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The Singular Value Decomposition in the Extended Max Algebra*

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ABSTRACT

First we establish a connection between the field of the real numbers and the extended max algebra, based on asymptotic equivalences. Next we propose a further extension of the extended max algebra that will correspond to the field of the complex numbers. Finally we use the analogy between the field of the real numbers and the extended max algebra to define the singular value decomposition of a matrix in the extended max algebra and to prove its existence.

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1. INTRODUCTION

1.1. Overview

One of the possible frameworks to describe and analyze discrete event systems (such as flexible manufacturing processes, railroad traffic networks, telecommunication networks, ...) is the max algebra [1, 3, 4]. A class of discrete event systems, the timed event graphs, can be described by a state

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space model that is linear in the max algebra. There exists a remarkable analogy between max-algebraic system theory and system theory for linear systems. However, in contrast to linear system theory the mathematical foundations of the max-algebraic system theory are not as fully developed as those of the classical linear system theory, although some of the properties and concepts of linear algebra, such as Cramer's rule, the Cayley-Hamilton theorem, eigenvalues, eigenvectors, ... also have a max-algebraic equivalent. In [14] Olsder and Roos have used a kind of link between the field of the real numbers and the max algebra based on asymptotic equivalences to show that every matrix has at least one max-algebraic eigenvalue and to prove a max-algebraic version of Cramer's rule and of the Cayley-Hamilton theorem. We shall extend this link and use it to define the singular value decomposition in the extended max algebra [9, 13], which is a kind of symmetrization of the max algebra. We also propose a further extension of the max algebra that will correspond to the field of the complex numbers.

In Section 1 we explain the notations we use in this paper and give some definitions and properties. We also include a short introduction to the max algebra and the extended max algebra. In Section 2 we establish a link between the field of the real numbers and the extended max algebra and we introduce the max-complex numbers, which yields a further extension of the max algebra. In Section 3 we use the correspondence between the field of the real numbers and the extended max algebra to define the singular value decomposition (SVD) in the extended max algebra and to prove its existence. We conclude with a possible application of the max-algebraic SVD and an example.

1.2. Notations and definitions

We use f or $f(\cdot)$ to represent a function. The value of f at x is denoted by $f(x)$. The set of all reals except for 0 is represented by $\mathbb{R}_0$ ($\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$). The set of all nonnegative real numbers is denoted by $\mathbb{R}^+$.

In this paper we use "vector" as a synonym for "n-tuple". Furthermore, all vectors are assumed to be column vectors. If a is a vector, then a_i is the ith component of a. If A is a matrix, then a_{ij} or $(A)_{ij}$ is the entry on the ith row and the jth column. The n by n identity matrix is denoted by I_n. A matrix $A \in \mathbb{R}^{n\times n}$ is called orthogonal if $A^TA = I_n$. The Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$ is represented by $\|A\|_\mathrm{F} = \sqrt{\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$. The 2-norm of the vector a is defined as $\|a\|_2 = \sqrt{a^Ta}$ and the 2-norm of the matrix A is defined as $\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$. We have

$$\frac{1}{\sqrt{n}} \|A\|_{\mathrm{F}} \leq \|A\|_2 \leq \|A\|_{\mathrm{F}} \tag{1}$$

for an arbitrary m by n matrix A.

Theorem 1. (Singular Value Decomposition) *Let $A \in \mathbb{R}^{m\times n}$ and let $r = \min(m,n)$. Then there exists a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ and two orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that*

$$A = U\Sigma V^T \tag{2}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$ where $\sigma_i = (\Sigma)_{ii}$.
Factorization (2) is called the singular value decomposition (SVD) of A. The diagonal entries of Σ are the singular values of A. The columns of U are the left singular vectors and the columns of V are the right singular vectors.

Proof. See e.g. [11] or [12]. ■

We represent the ith column of U by u_i and the ith column of V by v_i. The singular values of a matrix $A \in \mathbb{R}^{m\times n}$ are unique. Singular vectors corresponding to simple singular values are also uniquely determined (up to the sign). If two or more singular values coincide, only the subspace generated by the corresponding singular vectors is well determined: any choice of orthonormal basis vectors that satisfy $A^T u_i = \sigma_i v_i$ and $Av_i = \sigma_i u_i$ is a valid set of singular vectors. If σ_1 is the largest singular value of A then $\sigma_1 = \|A\|_2$.

Definition 2. A real function f is analytic at a point $\alpha \in \mathbb{R}$ if the Taylor series of f with center α exists and if there is a neighborhood of α where the Taylor series converges to f.
A real function f is analytic in an interval $[\alpha, \beta]$ if it is analytic at every point of that interval.
A real matrix-valued function is analytic in $[\alpha, \beta]$ if all its entries are analytic in $[\alpha, \beta]$.

Note that if f is analytic in $[\alpha, \beta]$ then f is also continuous on $[\alpha, \beta]$.

Theorem 3. (Analytic Singular Value Decomposition) *Let $A(\cdot)$ be a real m by n matrix-valued function with entries that are analytic in the interval $[a,b]$. Then there exist real matrix-valued functions $U(\cdot)$, $\Sigma(\cdot)$ and $V(\cdot)$ that are analytic in $[a,b]$, such that $U(s)$ is an m by m orthogonal matrix, $\Sigma(s)$ an m by n diagonal matrix, $V(s)$ an n by n*

orthogonal matrix and $A(s) = U(s)\Sigma(s)V^T(s)$ *for all* $s \in [a, b]$.
We call this factorization the analytic singular value decomposition (ASVD) of $A(\cdot)$ *on* $[a, b]$.

Proof. See [2]. ■

Note that the diagonal entries of $\Sigma(s)$ are not necessarily nonnegative and ordered.
Let $A(\cdot)$ by a real m by n matrix-valued function that is analytic in the interval $[a, b]$. Consider an arbitrary ASVD of $A(\cdot)$ on $[a, b]$ with singular values $\sigma_1(\cdot)$, $\sigma_2(\cdot)$, ..., $\sigma_r(\cdot)$. In [2] it is shown that these analytic singular values are unique up to the ordering and the signs. Some of the analytic singular values can be identically 0. It is also possible that some of the analytic singular values are identical (up to the sign) in $[a, b]$. Consider two analytic singular values $\sigma_i(\cdot)$ and $\sigma_j(\cdot)$ such that $\sigma_i(\cdot)$ is identical to neither $\sigma_j(\cdot)$ nor $-\sigma_j(\cdot)$. Then $\sigma_i(\cdot)$ and $\pm\sigma_j(\cdot)$ can only intersect at isolated points. These points are called non-generic. The zeros of an analytic singular value that is not identically 0 are also non-generic points. The other points are called *generic*.
The following theorem links the ASVD of $A(\cdot)$ on $[a, b]$ to the (constant) SVD of $A(\alpha)$ where $\alpha \in [a, b]$.

Theorem 4. *Let* $A(\cdot)$ *by a real* m *by* n *matrix-valued function that is analytic in the interval* $[a, b]$. *If* $\alpha \in [a, b]$ *is a generic point of* $A(\cdot)$ *and if* $U_\alpha \, \Sigma_\alpha \, V_\alpha^T$ *is a (constant) SVD of* $A(\alpha)$ *then there exists an ASVD* $U(\cdot)\,\Sigma(\cdot)\,V^T(\cdot)$ *of* $A(\cdot)$ *on* $[a, b]$ *such that* $U(\alpha) = U_\alpha$, $\Sigma(\alpha) = \Sigma_\alpha$ *and* $V(\alpha) = V_\alpha$.

Proof. See [2]. ■

The ASVD that interpolates a constant SVD is not necessarily unique. However, if $A(\cdot)$ has only simple analytic singular values, then the ASVD of $A(\cdot)$ is uniquely determined by the condition $U(\alpha) = U_\alpha$, $\Sigma(\alpha) = \Sigma_\alpha$ and $V(\alpha) = V_\alpha$ at a generic point α.

