quad[\tilde{B}]_{jk}\ge 0\quad\forall j\in\overline{m}$$

where $n=n_{\rm t}+n_{\rm q}$ and $m=m_{\rm t}+m_{\rm q}$. This shows that both $\tilde{A}$ and $\tilde{B}$ have a specific structure. Each row of $\tilde{A}$ has at least one zero element, and all the nonzero elements are less than zero. Similarly, each row of $\tilde{B}$ has at least one zero element and all the non-zero elements are greater than zero.

For a general time-invariant MMPS system, multiple temporal growth rates might exist (Markkassery et al., 2024).

Hence, we can normalize the system (2), with respect to each growth rate and the corresponding equilibrium point. Let there be S possible temporal growth rates denoted as λ_θ, $\theta \in \{1,\ldots,S\}$ and let $\tilde{A}_\theta, \tilde{B}_\theta$ be the corresponding normalized matrices of the form (9). Then we define a pair of footprint matrices $(G_{A_\theta}, G_{B_\theta})$ as follows:

$$G_{A_\theta} = \begin{bmatrix} G_{A_{t\theta}} & 0 \\ 0 & G_{A_{q\theta}} \end{bmatrix}, \qquad G_{B_\theta} = \begin{bmatrix} G_{B_{t\theta}} & 0 \\ 0 & G_{B_{q\theta}} \end{bmatrix} \quad (12)$$

$$[G_{A_{t\theta}}]_{ij} = \begin{cases} 1 \text{ if } [\tilde{A}_{t\theta}]_{ij} = 0 \\ 0 \text{ if } [\tilde{A}_{t\theta}]_{ij} < 0 \end{cases}, [G_{B_{t\theta}}]_{jl} = \begin{cases} 1 \text{ if } [\tilde{B}_{t\theta}]_{jl} = 0 \\ 0 \text{ if } [\tilde{B}_{t\theta}]_{jl} > 0 \end{cases}$$

$$[G_{A_{q\theta}}]_{rs} = \begin{cases} 1 \text{ if } [\tilde{A}_{q\theta}]_{rs} = 0 \\ 0 \text{ if } [\tilde{A}_{q\theta}]_{rs} < 0 \end{cases}, [G_{B_{q\theta}}]_{st} = \begin{cases} 1 \text{ if } [\tilde{B}_{q\theta}]_{st} = 0 \\ 0 \text{ if } [\tilde{B}_{q\theta}]_{st} > 0 \end{cases}$$

Here $\mathbb{0}$ denotes the zero matrix of appropriate size. Each pair of footprint matrix defines the location of zeros in a normalized system. They can be used to define a linear programming problem to find a possible growth rate and a corresponding equilibrium point, as discussed in 13. Therefore, the total number of linear programming problems is equal to the total number of possible footprint matrix pairs, which will be less than or equal to $m_t^{n_t} p_t^{m_t} m_q^{n_q} p_q^{m_q}$. Let the variables λ and $(x_t, x_q, y_t, y_q, z_t, z_q)$ denote the unknown growth rate and equilibrium point of MMPS system (2). From (7) and (11), we formulate a linear programming problem (LPP) (13) similar to (Markkassery et al., 2024) for each pair of $(G_{A_\theta}, G_{B_\theta})$.

$$\begin{aligned}
\min_{\substack{\lambda, x_t, x_q, y_t, \\ y_q, z_t, z_q}} \quad & \lambda \\
\text{s.t.} \quad & -\lambda - [x_t]_i + [y_t]_j \le -[A_t]_{ij} && \text{if } [G_{A_{t\theta}}]_{ij} = 0 \\
& \lambda + [x_t]_i - [y_t]_j = [A_t]_{ij} && \text{if } [G_{A_{t\theta}}]_{ij} = 1 \\
& -[x_q]_r + [y_q]_s \le -[A_q]_{rs} && \text{if } [G_{A_{q\theta}}]_{rs} = 0 \\
& [x_q]_r - [y_q]_s = [A_q]_{rs} && \text{if } [G_{A_{q\theta}}]_{rs} = 1 \\
& [y_t]_j - [z_t]_l \le [B_t]_{jl} && \text{if } [G_{B_{t\theta}}]_{jl} = 0 \\
& -[y_t]_j + [z_t]_l = [B_t]_{jl} && \text{if } [G_{B_{t\theta}}]_{jl} = 1 \\
& [y_q]_s - [z_q]_t \le [B_q]_{st} && \text{if } [G_{B_{q\theta}}]_{st} = 0 \\
& -[y_q]_s + [z_q]_t = [B_q]_{st} && \text{if } [G_{B_{q\theta}}]_{st} = 1 \\
& z_t = C_{11} \cdot x_t + C_{12} \cdot x_q \\
& z_q = C_{21} \cdot x_t + C_{22} \cdot x_q
\end{aligned} \quad (13)$$

