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Technical report 24-024

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If you want to cite this report, please use the following reference instead:
C. Liu, S. Shi, and B. De Schutter, "Stability and performance analysis of model predictive control of uncertain linear systems," *Proceedings of the 63rd IEEE Conference on Decision and Control*, Milan, Italy, pp. 7356–7362, Dec. 2024. doi:10.1109/CDC56724.2024.10885798

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Stability and Performance Analysis of Model Predictive Control of Uncertain Linear Systems

Changrui Liu*, Shengling Shi† and Bart De Schutter*, *Fellow IEEE*

***Abstract*— Model mismatch often presents significant challenges in model-based controller design. This paper investigates model predictive control (MPC) for uncertain linear systems with input constraints, where the uncertainty is characterized by a parametric mismatch between the true system and its estimated model. The main contributions of this work are twofold. First, a theoretical performance bound is derived using relaxed dynamic programming. This bound provides a novel insight into how the prediction horizon and modeling errors affect the suboptimality of the MPC controller to the oracle infinite-horizon optimal controller, which has complete knowledge of the true system. Second, sufficient conditions are established under which the nominal MPC controller, which relies solely on the estimated system model, can stabilize the true system despite model mismatch. Numerical simulations are presented to validate these theoretical results, demonstrating the practical applicability of the derived conditions and bounds. These findings offer practical guidelines for achieving desired modeling accuracy and selecting an appropriate prediction horizon in designing certainty-equivalence MPC controllers for uncertain linear systems.**

***Index Terms*— Performance guarantees, Predictive control for linear systems, Optimal control, Uncertain systems**

I. INTRODUCTION

Model predictive control (MPC) is an optimization-based control strategy that computes inputs to optimize a specific performance metric over a given prediction horizon based on a system model. MPC has found widespread application in various fields, such as chemical processes [1], aerospace vehicles [2], and portfolio optimization [3]. Regardless of the application, the effectiveness of MPC heavily depends on the accuracy of the prediction model. However, obtaining a perfect model is impossible due to inherent modeling errors in practice.

To address the issues that arise from having an imperfect model, adaptive MPC is commonly employed, integrating MPC with a system identification module [4]. Typical approaches use the comparison error [5], neural networks [6], and Bayesian inference [7]. On the other hand, data-driven MPC [8] has also emerged as a promising method for handling model uncertainty by directly using input-output data. However, while much of the literature focuses on the stability, feasibility, and robustness of MPC, studies that provide performance analysis are limited.

Relaxed dynamic programming (RDP) is a notable framework for analyzing the performance of MPC controllers compared to that of the idealized infinite-horizon optimal control problem [9]. RDP quantifies the suboptimality gap by analyzing a general value function that describes the energy-decreasing characteristic along the closed-loop system trajectory [10]. However, most of the existing results using RDP only hold for MPC with a perfect model [11], [12]. Considering model mismatch, recent extensions are rather limited. One work considered MPC for nonlinear systems with disturbances [13], but the results requires restrictive assumptions. The other investigated adaptive MPC for a class of parameterized linear systems, where the true matrices of the system lie in a known polytope [14]. However, performance analysis of MPC without model update for general uncertain linear systems remains unexplored. Given that nominal MPC without adaptation is still being extensively applied in practice and linear models are frequently adopted, it is imperative to investigate the stability and performance issues in this context.

Contributions: This work presents a novel analysis of the infinite-horizon performance of MPC for uncertain linear systems. In contrast to the previous result in [14], the problem considered in the current paper does not assume any specific parametric structure of the system nor adapt the system model online, adhering to the classical framework of MPC with a prediction model that has been identified offline. Compared to [15], the results presented in this paper constitute the first work in this direction that handles input constraints, filling a fundamental theoretical gap in performance analysis of MPC. Using the RDP method, a theoretical performance bound is established to illustrate the joint impact of model mismatch and prediction horizon on the closed-loop performance of the MPC controller. Moreover, our performance bound is a *consistent* extension of the bounds derived in [10], [16], allowing it to recover the existing bounds without model mismatch. Furthermore, we provide a sufficient condition on the prediction horizon and the level of model mismatch such that the closed-loop system is stable.

II. PRELIMINARIES

A. Notation

Let x be a vector; then its transpose is denoted by $x^\top$, and its vector i-norm by $\|x\|_i$. For a matrix M, $M^\top$, $\|M\|_2$, $\rho(M)$ and $\|M\|_\mathrm{F}$ denote its transpose, matrix 2-norm (i.e., spectral norm), spectral radius and Frobenius norm, respectively. Moreover, $M \geq 0$ $(x \geq 0)$ indicates element-wise nonnegativity, while $M \succ (\succeq)\ 0$ indicates positive

This paper is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 101018826 - CLariNet).

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(semi)definiteness. For a *symmetric* matrix $M \succ 0$, its largest and smallest eigenvalues are, respectively, denoted by $\overline{\sigma}_M$ and $\underline{\sigma}_M$, and we further define $r_M := \frac{\overline{\sigma}_M}{\underline{\sigma}_M}$. For a vector x and a positive symmetric (semi)definite matrix M, $\|x\|_M$ stands for $(x^\top M x)^{1/2}$.

The set of natural numbers is denoted by $\mathbb{N}$, the set of positive integers up to n is denoted by $\mathbb{Z}_n^+$ with $\mathbb{Z}_\infty^+$ representing the set of all positive integers, and the set of real and non-negative real numbers are denoted, respectively, by $\mathbb{R}$ and $\mathbb{R}_+$. The *bold* letter $\mathbf{x}$ is used to represent concatenation of a sequence of vectors $\{x_i\}$ as $\mathbf{x} = [x_0^\top, x_1^\top, \ldots]^\top$, and $\mathbf{x}[i] := x_i$. For any two vectors (matrices) x and y, $x \otimes y$ stands for their Kronecker product. Moreover, $\mathbf{0}_n$, $\mathbb{1}_n$, and I_n are the zero vector, one vector, and identity matrix of dimension n, respectively. Finally, $\lceil\cdot\rceil$ is the standard ceiling operator (i.e., the least integer operator).