Definition 5. Let $\alpha \in \mathbb{R} \cup \{\infty\}$ and let f and g be real functions. The function f is asymptotically equivalent to g in the neighborhood of α, denoted by $f(x) \sim g(x)$, $x \to \alpha$, if $\displaystyle\lim_{x\to\alpha} \frac{f(x)}{g(x)} = 1$.
If $\beta \in \mathbb{R}$ and if $\exists \delta > 0, \forall x \in (\beta - \delta, \beta + \delta) \setminus \{\beta\} : f(x) = 0$ then $f(x) \sim 0$, $x \to \beta$.
We say that $f(x) \sim 0$, $x \to \infty$ if $\exists K \in \mathbb{R}, \forall x > K : f(x) = 0$.
If $F(\cdot)$ and $G(\cdot)$ are real m by n matrix-valued functions then $F(x) \sim G(x)$, $x \to \alpha$ if $f_{ij}(x) \sim g_{ij}(x)$, $x \to \alpha$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Note that the main difference with the classic definition of asymptotic equivalence is that Definition 5 also allows us to say that a function is asymptotically equivalent to 0.

1.3. The max algebra and the extended max algebra

In this section we give a short introduction to the max algebra. A complete overview of the max algebra can be found in [1, 4]. The basic max-algebraic operations are defined as follows:

$$a \oplus b = \max(a, b) \tag{3}$$

$$a \otimes b = a + b \tag{4}$$

where $a, b \in \mathbb{R} \cup \{-\infty\}$. The reason for using these symbols is that there is an analogy between $\oplus$ and $+$ and between $\otimes$ and $\times$ as will be shown in Section 2. The resulting structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the max algebra. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$. The zero element for $\oplus$ in $\mathbb{R}_\varepsilon$ is represented by $\varepsilon \overset{\text{def}}{=} -\infty$. So $\forall a \in \mathbb{R}_\varepsilon : a \oplus \varepsilon = a = \varepsilon \oplus a$.
Let $r \in \mathbb{R}$. The rth max-algebraic power of $a \in \mathbb{R}$ is denoted by $a^{\otimes^r}$ and corresponds to ra in linear algebra. If $a \in \mathbb{R}$ then $a^{\otimes^0} = 0$ and the inverse element of a w.r.t. $\otimes$ is $a^{\otimes^{-1}} = -a$. There is no inverse element for ε since ε is absorbing for $\otimes$: $\forall a \in \mathbb{R}_\varepsilon : a \otimes \varepsilon = \varepsilon = \varepsilon \otimes a$. If $r > 0$ then $\varepsilon^{\otimes^r} = \varepsilon$. If $r \leq 0$ then $\varepsilon^{\otimes^r}$ is not defined.

The max-algebraic operations are extended to matrices in the usual way. If $\alpha \in \mathbb{R}_\varepsilon$ and if A and B are m by n matrices with entries in $\mathbb{R}_\varepsilon$ then

$$(\alpha \otimes A)_{ij} = \alpha \otimes a_{ij} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

and

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \ .$$

If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \ .$$

The matrix E_n is the n by n max-algebraic identity matrix:

$$(E_n)_{ii} = 0 \qquad \text{for } i = 1, 2, \ldots, n \ ,$$

$$(E_n)_{ij} = \varepsilon \qquad \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n \text{ with } i \neq j \ .$$

The m by n max-algebraic zero matrix is represented by $\mathcal{E}_{m \times n}$: $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j. The off-diagonal entries of a max-algebraic diagonal matrix $D \in \mathbb{R}_\varepsilon^{m \times n}$ are equal to ε: $d_{ij} = \varepsilon$ for all i, j with $i \neq j$.

In contrast to linear algebra, there exist no inverse elements w.r.t. $\oplus$ in $\mathbb{R}_\varepsilon$: if $a \in \mathbb{R}_\varepsilon$ then there does not exist an element $b \in \mathbb{R}_\varepsilon$ such that

$a \oplus b = \varepsilon = b \oplus a$, except when $a = \varepsilon$. To overcome this problem we need the extended max algebra $\mathbb{S}_{\max}$ [1, 9, 13], which is a kind of symmetrization of the max algebra. This can be compared with the extension of $\mathbb{N}$ to $\mathbb{Z}$. In Section 2 we shall indeed show that $\mathbb{R}_{\max}$ corresponds to $(\mathbb{R}^+, \times, +)$ and that $\mathbb{S}_{\max}$ corresponds to $(\mathbb{R}, \times, +)$. However, since the $\oplus$ operation is idempotent, i.e. $\forall a \in \mathbb{R}_\varepsilon : a \oplus a = a$, we cannot use the classical symmetrization technique since every idempotent group reduces to a trivial group [1, 13]. Nevertheless, it is possible to adapt the method of the construction of $\mathbb{Z}$ from $\mathbb{N}$ to obtain "balancing" elements rather than inverse elements.

We shall restrict ourselves to a short introduction to the most important features of $\mathbb{S}_{\max}$, which is based on [1, 13]. First we introduce the "algebra of pairs". We consider the set of pairs $\mathbb{R}_\varepsilon^2$ with the following laws:

$$\begin{aligned}(a,b) \oplus (c,d) &= (a \oplus c,\, b \oplus d)\\ (a,b) \otimes (c,d) &= (a \otimes c \oplus b \otimes d,\, a \otimes d \oplus b \otimes c)\end{aligned}$$

where $(a,b), (c,d) \in \mathbb{R}_\varepsilon^2$ and where the operations $\oplus$ and $\otimes$ on the right hand sides correspond to maximization and addition as defined in (3) and (4). The reason for also using $\oplus$ and $\otimes$ on the left hand sides is that they correspond to $\oplus$ and $\otimes$ as defined in $\mathbb{R}_\varepsilon$ as we will see later on. It is easy to verify that in $\mathbb{R}_\varepsilon^2$ the $\oplus$ law is associative, commutative and idempotent, and its zero element is $(\varepsilon, \varepsilon)$; the $\otimes$ law is associative and its unit element is $(0, \varepsilon)$ and $\otimes$ is distributive w.r.t. $\oplus$. The structure $(\mathbb{R}_\varepsilon^2, \oplus, \otimes)$ is called the algebra of pairs.
If $x = (a,b) \in \mathbb{R}_\varepsilon^2$ then we define the operator $\ominus$ as $\ominus x = (b,a)$, the max-absolute value $|x|_\oplus = a \oplus b$ and the balance operator as $x^\bullet = x \oplus (\ominus x) = (|x|_\oplus, |x|_\oplus)$. We have $\forall x, y \in \mathbb{R}_\varepsilon^2$:

$$\begin{aligned}x^\bullet &= (\ominus x)^\bullet = (x^\bullet)^\bullet\\ \ominus(\ominus x) &= x\\ \ominus(x \oplus y) &= (\ominus x) \oplus (\ominus y)\\ \ominus(x \otimes y) &= (\ominus x) \otimes y\ .\end{aligned}$$

The last three properties allow us to write $x \ominus y$ instead of $x \oplus (\ominus y)$. So the $\ominus$ operator in the algebra of pairs could be considered as the equivalent of the - operator in linear algebra (see also Section 2).
In linear algebra we have $\forall x \in \mathbb{R} : x - x = 0$, but in the algebra of pairs we have $\forall x \in \mathbb{R}_\varepsilon^2 : x \ominus x = x^\bullet \neq (\varepsilon, \varepsilon)$ unless $x = (\varepsilon, \varepsilon)$, the zero element for $\oplus$ in $\mathbb{R}_\varepsilon^2$. Therefore, we introduce a new relation, the balance relation, represented by ∇.

Definition 6. Consider $x = (a,b)$, $y = (c,d) \in \mathbb{R}_\varepsilon^2$. We say that x balances y, denoted by $x \,\nabla\, y$, if $a \oplus d = b \oplus c$.