Remark 1. The size of each LPP increases in a quadratic manner as the size of system matrices A, B, C increases. But, if any element in A_t, A_q is ε or B_t, B_q is $\top$, the corresponding constraints can be omitted from the LPP. This reduces the computational complexity.

4. INTERNAL STABILITY OF MMPS SYSTEM AT DIFFERENT GROWTH RATES

In this section, we study the local internal stability of a DES modeled as an MMPS system. A DES is stable when all the time states of the system grows at the same rate, i.e. the buffer of the system stays bounded (bounded buffer stability (Gupta et al., 2020)) and the quantity states does not grow with events, 'k'.

Definition 9 (Gupta et al. (2020)). An autonomous discrete-event system is max-plus bounded buffer stable if for every initial time state, $x_{t0} \in \mathbb{R}^n$, there exist a bound $M(x_0) \in \mathbb{R}$ such that the time states are bounded in Hilbert's projective norm: $||x_t(k)||_{\mathbb{P}} \le M(x_0)\ \forall k \in \mathbb{Z}^+$

From Section 3, we see that a time-invariant MMPS system with multiple growth rates can be represented using a set of normalized MMPS systems:

$$\tilde{x}_\theta(k) = \tilde{A}_\theta \otimes \left(\tilde{B}_\theta \otimes' (C \cdot \tilde{x}_\theta(k-1))\right) \quad (14)$$

for $\theta \in \{1,\ldots,S\}$, and where $\tilde{A}_\theta \in \mathbb{R}^{n\times m}$, $\tilde{B}_\theta \in \mathbb{R}^{m\times p}$, $C \in \mathbb{R}^{p\times n}$ have the same structure as $\tilde{A}, \tilde{B}, C$ in (9). Let Ω_θ be a region containing all the vectors $x \in \mathbb{R}^n$ such that

$$\tilde{A}_\theta \otimes \left(\tilde{B}_\theta \otimes' (C \cdot x)\right) = G_{A_\theta} \cdot G_{B_\theta} \cdot C \cdot x \quad (15)$$

This means that $y := \tilde{B}_\theta \otimes' (C \cdot x)$ selects the $i_{0,j}$-th component of the vector $C \cdot x$ where $i_{0,j}$ denotes the position of 0 in row j of $\tilde{B}_\theta$, and $\tilde{A}_\theta \otimes y$ selects the $l_{0,i}$-th component of the vector y where $l_{0,i}$ denotes the position of 0 in row, i, of $\tilde{A}_\theta$.

Proposition 2. Any normalized MMPS system can be reformulated as a linear system in conventional algebra for all $\tilde{x}_\theta(k) \in \Omega_\theta$, $k \in \mathbb{Z}^+$ as follows,

$$\tilde{x}_\theta(k) = D_\theta \cdot \tilde{x}_\theta(k-1), \quad D_\theta = G_{A_\theta} \cdot G_{B_\theta} \cdot C \quad (16)$$

Remark 2. Here we assume that the matrices G_{A_θ} and G_{B_θ} have exactly one '1' in each row. When there are multiple 1's in the same row, the equilibrium point is on the boundary of different regions. The analysis of this case is outside the scope of this paper.