B. System Description & Definitions

Consider discrete-time linear time-invariant (LTI) systems given by

$$x(t+1) = Ax(t) + Bu(t), \quad t \in \mathbb{N}, \tag{1}$$

where $x(t) \in \mathcal{X} = \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ is the input with $\mathcal{U}$ being the input constraint set, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the matrices of the *true* system. $x(t)$ ($u(t)$) denotes the true state (input) at time step t, whereas x_t (u_t) represents the predicted state (input) and/or decision variables in optimization problems. The set $\mathcal{U}$ is described using a set of linear constraints as

$$\mathcal{U} = \{u \in \mathbb{R}^m \mid F_u u \leq \mathbb{1}_{c_u}\}, \tag{2}$$

where $F_u \in \mathbb{R}^{c_u \times m}$ with c_u the number of input constraints. Note that $\mathbf{0}_m \in \mathcal{U}$, and hence $\mathcal{U}$ is nonempty. For the model given in as (1), the *open-loop* predicted state, starting from any state $x \in \mathcal{X}$ and being predicted k steps forward under the control sequence $\mathbf{u} = [u_0^\top, u_1^\top, \ldots, u_{k-1}^\top]^\top$, is denoted as $\psi_x(k, x, \mathbf{u})$. For linear systems characterized by (A, B), it is commonly known that

$$\psi_x(k, x, \mathbf{u}) = A^k x + \sum_{i=0}^{k-1} A^{k-1-i} B\mathbf{u}[i]. \tag{3}$$

On the other hand, given a state-feedback control law $\mu : \mathcal{X} \to \mathcal{U}$, the corresponding *closed-loop* predicted state, starting from any state $x \in \mathcal{X}$ and being predicted k steps forward, is denoted as $\phi_x^{[\mu]}(k, x)$. For the system described in (1), the controller only has access to an estimated system governed by the matrices $\hat{A} \in \mathcal{A}(A, \delta_A) \subseteq \mathbb{R}^{n\times n}$ and $\hat{B} \in \mathcal{B}(B, \delta_B) \subseteq \mathbb{R}^{n\times m}$, where the uncertainty sets $\mathcal{A}(A, \delta_A)$ and $\mathcal{B}(B, \delta_B)$ are defined as

$$\mathcal{A}(A, \delta_A) = \{M \in \mathbb{R}^{n\times n} \mid \|A - M\|_\mathrm{F} \leq \delta_A\}, \tag{4a}$$

$$\mathcal{B}(B, \delta_B) = \{M \in \mathbb{R}^{n\times m} \mid \|B - M\|_\mathrm{F} \leq \delta_B\}, \tag{4b}$$

where the parameters $\delta_A \geq 0$ and $\delta_B \geq 0$ are characterized using system identification or machine learning techniques before initiating the control task. For the open-loop and closed-loop predicted state of the estimated system, we use the notation $\widehat{\psi}_x(k, x, \mathbf{u})$ and $\widehat{\phi}_x^{[\mu]}(k, x)$, respectively. In addition, the following standard assumption [14], [17] for linear systems is imposed.

Assumption 1: The pairs (A, B) and $(\widehat{A}, \widehat{B})$ are both stabilizable.

Finally, for convenience in the analysis in Section IV, the following definition is provided.

Definition 1 (Error-consistent function): We call a function $\alpha(\cdot,\cdot) : \mathbb{R}_+^2 \to \mathbb{R}_+$ to be error-consistent if the following conditions hold:

1) $\alpha(\delta_1, \cdot)$ is non-decreasing for any $\delta_1 \in \mathbb{R}^+$,
2) $\alpha(\cdot, \delta_2)$ is non-decreasing for any $\delta_2 \in \mathbb{R}^+$,
3) $\alpha(\delta_1, \delta_2) = 0$ if and only if $\delta_1 = \delta_2 = 0$.

III. Problem Formulation

Consider an infinite-horizon optimal control problem, in which the controller aims to generate an input sequence $\mathbf{u}$ that stabilizes the system (i.e., steers the state to the origin) while minimizing the performance metric

$$J_\infty(x, \mathbf{u}_\infty) := \sum_{t=0}^{\infty} l(x_t, u_t), \tag{5}$$

where $x_0 = x \in \mathcal{X}$ and $\mathbf{u}_\infty = [u_0^\top, u_1^\top, \ldots, u_\infty^\top]^\top$. This work focuses on a quadratic stage cost given by $l(x, u) = \|x\|_Q^2 + \|u\|_R^2$, where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric matrices satisfying $Q, R \succ 0$, and $l^*(x) = \min_{u\in\mathcal{U}} l(x, u) = \|x\|_Q^2$. In addition, for any linear control law $u = \kappa(x) = Kx$, the local region Ω_K is defined as $\Omega_K = \{x \in \mathcal{X} \mid l^*(x) \leq \varepsilon_K\}$ with $\varepsilon_K > 0$ such that $Kx \in \mathcal{U}$ for all $x \in \Omega_K$. The maximum ϵ_K can be derived analytically, and the explicit form of which is given as

$$\varepsilon_K = \min_i \frac{1}{\|[F_u K]_{(i,:)}^\top\|_{Q^{-1}}^2}, \tag{6}$$

where $[F_u K]_{(i,:)}$ denotes the i-th row of the matrix $F_u K$. Given the dynamics in (1) and the input constraint set in (2), the optimization problem is given as

$$\begin{aligned} \mathrm{P}_{\mathrm{IH-OCP}} : \min_{\{u_t\}_{t=0}^\infty} & \sum_{t=0}^{\infty} \|x_t\|_Q^2 + \|u_t\|_R^2 \\ \text{s.t. } & x_{t+1} = Ax_t + Bu_t, \forall t \in \mathbb{N}, \\ & u_t \in \mathcal{U}, \forall t \in \mathbb{N}, \\ & x_0 = x(0). \end{aligned}$$