Since $\forall x \in \mathbb{R}_\varepsilon^2 : x \ominus x = x^\bullet = (|x|_\oplus, |x|_\oplus) \nabla (\varepsilon, \varepsilon)$, we could say that the balance relation in the algebra of pairs is the counterpart of the equality relation in linear algebra. The balance relation is reflexive and symmetric but it is not transitive since e.g. $(2,1) \nabla (2,2)$ and $(2,2) \nabla (1,2)$ but $(2,1) \not\nabla (1,2)$. Hence, the balance relation is not an equivalence relation and we cannot use it to define the quotient set of $\mathbb{R}_\varepsilon^2$ by ∇ (as opposed to linear algebra where $\mathbb{N}^2/=$ yields $\mathbb{Z}$). Therefore, we introduce another relation $\mathcal{B}$ that is closely related to the balance relation ∇ and that is defined as follows:

$$(a,b)\, \mathcal{B}\, (c,d) \quad \text{if} \quad \begin{cases} (a,b) \nabla (c,d) & \text{if } a \neq b \text{ and } c \neq d\,, \\ (a,b) = (c,d) & \text{otherwise}\,, \end{cases}$$

with $(a,b),(c,d) \in \mathbb{R}_\varepsilon^2$. Note that if $x \in \mathbb{R}_\varepsilon^2$ then $x \ominus x = (|x|_\oplus, |x|_\oplus) \not\mathcal{B} (\varepsilon, \varepsilon)$ unless $x = (\varepsilon, \varepsilon)$. It is easy to verify that the relation $\mathcal{B}$ is an equivalence relation that is compatible with the $\oplus$ and $\otimes$ laws defined in $\mathbb{R}_\varepsilon^2$, with the balance relation ∇ and with the $\ominus$, $|\cdot|_\oplus$ and $(\cdot)^\bullet$ operators. We can distinguish three kinds of equivalence classes generated by $\mathcal{B}$:

- $\overline{(a, -\infty)} = \{(a,x) \mid x < a\}$, called max-positive;
- $\overline{(-\infty, a)} = \{(x,a) \mid x < a\}$, called max-negative;
- $\overline{(a,a)} = \{(a,a)\}$, called balanced.

The class $\overline{(\varepsilon, \varepsilon)}$ is called the zero class.
Now we define the quotient set $\mathbb{S} = (\mathbb{R}_\varepsilon^2)/\mathcal{B}$. The resulting structure $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$ is called the extended max algebra. By associating $\overline{(a, -\infty)}$ with $a \in \mathbb{R}_\varepsilon$, we can identify $\mathbb{R}_\varepsilon$ with the set of max-positive or zero classes denoted by $\mathbb{S}^\oplus$. The set of max-negative or zero classes $\{\ominus a \mid a \in \mathbb{S}^\oplus\}$ will be denoted by $\mathbb{S}^\ominus$ and the set of balanced classes $\{a^\bullet \mid a \in \mathbb{S}^\oplus\}$ by $\mathbb{S}^\bullet$. This yields the decomposition $\mathbb{S} = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$. The max-positive and max-negative elements and the zero element are called signed ($\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$). Note that $\mathbb{S}^\oplus \cap \mathbb{S}^\ominus \cap \mathbb{S}^\bullet = \{\overline{(\varepsilon, \varepsilon)}\}$ and $\varepsilon = \ominus\varepsilon = \varepsilon^\bullet$.
These notations allow us to write e.g. $2 \oplus (\ominus 4)$ instead of $\overline{(2, -\infty)} \oplus \overline{(-\infty, 4)}$. Since $\overline{(2, -\infty)} \oplus \overline{(-\infty, 4)} = \overline{(2,4)} = \overline{(-\infty, 4)}$ we have $2 \oplus (\ominus 4) = \ominus 4$. In general, if $x, y \in \mathbb{R}_\varepsilon$ then

$$x \oplus (\ominus y) = x \qquad \text{if } x > y\,, \tag{5}$$

$$x \oplus (\ominus y) = \ominus y \qquad \text{if } x < y\,, \tag{6}$$

$$x \oplus (\ominus x) = x^\bullet\,. \tag{7}$$

Now we give some extra properties of balances that will be used in the next sections. We shall explicitly prove two of these properties to illustrate how the other properties of this section can be proved.

An element with a $\ominus$ sign can be transferred to the other side of a balance as follows:

PROPOSITION 7. $\forall a, b, c \in \mathbb{S} : a \ominus c \nabla b$ *if and only if* $a \nabla b \oplus c$.

Proof. Let (a', a''), (b', b'') and $(c', c'') \in \mathbb{R}_\varepsilon^2$ belong to the equivalence classes that correspond to a, b and c respectively. We have

$$
\begin{array}{lll}
(a', a'') \ominus (c', c'') \nabla (b', b'') & & \\
\quad \Leftrightarrow \quad (a', a'') \oplus (c'', c') \nabla (b', b'') & & \\
\quad \Leftrightarrow \quad (a' \oplus c'',\, a'' \oplus c') \nabla (b', b'') & & \\
\quad \Leftrightarrow \quad (a' \oplus c'') \oplus b'' = (a'' \oplus c') \oplus b' & & \text{(by Definition 6)} \\
\quad \Leftrightarrow \quad a' \oplus (b'' \oplus c'') = a'' \oplus (b' \oplus c') & & \text{(since } \oplus \text{ is associative} \\
 & & \text{and commutative in } \mathbb{R}_\varepsilon) \\
\quad \Leftrightarrow \quad (a', a'') \nabla (b' \oplus c',\, b'' \oplus c'') & & \text{(by Definition 6)} \\
\quad \Leftrightarrow \quad (a', a'') \nabla (b', b'') \oplus (c', c'') \ . & &
\end{array}
$$

Hence, $a \ominus c \nabla b$ if and only if $a \nabla b \oplus c$. ■

If both sides of a balance are signed, we can replace the balance by an equality:

PROPOSITION 8. $\forall a, b \in \mathbb{S}^\vee : a \nabla b \Rightarrow a = b$.

Proof. Let (a', a'') and $(b', b'') \in \mathbb{R}_\varepsilon^2$ belong to the equivalence classes that correspond to a and b respectively. If $a \nabla b$ then

$$a' \oplus b'' = a'' \oplus b' \ . \tag{8}$$

If $(a', a'') = (\varepsilon, \varepsilon)$ then (8) can only hold if $b' = b''$. Since b is signed this is only possible if $b' = b'' = \varepsilon$ and thus $(a', a'') = (b', b'')$. Hence, $a = b$.
If $(a', a'') \neq (\varepsilon, \varepsilon)$ then either $a' < a''$ or $a' > a''$ since a is signed. First we assume that $a' < a''$ and thus $a'' \neq \varepsilon$. Equation (8) then leads to

$$b'' = a'' \oplus b' \tag{9}$$

and since $a'' \neq \varepsilon$, we have $b'' \neq \varepsilon$. Since b is signed, this means that $b' < b''$. So (9) can only hold if $b'' = a''$. Hence, $(a', a'') \in \overline{(\varepsilon, a'')}$ and $(b', b'') \in \overline{(\varepsilon, b'')} = \overline{(\varepsilon, a'')}$ and this results in $a = b$.
If $a' > a''$ then analogous reasoning also leads to the conclusion that $a = b$. ■

Let $a \in \mathbb{S}$. The max-positive part $a^\oplus$ and the max-negative part $a^\ominus$ of a are defined as follows:

- if $a \in \mathbb{S}^\oplus$ then $a^\oplus = a$ and $a^\ominus = \varepsilon$,

- if $a \in \mathbb{S}^{\ominus}$ then $a^{\oplus} = \varepsilon$ and $a^{\ominus} = \ominus a$,
- if $a \in \mathbb{S}^{\bullet}$ then $\exists b \in \mathbb{R}_{\varepsilon}$ such that $a = b^{\bullet}$ and then $a^{\oplus} = a^{\ominus} = b$.