Note that $G_{A_\theta} \cdot G_{B_\theta}$ is a selection matrix and that it has the same block diagonal structure as in (12). So, D_θ will preserve the properties of C (Proposition 1) and hence D_θ will be time-invariant. Then we have,

$$D_\theta = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \sum_j [D_{11}]_{ij} = 1, \sum_j [D_{21}]_{ij} = 0 \quad (17)$$

Therefore, the matrix D_θ will have at least an eigenvalue equal to one with eigenvector $v = [\mathbf{1}_{n_t}^\intercal\ \mathbf{0}_{n_q}^\intercal]^\intercal$, i.e. $D_\theta \cdot v = 1 \cdot v$.

The max-plus bounded buffer stability of (16) can be assessed using the stability criteria of discrete-time systems in the conventional algebra, as stated in Proposition 3.

Proposition 3. The linearized system (16) for $\theta \in \{1,\ldots,S\}$ (recall that S is the total number of growth rates that exist) is

- max-plus bounded buffer stable if the system matrix D_θ has eigenvalues less than or equal to one and all Jordan blocks corresponding to magnitude one are 1×1 (Hespanha, 2018)
- Unstable if it has at least one eigenvalue greater than one or at least one of the Jordan blocks corresponding to magnitude one are not of size, 1×1 (Hespanha, 2018)

Note that the states of a stable linearized system will not keep growing. But neither do they always converge back to the equilibrium point $\mathbf{0}$. This is because there always exists an eigenvalue equal to 1. However, the states never diverge from each other. As a result, Hilbert's projective norm of the time-state vector $||\tilde{x}_{t\theta}||_{\mathbb{P}}$ will always be bounded. From (10),

$$\begin{aligned}
||x_{t\theta}(k)||_{\mathbb{P}} &= ||\tilde{x}_{t\theta}(k) + x_{te\theta} + \lambda_\theta k \mathbf{1}||_{\mathbb{P}} \\
&= ||\tilde{x}_{t\theta}(k) + x_{te\theta}||_{\mathbb{P}} \le ||\tilde{x}_t(k)||_{\mathbb{P}} + ||x_{te\theta}||_{\mathbb{P}}
\end{aligned}$$

where $x_{te\theta}$ is the equilibrium point of the original system (2) when the growth rate is λ_θ. Hence, the MMPS system (2) is max-plus bounded buffer stable at the temporal growth rate, λ_θ in the region Ω_θ. Also, a stable linearized system guarantees that none of the states are growing, hence the quantity states, $\tilde{x}_{\mathrm{q}}(k)$ remain bounded as well. From 10, we can see that if $\tilde{x}_{\mathrm{q}}(k)$ is bounded, $x_{\mathrm{q}}(k)$ is also bounded.

5. INVARIANT REGIONS OF THE STABLE MMPS SYSTEM

The mapping between the systems (14) and (16) is valid for all $\tilde{x}_\theta \in \Omega_\theta$. Later in this section, we will find an invariant subset of Ω_θ, such that if the stable equivalent linear system (16) is initialized in this region it will stay there.

Proposition 4. The region, Ω_θ associated to (15) is a polyhedron given by the set of inequalities

$$U \cdot x \leq \tilde{b} \ \text{ and } \ L \cdot x \geq \tilde{a}$$

with, $U = \big((G_{B_u} \boxtimes \mathbf{1}_p) - (\mathbf{1}_m \boxtimes I_p)\big) \cdot C, \quad \tilde{b} = \mathrm{vec}(\tilde{B}_u)$

$$L = \big((G_{A_u} \boxtimes \mathbf{1}_m) - (\mathbf{1}_n \boxtimes I_m)\big) \cdot G_{B_\theta} \cdot C, \ \tilde{a} = \mathrm{vec}(\tilde{A}_u)$$

where $x \in \mathbb{R}^n$, $\boxtimes$ is the Kronecker product (see Definition 1) and $\mathrm{vec}(\cdot)$ is the vector constructed in the row major order (see Definition 2) of the matrix.