The optimal solution to the problem $\mathrm{P}_{\mathrm{IH-OCP}}$ is denoted by $\mathbf{u}_\infty^*(x(0))$; the associated optimal value of the cost function is $V_\infty(x(0)) := J_\infty(x(0), \mathbf{u}_\infty^*(x(0)))$.

Assumption 2: The initial state $x(0)$ lies in the region of attraction $\mathcal{X}_{\mathrm{ROA}}$ of the considered system in (1) such that, given the input constraints in (2), $V_\infty(x(0))$ is *finite* for all $x(0) \in \mathcal{X}_{\mathrm{ROA}}$.

Corollary 1: Under Assumption 2, $\mathcal{X}_{\mathrm{ROA}}$ is a control invariant set, i.e., for all $x \in \mathcal{X}_{\mathrm{ROA}}$, there exists $u \in \mathcal{U}$ such that $Ax + Bu \in \mathcal{X}_{\mathrm{ROA}}$.

However, due to the infinite nature of $\mathrm{P}_{\mathrm{IH-OCP}}$, computing the optimal input is intractable. Therefore, at each time step t, an truncated performance metric follows as

$$J_N(x(t), \mathbf{u}_N) = \sum_{k=0}^{N} \|x_{k|t}\|_Q^2 + \|u_{k|t}\|_R^2, \tag{7}$$

where $N \geq 1$ is the prediction horizon, $x_{k|t}$ and $u_{k|t}$ are, respectively, the k-step forward predicted state and input with $x_{0|t} = x(t)$ being the true state at time step t, and $\mathbf{u}_N = \{u_{0|t}^\top, u_{1|t}^\top, \ldots, u_{N-1|t}^\top\}^\top$. In this work, we consider the formulation without a terminal cost, as also done in [16], [18]. At each time step t, the *ideal* MPC controller that has access to the true system solves the following optimization problem:

$$\begin{aligned}
\mathrm{P}_{\mathrm{ID-MPC}}: \min_{\{u_{k|t}\}_{k=0}^{N}} & \sum_{k=0}^{N} \left(\|x_{k|t}\|_Q^2 + \|u_{k|t}\|_R^2\right) \\
\text{s.t. } & x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \forall k \in \mathbb{Z}_{N-1}^+, \\
& u_{k|t} \in \mathcal{U}, \forall k \in \mathbb{Z}_N^+, \\
& x_{0|t} = x(t).
\end{aligned}$$

By solving the problem $\mathrm{P}_{\mathrm{ID-MPC}}$, the optimal value of $\mathbf{u}_N$ is denoted by $\mathbf{u}_N^*(x(t))$, and the corresponding value of the cost function is $V_N(x(t)) := J_N(x(t), \mathbf{u}_N^*(x(t)))$. Moreover, the problem $\mathrm{P}_{\mathrm{ID-MPC}}$ implicitly defines an ideal MPC control law as $\mu_N(x(t)) := \mathbf{u}_N^*(x(t))[0]$. In practice, however, a nominal MPC controller can only rely on the *estimated* system, and it instead solves the following optimization problem:

$$\begin{aligned}
\mathrm{P}_{\mathrm{NM-MPC}}: \min_{\{u_{k|t}\}_{k=0}^{N}} & \sum_{k=0}^{N} \left(\|x_{k|t}\|_Q^2 + \|u_{k|t}\|_R^2\right) \\
\text{s.t. } & x_{k+1|t} = \widehat{A}x_{k|t} + \widehat{B}u_{k|t}, \forall k \in \mathbb{Z}_{N}^+, \\
& u_{k|t} \in \mathcal{U}, \forall k \in \mathbb{Z}_N^+, \\
& x_{0|t} = x(t).
\end{aligned}$$

Likewise, the optimal solution to the problem $\mathrm{P}_{\mathrm{NM-MPC}}$ is denoted by $\widehat{\mathbf{u}}_N^*(x(t))$, the value of the cost function follows as $\widehat{V}_N(x(t)) := J_N(x(t), \widehat{\mathbf{u}}_N^*(x(t)))$, and the real MPC control law is defined as $\hat{\mu}_N(x(t)) := \widehat{\mathbf{u}}_N^*(x(t))[0]$. Starting from $x(0) = x$, the nominal MPC controller is applied recursively, and the resulting performance is

$$J_\infty^{[\widehat{\mu}_N]}(x) = \sum_{t=0}^{\infty} l\left(\phi_x^{[\widehat{\mu}_N]}(t,x), \widehat{\mu}_N\left(\phi_x^{[\widehat{\mu}_N]}(t,x)\right)\right) \tag{8}$$

The main objectives of this paper are twofold: (i) to investigate under which conditions the MPC control law $\widehat{\mu}_N$ can stabilize the system and (ii) to quantify the closed-loop performance of the real MPC controller $J_\infty^{[\widehat{\mu}_N]}(x)$ relative to the optimal infinite-horizon cost $V_\infty(x)$ (i.e., the optimal value function V_∞ evaluated at $x(0) = x$).