So $a = a^{\oplus} \ominus a^{\ominus}$ and $a^{\oplus}, a^{\ominus} \in \mathbb{R}_{\varepsilon}$. Note that a decomposition of the form $a = \alpha \ominus \beta$ with $\alpha, \beta \in \mathbb{R}_{\varepsilon}$ is unique if it is required that either $\alpha \neq \varepsilon$ and $\beta = \varepsilon$; $\alpha = \varepsilon$ and $\beta \neq \varepsilon$; or $\alpha = \beta$. Hence, the decomposition $a = a^{\oplus} \ominus a^{\ominus}$ is unique. We also have $|a|_{\oplus} = a^{\oplus} \oplus a^{\ominus}$.
Now we can reformulate Definition 6 as follows:

Proposition 9. *$\forall a, b \in \mathbb{S}: a \nabla b$ if $a^{\oplus} \oplus b^{\ominus} = a^{\ominus} \oplus b^{\oplus}$.*

The balance relation is extended to matrices in the usual way: if $A, B \in \mathbb{S}^{m \times n}$ then $A \nabla B$ if $a_{ij} \nabla b_{ij}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Propositions 7 and 8 can now be extended to the matrix case as follows:

Proposition 10. *$\forall A, B, C \in \mathbb{S}^{m \times n} : A \ominus C \nabla B$ if and only if $A \nabla B \oplus C$.*

Proposition 11. *$\forall A, B \in (\mathbb{S}^{\vee})^{m \times n} : A \nabla B \Rightarrow A = B$.*

We conclude this section with a few extra examples to illustrate the concepts defined above and their properties.

Example 12. By Proposition 9 we have $3 \nabla 4^{\bullet}$ since $3^{\oplus} = 3$, $3^{\ominus} = \varepsilon$, $(4^{\bullet})^{\oplus} = (4^{\bullet})^{\ominus} = 4$ and $3 \oplus 4 = 4 = \varepsilon \oplus 4$.

Example 13. Consider the balance

$$x \oplus 4 \nabla 3 \ . \tag{10}$$

Using Proposition 7 this balance can be rewritten as $x \nabla 3 \ominus 4$ or $x \nabla \ominus 4$ since $3 \ominus 4 = \ominus 4$ by (6).
If we want a signed solution the latter balance becomes an equality by Proposition 8. This yields $x = \ominus 4$.
The balanced solutions are of the form $x = t^{\bullet}$ with $t \in \mathbb{R}_{\varepsilon}$. We have $t^{\bullet} \nabla \ominus 4$ or equivalently $t \oplus 4 = t$ if and only if $t \geq 4$.
So the solution set of balance (10) is given by $\{ \ominus 4 \} \cup \{ t^{\bullet} \mid t \in \mathbb{R}_{\varepsilon}, t \geq 4 \}$.

Definition 14. The max-algebraic norm of a vector $a \in \mathbb{S}^n$ is defined as

$$\|a\|_{\oplus} = \bigoplus_{i=1}^{n} |a_i|_{\oplus} = \bigoplus_{i=1}^{n} (a_i^{\oplus} \oplus a_i^{\ominus}) \ .$$

The max-algebraic norm of a matrix $A \in \mathbb{S}^{m \times n}$ is defined as

$$\|A\|_\oplus = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_\oplus \quad .$$

Note that the max-algebraic vector norm corresponds to the p-norms in linear algebra since $\|a\|_\oplus = \left(\bigoplus_{i=1}^{n} |a_i|_\oplus^{\otimes^p} \right)^{\otimes^{\frac{1}{p}}}$ for every vector $a \in \mathbb{S}^n$. The max-algebraic matrix norm corresponds to both the Frobenius norm and the p-norms since we have for every matrix $A \in \mathbb{S}^{m \times n}$: $\|A\|_\oplus = \left(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_\oplus^{\otimes^2} \right)^{\otimes^{\frac{1}{2}}}$ and also $\|A\|_\oplus = \max\limits_{\|x\|_\oplus = 0} \|A \otimes x\|_\oplus$ by taking $x \in \mathbb{S}^n$ equal to $[\,0 \;\; 0 \; \ldots \; 0\,]^T$.

2. A LINK BETWEEN THE FIELD OF THE REAL NUMBERS AND THE EXTENDED MAX ALGEBRA

Consider the following correspondences for $x, y, z \in \mathbb{R}_\varepsilon$:

$$\begin{array}{lcl} x \oplus y = z & \leftrightarrow & e^{xs} + e^{ys} \sim e^{zs} \ , \ s \to \infty \\ x \otimes y = z & \leftrightarrow & e^{xs} \cdot e^{ys} = e^{zs} \qquad \text{for all } s \in \mathbb{R} \ . \end{array}$$

We shall extend this link between $(\mathbb{R}^+, +, \times)$ and $\mathbb{R}_{\max}$ that was already used in [14] – and under a slightly different form in [5] – to $\mathbb{S}_{\max}$. First we define the following mapping for $x \in \mathbb{R}_\varepsilon$:

$$\begin{aligned} \mathcal{F}(x, s) &= \mu e^{xs} \\ \mathcal{F}(\ominus x, s) &= -\mu e^{xs} \\ \mathcal{F}(x^\bullet, s) &= \nu e^{xs} \end{aligned}$$

where μ is an arbitrary positive real number or parameter and ν is an arbitrary real number or parameter different from 0 and s is a real parameter. Note that $\mathcal{F}(\varepsilon, s) = 0$.

To reverse the mapping we have to take $\lim\limits_{s \to \infty} \dfrac{\log(\,|\,\mathcal{F}(x, s)\,|\,)}{s}$ and adapt the max-sign depending on the sign of the coefficient of the exponential.

So if f is a real function, if $x \in \mathbb{R}_\varepsilon$ and if μ is a positive real number or if μ is a parameter that can only take on positive real values then we have

$$f(s) \sim \mu e^{xs}, \; s \to \infty \quad \Rightarrow \quad \mathcal{R}(f(\cdot)) = x$$
$$f(s) \sim -\mu e^{xs}, \; s \to \infty \quad \Rightarrow \quad \mathcal{R}(f(\cdot)) = \ominus x$$

where $\mathcal{R}$ is the reverse mapping of $\mathcal{F}$. If ν is a parameter that can take on both positive and negative real values then we have

$$f(s) \sim \nu e^{xs}, \; s \to \infty \quad \Rightarrow \quad \mathcal{R}(f(\cdot)) = x^{\bullet} \; .$$

Note that if the coefficient of e^{xs} is a number then the reverse mapping always yields a signed result.
Now we have for $a, b, c \in \mathbb{S}$:

$$a \oplus b = c \quad \to \quad \mathcal{F}(a,s) + \mathcal{F}(b,s) \sim \mathcal{F}(c,s), \; s \to \infty \qquad (11)$$

$$\mathcal{F}(a,s) + \mathcal{F}(b,s) \sim \mathcal{F}(c,s), \; s \to \infty \quad \to \quad a \oplus b \,\nabla\, c \qquad (12)$$

$$a \otimes b = c \quad \leftrightarrow \quad \mathcal{F}(a,s) \cdot \mathcal{F}(b,s) = \mathcal{F}(c,s) \qquad \text{for all } s \in \mathbb{R} \qquad (13)$$

for an appropriate choice of the μ's and ν's in $\mathcal{F}(c,s)$ in (11) and in (13) from the left to the right. The balance in (12) results from the fact that we can have cancellation of equal terms with opposite sign in $(\mathbb{R}, +, \times)$ whereas this is in general not possible in the extended max algebra since $\forall a \in \mathbb{S} \setminus \{\varepsilon\} : a \ominus a \neq \varepsilon$. So we have the following correspondences:

$$(\mathbb{R}^+, +, \times) \quad \leftrightarrow \quad (\mathbb{R}_\varepsilon, \oplus, \otimes) = \mathbb{R}_{\max}$$
$$(\mathbb{R}, +, \times) \quad \leftrightarrow \quad (\mathbb{S}, \oplus, \otimes) = \mathbb{S}_{\max} \; .$$