Proof. The condition (15) can be reformulated as

$$\tilde{B}_\theta \otimes' z = G_{B_\theta} \cdot z, \quad z = C \cdot x \tag{18}$$

$$\tilde{A}_\theta \otimes y = G_{A_\theta} \cdot y, \quad y = G_{B_\theta} \cdot z \tag{19}$$

for some $z \in \mathbb{R}^p$, $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$. From (18) we have,

$$\min_{k \in \overline{p}} ([\tilde{B}_\theta]_{jk} + z_k) = [G_{B_\theta} \cdot z]_j, \quad j \in \overline{m}$$

$$\text{So, } [\tilde{B}_\theta]_{jk} + \boldsymbol{e}_k \cdot z \geq [G_{B_\theta}]_j \cdot z, \quad \forall j, k$$

$$\text{and thus, } \ ([G_{B_\theta}]_j - \boldsymbol{e}_k) \cdot z \leq [\tilde{B}_\theta]_{jk}, \quad \forall j, k \tag{20}$$

Here, $\boldsymbol{e}_k$ is the standard basis vector (see section 2). Therefore, (20) can be written as

$$(G_{B_\theta} \boxtimes \mathbf{1}_p - \mathbf{1}_m \boxtimes I_p) \cdot C \cdot x \leq \mathrm{vec}(\tilde{B}_\theta)$$

Similarly, from (19), we have

$$\max_{j \in \overline{m}} ([\tilde{A}_\theta]_{ij} + y_j) = [G_{A_\theta} \cdot y]_i, \quad i \in \overline{n}$$

$$[\tilde{A}_\theta]_{ij} + \boldsymbol{e}_j \cdot y \leq [G_{A_\theta}]_i \cdot y, \quad \forall i, j$$

$$([G_{A_\theta}]_i - \boldsymbol{e}_j) \cdot y \geq [\tilde{A}_\theta]_{ij}, \quad \forall i, j \tag{21}$$

Equation (21) can be written as

$$(G_{A_\theta} \boxtimes \mathbf{1}_m - \mathbf{1}_n \boxtimes I_m) \cdot G_{B_\theta} \cdot C \cdot x \leq \mathrm{vec}(\tilde{A}_\theta)$$ □

Remark 3. The constraints with both time states x_{t} and quantity states x_{q}, i.e. constraints of the form $[U]_i \cdot [x_{\mathrm{t}}^\intercal \ \ x_{\mathrm{q}}^\intercal]^\intercal \leq [\tilde{b}]_i$ (and $[L]_i \cdot [x_{\mathrm{t}}^\intercal \ \ x_{\mathrm{q}}^\intercal]^\intercal \geq [\tilde{a}]_i$), with nonzero coefficients for x_{t} as well as x_{q}, can be eliminated as their upper bound in $\tilde{b}$ (lower bound in $\tilde{a}$) will be ε ($\top$) due to the block diagonal structure (9) of $\tilde{B}_\theta$ ($\tilde{A}_\theta$).

The linearization (16) is valid only when $\tilde{x}_\theta(k)$ lies in Ω_θ. However, in general it is not guaranteed that if $\tilde{x}_\theta(k)$ lies in Ω_θ, $\tilde{x}_\theta(k+1)$ also lies in Ω_θ. Hence, we seek an invariant subset of Ω_θ such that the states initialized in this subset will not leave the subset.

Let $v_1, v_2, \ldots, v_n$ be the eigenvectors of D_θ associated with eigenvalues $\mu_1, \mu_2, \ldots \mu_n$ of the linearized system (16). Using $T = [v_1 \ v_2 \ \ldots \ v_n]$ as the transformation matrix, the linearized system is rewritten with the system matrix in the Jordan normal form of D (Hespanha, 2018) as,

$$w(k+1) = \begin{bmatrix} w_1(k+1) \\ w_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix} \tag{22}$$

where $w(k) = T^{-1} x(k)$ and J is the Jordan block (Hespanha, 2018) related to the eigenvalues with magnitude less than 1.

Proposition 5. The region Ω_θ is unbounded in the direction of the eigenvector $v_1 = [\mathbf{1}_{n_{\mathrm{t}}}^\intercal \ \mathbf{0}_{n_{\mathrm{q}}}^\intercal]^\intercal$.