Remark 1 (Nullified input at stage N): For both the optimization problems $\mathrm{P}_{\mathrm{ID-MPC}}$ and $\mathrm{P}_{\mathrm{RE-MPC}}$, the optimal input satisfies $u_{N|t}^* = 0$ due to the positive definiteness of the quadratic cost. $u_{N|t}$ is included in the formulation to be consistent with the one without a terminal cost.

Remark 2 (State constraints & region of attraction): In this work, we do not consider an explicit state constraint set $\mathcal{X}$. However, in the presence of hard input constraints, stabilizing the system may not be possible for any arbitrary initial state $x(0) = x$, especially for unstable systems with $\rho(A) \geq 1$. Therefore, Assumption 2 is imposed for the validity of our work. Similar assumptions have been made in [19] and [11] using the notion of cost controllability. Characterizing $\mathcal{X}_{\mathrm{ROA}}$ without knowing the true system is an open issue and is out of the scope of this paper.

IV. Theoretical Analysis

In this section, we provide an analysis of the stability and the closed-loop performance of the MPC controller with model mismatch. In Section IV-A, a relation between the MPC value function $\widehat{V}_N$ and the infinite-horizon optimal value function V_∞ is established, leveraging the sensitivity analysis of quadratic programs (QPs). In Section IV-B, a performance bound is derived theoretically using the RDP inequality.

A. Evaluation of the MPC value function

An upper bound of $\widehat{V}_N$ in terms of V_∞ is established first. Note that these two value functions are constructed using different dynamic models. We first state the main result and highlight its interpretations.

Proposition 1: There exist two error-consistent functions $\alpha_N(\delta_A, \delta_B)$ and $\beta_N(\delta_A, \delta_B)$ such that, for all $x \in \mathcal{X}_{\mathrm{ROA}}$, $\widehat{V}_N$ and V_∞ satisfy the following inequality[1]:

$$\widehat{V}_N(x) \leq \left(1 + \alpha_N(\delta_A, \delta_B)\right)V_\infty(x) + \beta_N(\delta_A, \delta_B), \tag{9}$$

where functions α_N and β_N relate to the eigenvalues and matrix norms of $\widehat{A}$, $\widehat{B}$, Q, R, the input constraint set $\mathcal{U}$, but not to quantities derived from A or B. Explicit expressions of α_N and β_N are given in Appendix A.

The upper bound in (9) is consistent with the bound without model mismatch: $V_N(x) \leq V_\infty(x)$. In fact, if $\delta_A = \delta_B = 0$, $\widehat{V}_N$ is identical to V_N and (9) degenerates to $V_N(x) \leq V_\infty(x)$ since both α_N and β_N are error-consistent. Furthermore, α_N and β_N are *computable* because they do not depend on the matrix pair (A, B) of the true system. The proof of Proposition 1 can be found in [20]. The main challenge lies in expressing $\widehat{V}_N(x)$ in terms of $\mathbf{u}_N^*(x)$ and $\psi_x(\cdot, x, \mathbf{u}_N^*(x))$, which can lead to a relationship between $\widehat{V}_N(x)$ and $V_N(x)$. Consequently, by using the fact that $V_N(x) \leq V_\infty(x)$, we can arrive at the desired inequality as in (9).

B. Closed-loop Stability and Performance Analysis

Under the MPC control law $\widehat{\mu}_N$, at a given state x, the next-step state is computed as $x_{\mathrm{re}}^+ = Ax + B\widehat{\mu}_N(x)$. If there

[1]The explicit dependence of α_N and β_N on the prediction horizon is highlighted using the subscript N.

exists a so-called energy function $\tilde{V}: \mathbb{R}^n \to \mathbb{R}_+$ such that the RDP inequality

$$\tilde{V}(x_{\text{re}}^+) - \tilde{V}(x) \leq -\epsilon l(x, \widehat{\mu}_N(x)) \tag{10}$$

holds for all $x \in \mathcal{X}_{\text{ROA}}$, where $\epsilon \in (0,1]$, then the controlled system is asymptotically stable [9], [12]. In this paper, the case where $\tilde{V} = \widehat{V}_N$ is investigated and the coefficient ϵ is derived as a function of the prediction horizon N and the parameters δ_A and δ_B that quantify the model mismatch.

We first present a preliminary result that can be applied to both the true system and the estimated system.

Lemma 1: Given a positive-definite quadratic stage cost $l(x,u) = \|x\|_Q^2 + \|u\|_R^2$ and a local linear stabilizing control law $u = \kappa(x) = Kx$ for the system pair $(\widehat{A}, \widehat{B})$, there exist scalars $\lambda_K \geq 1$ and $\rho_K \in (0,1)$ such that for all $x \in \Omega_K$ and $k \in \mathbb{N}$, it holds that

$$l(\widehat{\phi}_x^{[\kappa]}(k,x), K\widehat{\phi}_x^{[\kappa]}(k,x)) \leq C_K^*(\rho_K)^k l^*(x), \tag{11}$$

where $C_K^* = \big(1 + \underline{\sigma}_Q^{-1}\overline{\sigma}_R\|K\|_2^2\big) r_Q (\lambda_K)^2$.

Proof: The proof is closely related to the common results on exponential stability of linear systems, and it is given in the full version [20] for completeness. □

Based on Lemma 1, the following result about the properties of the open-loop system can be obtained.