We extend this mapping to matrices such that if $A \in \mathbb{S}^{m \times n}$ then $\tilde{A}(\cdot) = \mathcal{F}(A, \cdot)$ is a real m by n matrix-valued function with $\tilde{a}_{ij}(s) = \mathcal{F}(a_{ij}, s)$ for some choice of the μ's and ν's. Note that the mapping is performed entrywise — it is not a matrix exponential! The reverse mapping $\mathcal{R}$ is extended to matrices in a similar way: if $\tilde{A}(\cdot)$ is a real matrix-valued function then $(\mathcal{R}(\tilde{A}(\cdot)))_{ij} = \mathcal{R}(\tilde{a}_{ij}(\cdot))$ for all i, j. If A ,B and C are matrices with entries in $\mathbb{S}$, we have

$$A \oplus B = C \quad \to \quad \mathcal{F}(A,s) + \mathcal{F}(B,s) \sim \mathcal{F}(C,s), \; s \to \infty \qquad (14)$$
$$\mathcal{F}(A,s) + \mathcal{F}(B,s) \sim \mathcal{F}(C,s), \; s \to \infty \quad \to \quad A \oplus B \,\nabla\, C \qquad (15)$$
$$A \otimes B = C \quad \to \quad \mathcal{F}(A,s) \cdot \mathcal{F}(B,s) \sim \mathcal{F}(C,s), \; s \to \infty \qquad (16)$$
$$\mathcal{F}(A,s) \cdot \mathcal{F}(B,s) \sim \mathcal{F}(C,s), \; s \to \infty \quad \to \quad A \otimes B \,\nabla\, C \qquad (17)$$

for an appropriate choice of the μ's and ν's in $\mathcal{F}(C,s)$ in (14) and (16).

EXAMPLE 15. Let $A = \begin{bmatrix} 1^{\bullet} & 0 \\ 2 & \ominus 3 \end{bmatrix}$ and $B = \begin{bmatrix} \varepsilon & 2 \\ \ominus 2 & 1 \end{bmatrix}$. Hence, $A \otimes B = \begin{bmatrix} \ominus 2 & 3^{\bullet} \\ 5 & 4^{\bullet} \end{bmatrix}$. In general we have $\mathcal{F}(A,s) = \begin{bmatrix} \nu_1 \, e^{s} & \mu_1 \\ \mu_2 \, e^{2s} & -\mu_3 \, e^{3s} \end{bmatrix}$, $\mathcal{F}(B,s) = \begin{bmatrix} 0 & \mu_4 \, e^{2s} \\ -\mu_5 \, e^{2s} & \mu_6 \, e^{s} \end{bmatrix}$ and $\mathcal{F}(A \otimes B, s) = \begin{bmatrix} -\mu_7 \, e^{2s} & \nu_2 \, e^{3s} \\ \mu_8 \, e^{5s} & \nu_3 \, e^{4s} \end{bmatrix}$ with $\mu_i > 0$ and $\nu_i \in \mathbb{R}_0$. So

$$\mathcal{F}(A,s) \cdot \mathcal{F}(B,s) = \begin{bmatrix} -\mu_1 \, \mu_5 \, e^{2s} & \nu_1 \, \mu_4 \, e^{3s} \, + \, \mu_1 \, \mu_6 \, e^{s} \\ \mu_3 \, \mu_5 \, e^{5s} & \mu_2 \, \mu_4 \, e^{4s} \, - \, \mu_3 \, \mu_6 \, e^{4s} \end{bmatrix} .$$

If we take

$$\mu_7 = \mu_1 \, \mu_5 \, , \quad \mu_8 = \mu_3 \, \mu_5 \, , \quad \nu_2 = \nu_1 \, \mu_4 \quad \text{and} \quad \nu_3 = \mu_2 \, \mu_4 - \mu_3 \, \mu_6$$

then $\mathcal{F}(A,s) \cdot \mathcal{F}(B,s) \sim \mathcal{F}(A \otimes B, s) \, , \ s \to \infty$.
If we take all the μ_i's and ν_1 equal to 1 we get

$$\mathcal{F}(A,s) \cdot \mathcal{F}(B,s) \sim \begin{bmatrix} -e^{2s} & e^{3s} \\ e^{5s} & 0 \end{bmatrix} \stackrel{\text{def}}{=} \tilde{C}(s) \, , \ s \to \infty \ .$$

The reverse mapping then results in $C = \mathcal{R}\Big(\tilde{C}(\cdot)\Big) = \begin{bmatrix} \ominus 2 & 3 \\ 5 & \varepsilon \end{bmatrix}$ and we see that $A \otimes B \, \nabla \, C$.
Taking $\mu_i = i$ for $i = 1, 2, \ldots, 6$ and $\nu_1 = -1$ leads to

$$\mathcal{F}(A,s) \cdot \mathcal{F}(B,s) \sim \begin{bmatrix} -5e^{2s} & -4e^{3s} \\ 15e^{5s} & -10e^{4s} \end{bmatrix} \stackrel{\text{def}}{=} \tilde{D}(s) \, , \ s \to \infty \ .$$

The reverse mapping now results in $D = \mathcal{R}\Big(\tilde{D}(\cdot)\Big) = \begin{bmatrix} \ominus 2 & \ominus 3 \\ 5 & \ominus 4 \end{bmatrix}$ and again we have $A \otimes B \, \nabla \, D$.

We can extend the link between $(\mathbb{R}, +, \times)$ and $\mathbb{S}_{\max}$ even further by introducing the "max-complex" numbers. First we define $\bar{k}$ such that $\bar{k} \otimes \bar{k} = \ominus 0$. This yields $\mathbb{T} = \{ a \oplus b \otimes \bar{k} \mid a, b \in \mathbb{S} \}$, the set of the max-complex numbers. The set $\mathbb{S} \subset \mathbb{T}$ is the subset of the max-real numbers and $\mathbb{R}_\varepsilon \subset \mathbb{S} \subset \mathbb{T}$ is the subset of the max-positive max-real numbers. Using a method that is analogous to the method used to construct $\mathbb{C}$ from $\mathbb{R}$ we get the following calculation rules:

$$\begin{aligned} (a \oplus b \otimes \bar{k}) \ \oplus \ (c \oplus d \otimes \bar{k}) &= (a \oplus c) \ \oplus \ (b \oplus d) \otimes \bar{k} \\ (a \oplus b \otimes \bar{k}) \ \otimes \ (c \oplus d \otimes \bar{k}) &= (a \otimes c \ominus b \otimes d) \ \oplus \ (a \otimes d \oplus b \otimes c) \otimes \bar{k} \end{aligned}$$

where a, b, c and $d \in \mathbb{S}$. This results in the structure $\mathbb{T}_{\max} = (\mathbb{T}, \oplus, \otimes)$.
If $a, b \in \mathbb{S}$ and if f and g are real functions that are asymptotically equivalent to an exponential in the neighborhood of ∞, we define

$$\begin{aligned} \mathcal{F}(a \oplus b \otimes \bar{k}, \cdot) &= \mathcal{F}(a, \cdot) + \mathcal{F}(b, \cdot) i \\ \mathcal{R}(f(\cdot) + g(\cdot)\, i) &= \mathcal{R}(f(\cdot)) \oplus \mathcal{R}(g(\cdot)) \otimes \bar{k} \end{aligned}$$

where i is the imaginary unit ($i^2 = -1$). This leads to the following correspondence:

$$(\mathbb{C}, +, \times) \quad \leftrightarrow \quad (\mathbb{T}, \oplus, \otimes) \;=\; \mathbb{T}_{\max} \; .$$

We shall not further elaborate this correspondence between the field of complex numbers and $\mathbb{T}_{\max}$ since it will not be needed in the remainder of this paper.

3. THE SINGULAR VALUE DECOMPOSITION IN THE EXTENDED MAX ALGEBRA

We shall now use the mapping from $(\mathbb{R}, +, \times)$ to $\mathbb{S}_{\max}$ and the reverse mapping to prove the existence of a kind of singular value decomposition in $\mathbb{S}_{\max}$. But first we need some extra properties.