Proof. Sufficient conditions for the region Ω_θ to be unbounded in the direction of v_1 are $U \cdot v_1 = 0$ and $L \cdot v_1 = 0$. From the properties (Proposition 1) of C, we have $C \cdot v_1 = v_1$. So,

$$U \cdot v_1 = \underbrace{\big(G_{B_\theta} \boxtimes \mathbf{1}_p) - (\mathbf{1}_m \boxtimes I_p)\big)}_{\hat{U}} \cdot v_1$$

$$L \cdot v_1 = \underbrace{\big(G_{A_\theta} \boxtimes \mathbf{1}_m) - (\mathbf{1}_n \boxtimes I_m)\big) \cdot G_{B_\theta}}_{\hat{L}} \cdot v_1$$

From Remark 3, it is evident that the finite constraints generated by U and L involve either x_{t} or x_{q}. Without loss of generality, we consider a nonzero row i of $\hat{U}$ with finite $[\tilde{b}]_i$ as below.

$$[\hat{U}]_i = \boldsymbol{e}_j - \boldsymbol{e}_k, \quad j \neq k$$

and either $j, k \leq n_t$ or $j, k \geq n_t$. Therefore $[\hat{U}]_i \cdot v_1 = 0$. As every row of $\hat{U}$ has the same structure except for the values of j and k, $\hat{U} \cdot v_1 = 0$. The condition $\hat{L} \cdot v_1 = 0$ can be proved similarly. □

Now, we find a solution $P \geq 0$ for the Lyapunov inequality (Hespanha, 2018),

$$J^\intercal \cdot P \cdot J - P < 0.$$

Then the ellipsoid, $w_2^\intercal \cdot P \cdot w_2 \leq 1$ will be an invariant set for the system (22) (Hespanha, 2018). Let $T_2 = [v_2 \ \ldots \ v_n]$. Then, $T = [v_1 \ T_2]$ and the constraints $U \cdot T \cdot w \leq \tilde{b}$ and $L \cdot T \cdot w \geq \tilde{a}$ will reduce to $U \cdot T_2 \cdot w_2 \leq \tilde{b}$ and $L \cdot T_2 \cdot w_2 \geq \tilde{a}$. The maximal ellipsoid that fits inside the region Ω_θ will be an invariant set for the system (22). This can be found using the optimization problem (23) (Boyd and Vandenberghe, 2004, Section 8.4.2) [1].

$$\begin{aligned} \min_{P} \quad & \mathrm{logdet}\ P \\ \text{s.t.} \quad & J^\intercal \cdot P \cdot J - P < 0, \\ & [U \cdot T_2]_i \cdot P^{-1} \cdot [U \cdot T_2]_i^\intercal \leq \tilde{b}_i^2 \quad \forall i, \\ & [L \cdot T_2]_j \cdot P^{-1} \cdot [L \cdot T_2]_j^\intercal \geq \tilde{a}_j^2 \quad \forall j, \\ & P > 0, P = P^\intercal \end{aligned} \tag{23}$$

Note that optimization problem (23) can be reformulated as a linear matrix inequality (LMI) problem (Scherer and Weiland, 2000). Let P^* be the solution to the optimization problem (23). Then, the elliptical cylinder

$$\Omega_{\mathrm{i}u} : x^\intercal \cdot P_{\mathrm{ell}} \cdot x \leq 1, \quad P_{\mathrm{ell}} = (T^{-1})^\intercal \cdot \begin{bmatrix} 0 & 0 \\ 0 & P^* \end{bmatrix} \cdot T^{-1}$$

with its axis along the eigenvector v_1 is an invariant set for the linearized system (16). Hence, the linearized system initialized in this set will stay in the set.

[1] The variable P used in problem (23) is equivalent to $(B \cdot B^\intercal)^{-1}$ from the problem formulation of Boyd and Vandenberghe (2004, Section 8.4.2).