Lemma 2: Given an upper bound $M_{\widehat{V}}$ of $\widehat{V}_N$ for all $x = x(0)$ and a stabilizing control law $u = Kx$ for the estimated system, there exist constants $L_{\widehat{V}} := \max\{\gamma, \frac{M_{\widehat{V}}}{\varepsilon_K}\}$ and $N_0 := \big\lceil \max\{0, \frac{M_{\widehat{V}} - \gamma\varepsilon_K}{\varepsilon_K}\}\big\rceil$ such that for $N \geq N_0$, it holds that

$$\widehat{V}_N(x) \leq L_{\widehat{V}} l^*(x) \leq L_{\widehat{V}} l(x, \widehat{\mu}_N(x)) \tag{12a}$$

$$\|\widehat{\psi}_x(N, x, \hat{\mathbf{u}}_N^*(x))\|_Q^2 \leq \gamma \rho_\gamma^{N-N_0} l(x, \widehat{\mu}_N(x)), \tag{12b}$$

where ε_K is defined as in (6) using the gain K, $\gamma = C_K^*(1-\rho_K)^{-1}$ with C_K^* and ρ_K given as in Lemma 1, and $\rho_\gamma = \gamma^{-1}(\gamma - 1)$.

Proof: The proof follows a procedure similar to that in [19, Theorem 5] and is based on the lemma 1. □

For a stabilizable system, (12a) and (12b) provides an upper bound, respectively, for the value function $\widehat{V}_N$ and the cost of the final state, both in terms of the stage cost. Moreover, (12b) implies that, given a sufficiently long horizon, the cost of the final state exponentially decays as the horizon increases. The *critical horizon* N_0 quantifies the required number of steps such that the final state lies in the local region Ω_K.

Proposition 2: There exist a constant η_N and an error-consistent function ξ_N satisfying $\xi_N(\delta_A, \delta_B) + \eta_N < 1$, and for all $x \in \mathcal{X}_{\text{ROA}}$ the following holds:

$$\widehat{V}_N(x_{\text{re}}^+) - \widehat{V}_N(x) \leq \\ -(1 - \xi_N(\delta_A, \delta_B) - \eta_N) l(x, \hat{\mu}_N(x)), \tag{13}$$

where the function ξ_N and constant η_N relate to the eigenvalues and matrix norms of $\widehat{A}$, $\widehat{B}$, Q, R, the input constraint set $\mathcal{U}$, but not to quantities derived from A or B. The explicit expressions of ξ_N and η_N are given, respectively, in (28) and (27) in Appendix B.

In Proposition 2, the relation in (13), serving as the RDP inequality for the closed-loop systems (cf. (10)), provides a lower bound of the energy decrease in terms of the stage cost. Moreover, based on $\xi_N(\delta_A, \delta_B) + \eta_N < 1$, a *sufficient* condition on the prediction horizon and modeling error can be derived such that the closed-loop system is stable. This condition is summarized in the following corollary:

Corollary 2: Given a sufficiently long prediction horizon N satisfying

$$N > N_0 + \frac{\log\left(\|\widehat{A}\|_2^2 r_Q \gamma\right)}{\log(\rho_\gamma^{-1})}, \tag{14}$$

where N_0, γ and ρ_γ are defined as in Lemma 2 using gain K, the closed-loop system is asymptotically stable if the modeling error is small enough such that

$$h(\delta_A, \delta_B) < \left\{ \frac{-\omega_{N,(\frac{1}{2})} + [\omega_{N,(\frac{1}{2})}^2 + \omega_{N,(1)}(1-\eta_N)]^{\frac{1}{2}}}{\omega_{N,(1)}} \right\}^2, \tag{15}$$

where h is defined in (29), η_N is given as in (27), and $\omega_{N,(1)}$ and $\omega_{N,(\frac{1}{2})}$ are given, respectively, as in (31a) and (31b).

Informally, Corollary 2 indicates that the model mismatch should be small enough to ensure the stability of the closed-loop system if the MPC control law is derived from the estimated model instead of the true model. The complete proof of Proposition 2 is given in the full version of this work [20], and the main challenge is to establish an upper bound for $\widehat{V}_N(x_{\text{re}}^+) - \widehat{V}_N(x)$ in terms of $l(x, \widehat{\mu}_N(x))$, where Lemma 1 and Lemma 2 will be used.

The final infinite-horizon performance guarantee is stated in the following theorem:

Theorem 1: Given α_N and β_N as in Proposition 1, and ξ_N and η_N as in Proposition 2, it holds that

$$J_\infty^{[\widehat{\mu}_N]}(x) \leq \frac{1 + \alpha_N}{1 - \xi_N - \eta_N} V_\infty(x) + \frac{\beta_N}{1 - \xi_N - \eta_N}, \tag{16}$$

where $J_\infty^{[\widehat{\mu}_N]}(x)$ is defined in (8).

Proof: The proof resembles that of [10, Prop. 2.2], and it is provided for completeness. By performing a telescopic sum of (13), for any $T \in \mathbb{N}$, the following holds:

$$(1 - \xi_N - \eta_N) \sum_{t=0}^{T-1} l\left(\phi_x^{[\widehat{\mu}_N]}(t,x), \widehat{\mu}_N\left(\phi_x^{[\widehat{\mu}_N]}(t,x)\right)\right) \\ \leq \widehat{V}_N(x) - \widehat{V}_N(\phi_x^{[\widehat{\mu}_N]}(T,x)). \tag{17}$$

Taking $T \to \infty$, using the performance definition in (8) and $0 \leq \widehat{V}_N(\phi_x^{[\widehat{\mu}_N]}(\infty, x))$, (17) yields

$$(1 - \xi_N - \eta_N) J_\infty^{[\widehat{\mu}_N]}(x) \leq \widehat{V}_N(x). \tag{18}$$

Then, substituting the bound in (9) into (18) leads to

$$(1 - \xi_N - \eta_N) J_\infty^{[\widehat{\mu}_N]}(x) \leq \alpha_N V_\infty(x) + \beta_N, \tag{19}$$

which gives the final bound in (16) after dividing both sides by the constant $1 - \xi_N - \eta_N$. □

Given that α_N, β_N, and ξ_N are all error-consistent, the worst-case performance bound as in (16) will increase if the modeling error becomes larger (cf. the non-decreasing property of the error-consistent functions in Def. 1). Besides, if the model is perfect, the final performance bound will degenerate to

$$J_{\infty}^{[\mu_N]}(x) \leq \frac{1}{1-\eta_N} V_{\infty}(x). \tag{20}$$

Note that (20) is a variant of the bounds given in [10], [19] without modeling error.