PROPOSITION 16. *Every function f that is analytic in 0 is asymptotically equivalent to a power function in the neighborhood of* 0*: $\exists \alpha \in \mathbb{R}$, $\exists k \in \mathbb{N}$ such that $f(x) \sim \alpha x^k$, $x \to 0$.*

Proof. If f is analytic in 0 then there exists a neighborhood $(-\xi, \xi)$ of 0 where f can be written as a convergent Taylor series

$$f(x) \;=\; \sum_{i=0}^{\infty} \alpha_i x^i \qquad \text{for all } x \in (-\xi, \xi) \; .$$

Furthermore, this Taylor series converges absolutely in $(-\xi, \xi)$ and it converges uniformly to f in every interval $[-\rho, \rho]$ with $0 < \rho < \xi$.
First we consider the case where all the coefficients α_i are equal to 0. Then $\forall x \in (-\xi, \xi) : f(x) = 0$ and thus $f(x) \sim 0$, $x \to 0$ by Definition 5.
Now we assume that at least one coefficient α_i is different from 0. Let α_k be the first coefficient that is different from 0. Then we can rewrite $f(x)$ as

$$f(x) \;=\; \alpha_k x^k \left(1 + \sum_{i=k+1}^{\infty} \frac{\alpha_i}{\alpha_k}\, x^{i-k} \right) \;=\; \alpha_k x^k (1 + p(x))$$

with $p(x) \;=\; \sum_{j=1}^{\infty} \gamma_j x^j$ where $\gamma_j = \dfrac{\alpha_{j+k}}{\alpha_k} \in \mathbb{R}$. Let ρ be a real number such that $0 < \rho < \xi$. Since the Taylor series of f converges uniformly in $[-\rho, \rho]$, the series $\sum_{j=1}^{\infty} \gamma_j x^j$ also converges uniformly in $[-\rho, \rho]$. Therefore,

$$\lim_{x\to 0} p(x) \;=\; \lim_{x\to 0} \sum_{j=1}^{\infty} \gamma_j x^j \;=\; \sum_{j=1}^{\infty} \left(\lim_{x\to 0} \gamma_j x^j \right) \;=\; 0$$

where we have used the fact that the summation and the limit can be interchanged since the series $\sum_{j=1}^{\infty} \gamma_j x^j$ converges uniformly in $[-\rho, \rho]$. This leads to

$$\lim_{x\to 0} \frac{f(x)}{\alpha_k x^k} \;=\; \lim_{x\to 0} \frac{\alpha_k x^k (1 + p(x))}{\alpha_k x^k} \;=\; \lim_{x\to\infty} (1 + p(x)) \;=\; 1$$

and thus $f(x) \;\sim\; \alpha_k x^k$, $x \to 0$ where $\alpha_k \in \mathbb{R}$ and $k \in \mathbb{N}$. ■

PROPOSITION 17. *Let A, $B \in \mathbb{R}^{m\times n}$ and let $r = \min(m, n)$. Then*

$$|\, \sigma_i(A) - \sigma_i(B) \,| \;\leq\; \|\, A - B \,\|_{\mathrm{F}} \qquad \textit{for } i = 1, 2, \ldots, r$$

where $\sigma_i(A)$ is the ith singular value of A and $\sigma_i(B)$ is the ith singular value of B.

Proof. See e.g. [11] or [12]. ■

LEMMA 18. (SELECTION PRINCIPLE FOR ORTHOGONAL MATRICES) *Let $\{U_i\}_{i=0}^{\infty}$ with $U_i \in \mathbb{R}^{n\times n}$ be a given sequence of orthogonal matrices. Then there exists a subsequence $\{U_{k_i}\}_{i=0}^{\infty}$ such that all of the entries of U_{k_i} converge (as sequences of real numbers) to the entries of an orthogonal matrix U as i goes to ∞.*

Proof. See e.g. [12]. ■

LEMMA 19. *Consider $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha \leq \beta$. Let K be an arbitrary real number with $K \geq \dfrac{1}{\alpha}$. Then $\forall s \in \mathbb{R}$ such that $s \geq K$: $0 \leq e^{-\alpha s} - e^{-\beta s} \leq e^{-\alpha K} - e^{-\beta K}$.*

Proof. If $\alpha = \beta$ then the proof is trivial. So from now on we assume that $\alpha < \beta$. If we define a real function f such that $f(s) = e^{-\alpha s} - e^{-\beta s}$

then $f'(s) = -\alpha e^{-\alpha s} + \beta e^{-\beta s}$. The zero of f' is given by $e^{(\beta-\alpha)s^*} = \dfrac{\beta}{\alpha}$ or $s^* = \dfrac{\log \frac{\beta}{\alpha}}{\beta - \alpha}$. Note that $s^* > 0$ since $\beta > \alpha$.
We have $f'(0) = \beta - \alpha > 0$ and

$$\begin{aligned} f'(2s^*) &= -\alpha e^{-\alpha 2s^*} + \beta e^{-\beta 2s^*} \\ &= -\alpha e^{-\alpha s^*} e^{-\alpha s^*} + \beta e^{-\beta 2s^*} \\ &= -\beta e^{-\beta s^*} e^{-\alpha s^*} + \beta e^{-\beta 2s^*} \qquad (\text{since } \alpha e^{-\alpha s^*} = \beta e^{-\beta s^*}) \\ &= -\beta e^{-\beta s^*} (e^{-\alpha s^*} - e^{-\beta s^*}) \\ &< 0 \end{aligned}$$

since $\alpha < \beta$ and $s^* > 0$ lead to $-\alpha s^* > -\beta s^*$ and thus $e^{-\alpha s^*} > e^{-\beta s^*}$.
The function f' has only one zero and is defined and continuous on $\mathbb{R}$. Hence,

$$\forall s < s^* : f'(s) > 0 \quad \text{and} \quad \forall s > s^* : f'(s) < 0 \ .$$

So f reaches a maximum for $s = s^*$ and f is decreasing for $s > s^*$. Furthermore, $\lim_{s \to \infty} f(s) = 0$. Hence, if $K \geq s^*$ then $\forall s \geq K : 0 \leq f(s) \leq f(K)$.
Since $\forall s > 0 : \log(s) \leq s - 1$ we have $s^* = \dfrac{\log \frac{\beta}{\alpha}}{\beta - \alpha} \leq \dfrac{\frac{\beta}{\alpha} - 1}{\beta - \alpha} = \dfrac{1}{\alpha}$. So if $K \geq \dfrac{1}{\alpha}$ then also $K \geq s^*$ and thus $\forall s \geq K : 0 \leq f(s) \leq f(K)$. ■

Now we come to the main theorem of this paper:

THEOREM 20. (EXISTENCE OF THE SINGULAR VALUE DECOMPOSITION IN $\mathbb{S}_{\mathbf{max}}$) *Let $A \in \mathbb{S}^{m \times n}$ and let $r = \min(m, n)$. Then there exist a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m \times n}$ and matrices $U \in (\mathbb{S}^\vee)^{m \times m}$ and $V \in (\mathbb{S}^\vee)^{n \times n}$ such that*

$$A \ \nabla \ U \otimes \Sigma \otimes V^T \tag{18}$$

with

$$\begin{aligned} U^T \otimes U \quad &\nabla \quad E_m \\ V^T \otimes V \quad &\nabla \quad E_n \end{aligned}$$

and $\|A\|_\oplus \geq \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq \varepsilon$ where $\sigma_i = (\Sigma)_{ii}$.
Every decomposition of the form (18) that satisfies the above conditions is called a max-algebraic singular value decomposition of A.

Proof. If $A \in \mathbb{S}^{m \times n}$ has entries that are not signed we can always define a signed m by n matrix $\hat{A}$ such that

$$\begin{aligned} \hat{a}_{ij} \quad &= \quad a_{ij} \qquad \text{if } a_{ij} \text{ is signed,} \\ &= \quad a_{ij}^\oplus \qquad \text{if } a_{ij} \text{ is not signed.} \end{aligned}$$

Since $\forall i,j : |\hat{a}_{ij}|_{\oplus} = |a_{ij}|_{\oplus}$, we have $\|\hat{A}\|_{\oplus} = \|A\|_{\oplus}$. Furthermore, $\forall a,b \in \mathbb{S}$: $a \nabla b \Rightarrow a^{\bullet} \nabla b$, which means that $\hat{A} \nabla U \otimes \Sigma \otimes V^T$ would imply $A \nabla U \otimes \Sigma \otimes V^T$. Therefore, it is sufficient to prove this theorem for signed matrices A.