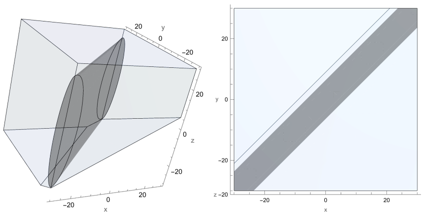

Fig. 1. The polyhedral region Ω_θ ("light gray") and the invariant elliptical cylinder $\Omega_{i\theta}$ ("dark gray") and their orthographic projection on to the $x-y$ plane

6. EXAMPLE

Consider the time-invariant MMPS system of the form (2) with matrices,

$$A=\begin{bmatrix}9&5&\varepsilon\\2&6&\varepsilon\\\varepsilon&\varepsilon&10\end{bmatrix},\ B=\begin{bmatrix}8&3&\top\\5&8&\top\\\top&\top&9\end{bmatrix},\ C=\begin{bmatrix}-0.75&1.75&4\\-0.75&1.75&0.2458\\-4&4&0.1\end{bmatrix}$$

This system has two time states and a quantity state. The C matrix is in accordance with Proposition 1. This system has two growth rates according to the LPP (13):

$$\lambda_1=19.0805,\quad \lambda_2=13.2289 \tag{24}$$

The normalized system matrices, $(\tilde{A}_\theta,\tilde{B}_\theta)$ associated with the growth rates λ_θ, $\theta\in\{1,2\}$ are:

$$\tilde{A}_1=\begin{bmatrix}-1&0&\varepsilon\\0&-9&\varepsilon\\\varepsilon&\varepsilon&0\end{bmatrix},\quad \tilde{B}_1=\begin{bmatrix}102.8918&0&\top\\94.8918&0&\top\\\top&\top&0\end{bmatrix}$$

$$\tilde{A}_2=\begin{bmatrix}0&-5.4374&\varepsilon\\-2.5626&0&\varepsilon\\\varepsilon&\varepsilon&0\end{bmatrix},\quad \tilde{B}_2=\begin{bmatrix}1.5626&0&\top\\0&6.4374&\top\\\top&\top&0\end{bmatrix}$$

The footprint matrix pair $(G_{A_\theta},G_{B_\theta})$ for these normalized systems has the same structure as the normalized matrix pair $(\tilde{A}_\theta,\tilde{B}_\theta)$ and can be constructed using (12). Then the linearized system (16) matrices D_θ and the associated eigenvalues μ_θ are

$$D_1=\begin{bmatrix}1.2&-0.2&0.2458\\1.2&-0.2&0.2458\\-4&4&0\end{bmatrix},\quad D_2=\begin{bmatrix}1.2&-0.2&0.2458\\-0.75&1.75&4\\-4&4&0.1\end{bmatrix}$$

$$\mu_1\in\{0,0.1,1\},\qquad \mu_2\in\{5.0090,1,-2.9590\}$$

It can be observed that the linear system with matrix D_2 is unstable as some of its eigenvalues have magnitude greater than one (Proposition 3). Now the stable linearized system $x(k+1)=D_1\cdot x(k)$ is transformed to the Jordan normal form to find the maximal invariant ellipsoid. By solving the optimization problem (23), we get

$$P^*=\begin{bmatrix}0.0013&0.0013\\0.0013&0.0014\end{bmatrix},\ P_{\text{ell}}=\begin{bmatrix}0.0253&-0.0253&-0.0011\\-0.0253&0.0253&0.0011\\-0.0011&0.0011&0.0016\end{bmatrix}$$

The polyhedral region Ω_θ is the half plane $1.95x_1-1.95x_2-3.7542x_3\leq 94.8918$ and the invariant region Ω_{iu} is the elliptical cylinder $x^\intercal\cdot P_{\text{ell}}\cdot x\leq 1$. Figure 1 shows the regions Ω_θ and Ω_{iu} with states x_1,x_2,x_3 plotted on the x,y,z axes, respectively, for the linearized system with system matrix D_1. It can be observed that the invariant set Ω_{iu} is an elliptical cylinder with its axis along the eigenvector $[1\ 1\ 0]^\intercal$.

7. CONCLUSIONS

This paper has proposed a novel framework to examine the stability of a general time-invariant MMPS system. A set of linear programming problems has been developed to find all the temporal growth rates and equilibrium points of a general MMPS system with both time and quantity states. Further, we have related the stability of a time-invariant MMPS system to a linear discrete-time system through normalization. This method establishes the stability analysis of any discrete-event system modeled as an MMPS system.

In our future work, we plan to study the region of attraction of the invariant set and formulate Lyapunov stability criterion for MMPS systems.

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