V. Numerical Example

Consider the true linear system of the form (1) specified by

$$A = \begin{bmatrix} 1 & 0.7 \\ 0.12 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix},$$

with the input constraint set given in (2) with $F_u = [10, -10]^\top$. It should be noted that this considered numerical example is challenging to analyze since it is unstable ($\rho(A) = 1.1171 > 1$), meaning that the allowed level of modeling error and the prediction horizon are restricted to ensure $\xi_N + \eta_N < 1$. Now its infinite-horizon performance is analyzed based on the theoretical results. For the quadratic objective, the matrices Q and R are given by

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = 1.$$

To simplify the simulation, the case $\delta_A = \delta_B = \delta$ is considered, and we investigate the behavior of the derived performance bound when δ varies between 10^{-3} and 10^{-2} for a fixed prediction horizon. For a given level of modeling error $\delta \in [10^{-3}, 10^{-2}]$, 100 different estimated systems have been simulated, all of which are generated randomly and satisfy (4). For each of the system, its corresponding feedback gain K as in Lemma 1 is designed using LQR with the tuple $(\widehat{A}, \widehat{B}, Q, R)$. The code used for the simulations in this paper is available on GitHub.[2] The behavior of α_N, β_N, ξ_N, and J_{bound} (the bound given in (16)) as a function of δ is shown in Fig. 1. The results in Fig. 1 indicate that an increased modeling error leads to a larger difference between the value functions $\widehat{V}_N$ and V_∞ (by having larger values of α_N and β_N), a mitigated energy-decreasing property (by having a larger value of ξ_N), and thus an increased worst-case performance bound.

On the other hand, given a specific value of δ, the behavior of the four quantities α_N, β_N, ξ_N, and J_{bound} when varying the prediction horizon has also been simulated, and the results are presented in Fig. 2. According to Fig. 2, as the prediction horizon is extended, the error accumulates, which is reflected in the increasing behavior of α_N, β_N, and ξ_N. However, the performance bound is not necessarily a monotonous function of the prediction horizon since η_N decreases as N increases. For $\delta = 5 \cdot 10^{-3}$, $N = 7$ is the prediction horizon that achieves the smallest worst-case performance bound. For this numerical example, the optimal cost $V_\infty = 0.20229$, and the true closed-loop MPC cost

[2]See `https://github.com/lcrekko/lq_mpc`

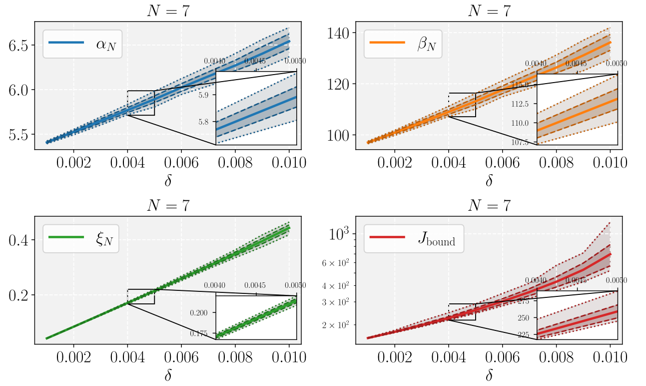

Fig. 1. The behavior of α_N (upper left), β_N (upper right), ξ_N (lower left), and J_{bound} (lower right) for a varying modeling error $\delta \in [10^{-3}, 10^{-2}]$. For a given error level δ, 100 estimated systems are simulated, and the mean, variance, and max (min) of each of the four quantities are shown, respectively, using a solid line, dashed lines, and dotted lines.

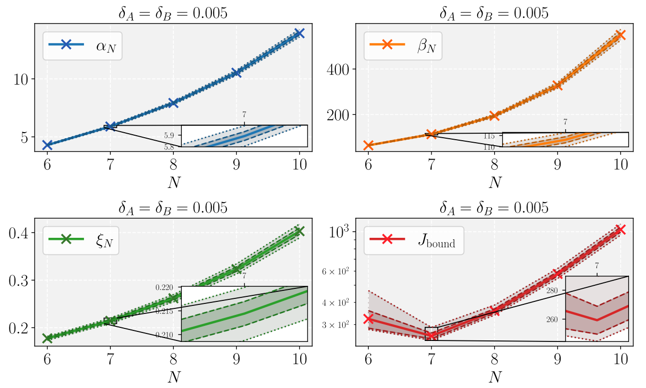

Fig. 2. The behavior of α_N (upper left), β_N (upper right), ξ_N (lower left), and J_{bound} (lower right) for a varying prediction horizon $N \in \{6, 7, 8, 9, 10\}$. For the specified $\delta = 5 \cdot 10^{-3}$, 100 estimated systems are simulated, and the mean, variance, and max (min) of each of the four quantities are shown, respectively, using a solid line, dashed lines, and dotted lines.