So from now on we assume that A is signed. First we define $c = \|A\|_{\oplus} = \max_{i,j} \{ |a_{ij}|_{\oplus} \}$.
If $c = \varepsilon$ then $A = \mathcal{E}_{m \times n}$. If we take $U = E_m$, $\Sigma = \mathcal{E}_{m\times n}$ and $V = E_n$, then we have $A = U \otimes \Sigma \otimes V^T$, $U^T \otimes U = E_m$, $V^T \otimes V = E_n$ and $\sigma_1 = \sigma_2 = \ldots = \sigma_r = \varepsilon = \|A\|_{\oplus}$. So $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A.

From now on we assume that $c \neq \varepsilon$. If we define a matrix-valued function $\tilde{A}(\cdot) = \mathcal{F}(A,\cdot)$ then $\tilde{a}_{ij}(s) = \gamma_{ij} e^{c_{ij}s}$ with $\gamma_{ij} \in \mathbb{R}_0$ and $c_{ij} = |a_{ij}|_{\oplus} \in \mathbb{R}_{\varepsilon}$. Now we define a matrix-valued function $\tilde{D}(\cdot)$ such that $\tilde{D}(s) = e^{-cs}\tilde{A}(s)$. The entries of $\tilde{D}(s)$ can then be written as $\tilde{d}_{ij}(s) = \delta_{ij} e^{-d_{ij}s}$ with

$$\begin{array}{llll} \delta_{ij} = \gamma_{ij} & \text{and} & d_{ij} = c - c_{ij} \geq 0 & \text{if } c_{ij} \neq \varepsilon \,, \\ \delta_{ij} = 0 & \text{and} & d_{ij} = 0 & \text{if } c_{ij} = \varepsilon \,. \end{array}$$

Hence, $\delta_{ij}, d_{ij} \in \mathbb{R}$ and $d_{ij} \geq 0$ for all i,j.
Let $I \subset \mathbb{R}$. Then $\tilde{U}(s)\,\tilde{\Sigma}(s)\,\tilde{V}^T(s)$ is a (constant) SVD of $\tilde{A}(s)$ for each $s \in I$ if and only if $\tilde{U}(s)\,\tilde{\Psi}(s)\,\tilde{V}^T(s)$ with $\tilde{\Psi}(s) = e^{-cs}\tilde{\Sigma}(s)$ is a (constant) SVD of $\tilde{D}(s)$ for each $s \in I$.
Now we have to distinguish between two different situations depending on whether or not all the d_{ij}'s are rational.

Case 1: all the d_{ij}'s are rational.
Then there exists a positive rational number β such that

$$\forall i,j : \exists n_{ij} \in \mathbb{N} \text{ such that } d_{ij} = n_{ij}\beta \,. \tag{19}$$

Now we apply the substitution $z = e^{-\beta s}$. So $z \to 0^+$ if $s \to \infty$. We define a real m by n matrix-valued function $\hat{D}(\cdot)$ such that $\hat{d}_{ij}(z) = \delta_{ij} z^{n_{ij}}$ for all i,j. The entries of $\hat{D}(\cdot)$ are analytic in $\mathbb{R}$ and by Theorem 3 there exists an ASVD of $\hat{D}(\cdot)$ on $\mathbb{R}$.
Consider an arbitrary ASVD $\hat{U}(\cdot)\,\hat{\Psi}(\cdot)\,\hat{V}^T(\cdot)$ of $\hat{D}(\cdot)$. The singular values and the entries of the singular vectors of this ASVD are analytic in $z = 0$. Let $\hat{\psi}_i(\cdot) = (\hat{\Psi}(\cdot))_{ii}$. The $\hat{\psi}_i(\cdot)$'s are asymptotically equivalent to a power function in the neighborhood of 0 by Proposition 16. So there exists a neighborhood $(-\xi, \xi)$ of 0 that — except for 0 itself — contains no zeros of the analytic singular values that are not identically zero. Hence, there exists a real number η with $0 < \eta < \xi$ such that η is a generic point of $\hat{D}(\cdot)$. Note that η depends on β.
Now we define $D_\eta = D(\eta)$ and we consider an SVD $U_\eta\,\Psi_\eta\,V_\eta^T$ of D_η.

By Theorem 4 we know that there exists an ASVD $\hat{U}(\cdot)\,\hat{\Psi}(\cdot)\,\hat{V}^T(\cdot)$ of $\hat{D}(\cdot)$ on $\mathbb{R}$ such that $\hat{U}(\eta) = U_\eta$, $\hat{\Psi}(\eta) = \Psi_\eta$ and $\hat{V}(\eta) = V_\eta$. Since the singular values of $\hat{D}(\eta) = D_\eta$ are ordered and nonnegative and since the analytic singular values $\hat{\psi}_i(\cdot)$ are asymptotically equivalent to a power function, the analytic singular values are also ordered and nonnegative in some interval $(0, \zeta)$ with $0 < \zeta < \xi$. Therefore, $\hat{U}(z)\,\hat{\Psi}(z)\,\hat{V}^T(z)$ corresponds to an SVD of $\hat{D}(z)$ for each $z \in (0, \zeta)$.
Now we replace z by $e^{-\beta s}$. We define three matrix-valued functions $\tilde{U}(\cdot)$, $\tilde{\Sigma}(\cdot)$ and $\tilde{V}(\cdot)$ such that $\tilde{U}(s) = \hat{U}(e^{-\beta s})$, $\tilde{\Psi}(s) = \hat{\Psi}(e^{-\beta s})$ and $\tilde{V}(s) = \hat{V}(e^{-\beta s})$. Since $\tilde{D}(s) = \hat{D}(e^{-\beta s})$ and since $\hat{U}(\cdot)$, $\hat{\Psi}(\cdot)$, $\hat{V}(\cdot)$ and the function defined by $z = e^{-\beta s}$ are analytic in $\mathbb{R}$ and since an analytic function of an analytic function is also analytic, $\tilde{U}(\cdot)\,\tilde{\Psi}(\cdot)\,\tilde{V}^T(\cdot)$ is an ASVD of $\tilde{D}(\cdot)$ on $\mathbb{R}$.
Let K be a real number such that $K > \dfrac{-\log\zeta}{\beta}$. Since $0 < z < \zeta$ corresponds to $e^{-\beta s} < \zeta$ or $-\beta s < \log\zeta$ or $s > \dfrac{-\log\zeta}{\beta}$, the analytic singular values $\tilde{\psi}_i(\cdot)$ are ordered and nonnegative on $[K, \infty)$. Hence, $\tilde{U}(s)\,\tilde{\Psi}(s)\,\tilde{V}^T(s)$ corresponds to a (constant) SVD of $\tilde{D}(s)$ for each $s \in [K, \infty)$.
Since the diagonal entries of $\hat{\Psi}(\cdot)$ and the entries of $\hat{U}(\cdot)$ and $\hat{V}(\cdot)$ are asymptotically equivalent to a power function in the neighborhood of 0 by Proposition 16, we have

$$\tilde{\psi}_i(s) \sim \psi_{i,k_i} e^{-k_i \beta s}\ ,\ s \to \infty \tag{20}$$
$$\tilde{u}_{ij}(s) \sim u_{ij,l_{ij}} e^{-l_{ij} \beta s}\ ,\ s \to \infty \tag{21}$$
$$\tilde{v}_{ij}(s) \sim v_{ij,m_{ij}} e^{-m_{ij} \beta s}\ ,\ s \to \infty \tag{22}$$

for some $k_i, l_{ij}, m_{ij} \in \mathbb{N}$. If $\psi_{i,k_i} = 0$ then we set ψ_{i,k_i} equal to 1 and k_i equal to ∞ (so that $-k_i\beta$ becomes ε). If we also redefine l_{ij}, $u_{i,l_{ij}}$, m_{ij} and $v_{i,m_{ij}}$ in an analogous way then we can say that all the analytic singular values and all the entries of the analytic singular vectors are asymptotically equivalent to an exponential of the form αe^{as} with $\alpha \in \mathbb{R}_0$ and $a \in \mathbb{R}_\varepsilon$ in the neighborhood of ∞. The redefined exponents satisfy $-k_1\beta \geq -k_2\beta \geq \ldots \geq -k_r\beta \geq \varepsilon$ since the $\tilde{\psi}_i(\cdot)$'s are ordered in $[K, \infty)$.
So if all the entries of D are rational then we have proved that there exists a real number K and an ASVD of $\tilde{D}(\cdot)$ that corresponds to a constant SVD for each $s \in [K, \infty)$ and for which the singular values and the entries of the singular vectors are asymptotically equivalent to an exponential in the neighborhood of ∞.