varies between from $0.20229 + 5 \cdot 10^{-5}$ to $0.20229 + 1.1 \cdot 10^{-4}$. In Fig. 3, the true cost of the certainty-equivalent MPC controller is shown, revealing that our performance bound is conservative. This conservatism primarily arises from repeatedly applying various inequalities, leading to the accumulation of relaxation errors. In general, RDP-based theoretical performance analysis suffers from conservatism (see also [11], [21]) since the RDP inequality only provides a lower bound on the decreased energy of the closed-loop system (cf. (10)). Furthermore, the optimal prediction horizon deduced using the worst-case performance bound is *not* necessarily the optimal prediction horizon N_{OPT} that achieves the best performance, which is reflected in the right subplot of Fig. 3 where it can be observed that the true performance does not vary significantly when the horizon changes. This discrepancy is due to that facts i) that N_{OPT} depends on the linear feedback gain K that is chosen subject to user preference and ii) that the derivation based on the energy-decreasing property in (10) is only sufficient but not

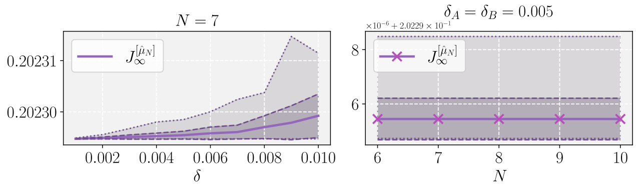

Fig. 3. The behavior of the true performance as a function of the modeling error (left) and prediction horizon (right). For a given error level δ, 100 estimated systems are simulated, and the mean, variance, and max (min) of each of the four quantities are shown, respectively, using a solid line, dashed lines, and dotted lines.

necessary. However, our theoretical analysis still highlights the tension between choosing a long horizon for optimality and a short horizon for a lower prediction error, and thus the conclusions can still provide design insights for choosing a proper prediction horizon of certainty-equivalence MPC.

VI. Conclusions

This paper has studied the stability and closed-loop performance of model predictive control (MPC) for uncertain linear systems. A significant contribution of this work is the derivation of a performance bound that quantifies the suboptimality gap between a certainty-equivalence MPC controller and the ideal infinite-horizon optimal controller that has full knowledge of the true system model. Additionally, a sufficient condition regarding the prediction horizon and the level of model mismatch has been established, which guarantees the stability of the closed-loop system. The analysis further reveals the interplay between the prediction horizon and model mismatch, offering crucial insights into their joint impact on the performance of MPC. These findings provide valuable guidance for the design and implementation of MPC controllers for uncertain linear systems, particularly in achieving an acceptable level of model identification error and selecting a suitable prediction horizon to ensure robust performance. Future research directions include extending the current analytical framework to incorporate MPC with terminal costs, developing alternative analysis tools and do comparisons, as well as investigating the performance of adaptive MPC and learning-based MPC approaches that dynamically learn the system model online.

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Appendix

A. More on Proposition 1

The detailed expressions of α_N and β_N are

$$\alpha_N = \max\Big\{p_1 E_{N,(\psi)}^{\frac{1}{2}} + p_3 E_{N,(\psi,u)}^{\frac{1}{2}} + p_1 p_3 (E_{N,(\psi)} E_{N,(\psi,u)})^{\frac{1}{2}},\ p_2 E_{N,(u)}^{\frac{1}{2}}\Big\} \tag{21a}$$

$$\beta_N = (1 + p_1 E_{N,(\psi)}^{\frac{1}{2}})(q_3 E_{N,(\psi,u)}^{\frac{1}{2}} + E_{N,(\psi,u)}) + q_2 E_{N,(u)}^{\frac{1}{2}} + E_{N,(u)} + q_1 E_{N,(\psi)}^{\frac{1}{2}} + E_{N,(\psi)}^{\frac{1}{2}}, \tag{21b}$$

where the pairs $(p_i, q_i) \in \mathbb{R}_+^2$ satisfy $p_i q_i = 1$ and the details of the terms $E_{N,(\psi)}$, $E_{N,(u)}$, and $E_{N,(\psi,u)}$ are given, respectively, in (24), (25), and (26) below. First, several preparatory definitions are presented to support those details. Given matrices A and B as in (1), define

$$\Phi_N{:=}\begin{bmatrix} I\\ A\\ A^2\\ \vdots\\ A^N\end{bmatrix}, \Gamma_N{:=}\begin{bmatrix} 0 & 0 & \cdots & 0\\ B & 0 & \cdots & 0\\ AB & B & \cdots & 0\\ \vdots & \vdots & & \vdots\\ A^{N-1}B & A^{N-2}B & \cdots & B\end{bmatrix},$$

and $\widehat{\Phi}_N$ and $\widehat{\Gamma}_N$ are defined similarly using $\widehat{A}$ and $\widehat{B}$. Further, $\overline{d}_u := \max_{u_1,u_2\in\mathcal{U}} \|u_1 - u_2\|_2^2$, $\overline{u} := \max_{u\in\mathcal{U}} \|u\|_2^2$, and error-consistent functions $g^{(n)}_{i,(x)}$ and $g^{(n)}_{i,(u)}$ are defined for all $n \in \mathbb{N}_+$ as

$$g^{(n)}_{i,(x)}(\delta_A,\cdot) = \left[(\delta_A + \|\widehat{A}\|_2)^i - (\|\widehat{A}\|_2)^i\right]^n, \tag{22a}$$

$$g^{(n)}_{i,(u)}(\delta_A,\delta_B){=}\Big\{(\delta_B + \|\widehat{B}\|_2)g^{(1)}_{i,(x)}(\delta_A,\cdot) + \delta_B(\|\widehat{A}\|_2)^i\Big\}^n. \tag{22b}$$

Lastly, define $\theta_{N,(u)}$ and $\theta_{N,(x,u)}$ as

$$\theta_{N,(u)}{:=}\overline{\sigma}_Q\left(2\|\widehat{\Gamma}_N\|_2\bar{g}_{(u)} + \bar{g}^2_{(u)}\right), \tag{23a}$$

$$\theta_{N,(x,u)}{:=}\overline{\sigma}_Q\left(\|\widehat{\Gamma}_N\|_2\bar{g}_{(x)} + \|\widehat{\Phi}_N\|_2\bar{g}_{(u)} + \bar{g}_{(x)}\bar{g}_{(u)}\right), \tag{23b}$$

where $\bar{g}_{(x)}$ and $\bar{g}_{(u)}$ are, respectively, defined as $\bar{g}_{(x)} := \sum_{i=1}^N g^{(1)}_{i,(x)}$ and $\bar{g}_{(u)} := \sum_{i=1}^N \sum_{j=0}^{i-1} g^{(1)}_{j,(u)}$ with $g^{(1)}_{i,(x)}$ and $g^{(1)}_{i,(u)}$ given as in (22) with $n = 1$.

(1) Details of $E_{N,(\psi)}$: The explicit expression of $E_{N,(\psi)}$ is

$$E_{N,(\psi)}{=}\overline{\sigma}_Q\Delta_{N,(\psi)}(x), \tag{24}$$

where $\Delta_{N,(\psi)}(x)$ is given by

$$\sum_{k=0}^{N}\left[\left(g^{(2)}_{k,(x)}{+}\sum_{i=0}^{k-1} g^{(2)}_{k-i-1,(u)}\right)\left(\|x\|_2^2{+}k\overline{u}\right)\right].$$

(2) Details of $E_{N,(u)}$ and $E_{N,(\psi,u)}$: The explicit form of $E_{N,(u)}$ is

$$E_{N,(u)} = \overline{\sigma}_R\left(\Delta_{N,(\delta\mathbf{u})}(x)\right)^2, \tag{25}$$

where $\Delta_{N,(\delta\mathbf{u})}(x)$ is given as

$$\min\left\{(N\overline{d}_u)^{\frac{1}{2}}, \frac{1}{\underline{\sigma}_{\widehat{H}_N}}\left((N\overline{u})^{\frac{1}{2}}\theta_{N,(u)}{+}\|x\|_2\theta_{N,(x,u)}\right)\right\},$$

in which $\theta_{N,(x)}$ and $\theta_{N,(x,u)}$ are defined as in (23), $\widehat{H}_N = \overline{R}_N{+}\widehat{\Gamma}_N^\top\overline{Q}_{N+1}\widehat{\Gamma}_N$ with $\overline{R}_N = I_N\otimes R$ and $\overline{Q}_{N+1} = I_{N+1}\otimes Q$. Next, the expression of $E_{N,(\psi,u)}$ follows as

$$E_{N,(\psi,u)}{=}\frac{\overline{\sigma}_Q}{\overline{\sigma}_R}\left(\|\widehat{\Gamma}_N\|_2 + \bar{g}_{(u)}\right)^2 E_{N,(u)}. \tag{26}$$

B. More on Proposition 2

The detailed expression of η_N is

$$\eta_N = \|\widehat{A}\|_2^2 r_Q\gamma\rho_\gamma^{N-N_0}, \tag{27}$$

where γ, ρ_γ and N_0 are defined in Lemma 2 using a linear feedback gain K. Finally, the detailed expression of ξ_N is

$$\xi_N = \omega_{N,(1)}h(\delta_A,\delta_B) + 2\omega_{N,(\frac{1}{2})}h^{\frac{1}{2}}(\delta_A,\delta_B), \tag{28}$$

where the function h is given by

$$h(\delta_A,\delta_B) = \underline{\sigma}_Q^{-1}\delta_A^2 + \underline{\sigma}_R^{-1}\delta_B^2, \tag{29}$$

and the terms $\omega_{N,(1)}$ and $\omega_{N,(\frac{1}{2})}$ are given, respectively, in (31a) and (31b) below.

Details of $\omega_{N,(1)}$ and $\omega_{N,(\frac{1}{2})}$: Define $G_N(\widehat{A}) = \sum_{i=1}^{N-1}\|\widehat{A}\|_2^{2(i-1)}$ as

$$G_N(\widehat{A}) = \begin{cases} N-1 & \text{if } \|\widehat{A}\|_2 = 1\\ \frac{1-(\|\widehat{A}\|_2^2)^{N-1}}{1-\|\widehat{A}\|_2^2} & \text{if } \|\widehat{A}\|_2 \neq 1,\end{cases} \tag{30}$$

and $\omega_{N,(1)}$ and $\omega_{N,(\frac{1}{2})}$ follows as

$$\omega_{N,(1)}{:=}\overline{\sigma}_Q\Big[(1{+}\|\widehat{A}\|_2^2 r_Q)(\|\widehat{A}\|_2^2)^{N-1}{+}G_N(\widehat{A})\Big], \tag{31a}$$

$$\omega_{N,(\frac{1}{2})}{:=}\Big[\overline{\sigma}_Q(L_{\hat{V}}-1)G_N(\widehat{A})\Big]^{\frac{1}{2}} + \frac{1{+}\|\widehat{A}\|_2^2 r_Q}{2}\Big[\overline{\sigma}_Q(\|\widehat{A}\|_2^2)^{N-1}\gamma\rho_\gamma^{N-N_0}\Big]^{\frac{1}{2}}, \tag{31b}$$

where $L_{\hat{V}}$ is given as in Lemma 2 and $G_N(\widehat{A})$ is given as in (30).