Case 2: not all the d_{ij}'s are rational.

In general it is now not possible anymore to find a positive real number β such that (19) holds. Since a real function f defined by $f(z) = z^r$ is only analytic in a neighborhood of 0 if $r \in \mathbb{N}$, this means that we cannot use the same reasoning as for the rational case. Therefore, we construct a sequence of m by n matrices Q_k and a corresponding sequence of matrix-valued functions $F_k(\cdot)$ such that

$$(Q_k)_{ij} \in \mathbb{Q} \tag{23}$$
$$(Q_k)_{ij} \geq d_{ij} \quad \text{if} \quad d_{ij} > 0 \tag{24}$$
$$(Q_k)_{ij} = 0 \quad \text{if} \quad d_{ij} = 0 \tag{25}$$
$$\lim_{k \to \infty} (Q_k)_{ij} \; = \; d_{ij} \tag{26}$$
$$(F_k(s))_{ij} \; = \; \delta_{ij} e^{-(Q_k)_{ij} s} \tag{27}$$

$F_k(\cdot)$ has the same generic rank as $\tilde{D}(\cdot)$, i.e. $F_k(s)$ and $\tilde{D}(s)$ have the same rank for almost all values of s. (28)

Note that $\lim_{s \to \infty} F_k(s) = \lim_{s \to \infty} \tilde{D}(s)$ by (24), (25) and (27). From the first part of this proof we know that for each $F_k(\cdot)$ there exists a real number K_k and an ASVD $U_k(\cdot)\, \Psi_k(\cdot)\, V_k^T(\cdot)$ that corresponds to a (constant) SVD of $F_k(s)$ for each $s \in [K_k, \infty)$.

First we prove that the sequence of functions $\{F_k(\cdot)\}_{k=0}^{\infty}$ converges uniformly to $\tilde{D}(\cdot)$ in some interval $[L, \infty)$.

If we define $L = \max_{i,j} \left\{ \frac{1}{d_{ij}} \,\middle|\, d_{ij} \neq 0 \right\}$ then $L \in \mathbb{R}$. If we take (24) and (25) into account then we have

$$\forall k \in \mathbb{N}, \forall s \geq L : \; \| F_k(s) - \tilde{D}(s) \|_{\mathrm{F}} \; \leq \; \| F_k(L) - \tilde{D}(L) \|_{\mathrm{F}} \tag{29}$$

by Lemma 19. Furthermore, the sequence $\{F_k(L)\}_{k=0}^{\infty}$ converges to $\tilde{D}(L)$, i.e.

$$\forall \delta > 0, \exists M \in \mathbb{N} \text{ such that}$$
$$\forall k \in \mathbb{N} \text{ with } k \geq M : \; \left\| F_k(L) - \tilde{D}(L) \right\|_{\mathrm{F}} < \delta \; .$$

If we combine this with (29) we get

$$\forall \delta > 0, \exists M \in \mathbb{N} \text{ such that}$$
$$\forall k \in \mathbb{N} \text{ with } k \geq M : \forall s \geq L : \; \left\| F_k(s) - \tilde{D}(s) \right\|_{\mathrm{F}} < \delta$$

which means that the sequence $\{F_k(\cdot)\}_{k=0}^{\infty}$ converges uniformly to $\tilde{D}(\cdot)$ in $[L, \infty)$. This also means that

$$\forall \delta > 0, \exists M \in \mathbb{N} \text{ such that } \forall k, l \in \mathbb{N} \text{ with } k, l \geq M,$$
$$\forall s \geq L : \; \|F_k(s) - F_l(s)\|_{\mathrm{F}} < \delta \; . \tag{30}$$

Now we show that there exists a subsequence $\{\Psi_{k_p}(\cdot)\}_{p=0}^{\infty}$ of the sequence $\{\Psi_k(\cdot)\}_{k=0}^{\infty}$ that also converges uniformly in some interval $[P,\infty)$. We already know that the functions $(\Psi_k(\cdot))_{ii}$ are nonnegative and ordered in some interval $[K_k,\infty)$. Note that all the $F_k(\cdot)$'s and $\tilde{D}(\cdot)$ have the same number of singular values that are identically zero since they all have the same generic rank. Proposition 17 gives us an upper bound for the change in the singular values if the entries of a matrix are perturbed. So if we take a fixed value of s then the differences between the (constant) singular values of $F_k(s)$ and $F_l(s)$ become smaller and smaller as k and l become larger. Furthermore, the (constant) singular values of a matrix are unique and the analytic singular values in s are equal to the (constant) singular values up to the ordering and the signs. Since there is only a finite number of possible permutations and sign changes, we can always construct a subsequence of ASVDs $\{\, U_{k_p}(\cdot)\, \Psi_{k_p}(\cdot)\, V_{k_p}^T(\cdot)\,\}_{p=1}^{\infty}$ for which the differences between the corresponding entries of $\Psi_{k_p}(\cdot)$ and $\Psi_{k_q}(\cdot)$ become smaller and smaller as p and q become larger. This also means that the difference between K_{k_p} and K_{k_q} becomes smaller and smaller as p and q become larger and that the sequence $\{K_{k_p}\}_{p=1}^{\infty}$ will have a finite limit K_∞. Let $P = \max(L, K_\infty)$.
Since each $\Psi_{k_p}(s)$ corresponds to a constant SVD for a fixed value of $s \in [P,\infty)$, we have

$$\forall p,q \in \mathbb{N} : \left| \left(\Psi_{k_p}(s)\right)_{ii} - \left(\Psi_{k_q}(s)\right)_{ii} \right| \;\leq\; \left\| F_{k_p}(s) - F_{k_q}(s) \right\|_{\mathrm{F}}$$

for $i = 1,2,\ldots,r$ by Proposition 17. If we combine this with (30), we can conclude that the sequence $\{\Psi_{k_p}(\cdot)\}_{p=0}^{\infty}$ converges uniformly to a matrix-valued function $\tilde{\Psi}(\cdot)$ on $[P,\infty)$. Since the functions $\Psi_{k_p}(\cdot)$ are continuous on $[P,\infty)$ this means that $\tilde{\Psi}(\cdot)$ is also continuous on $[P,\infty)$. Furthermore, since the analytic singular values $\left(\Psi_{k_p}(\cdot)\right)_{ii}$ are nonnegative, ordered and asymptotically equivalent to an exponential in the neighborhood of ∞, the diagonal entries of $\tilde{\Psi}(\cdot)$ are also nonnegative, ordered and asymptotically equivalent to an exponential in the neighborhood of ∞.

Now we consider the singular vectors. Unfortunately, for the singular vectors there does not exist a perturbation property similar to that of Proposition 17 since if there are multiple singular values a small perturbation of the entries of the matrix may cause radical changes in the singular vectors [12, 15].
Therefore, we first use the selection principle of Lemma 18 to construct a subsequence $\{U_{l_p}(\cdot)\}_{p=0}^{\infty}$ of $\{U_{k_p}(\cdot)\}_{p=0}^{\infty}$ and a subsequence $\{V_{l_p}(\cdot)\}_{p=0}^{\infty}$ of $\{V_{k_p}(\cdot)\}_{p=0}^{\infty}$ such that both $\{U_{l_p}(K)\}_{p=0}^{\infty}$ and $\{V_{l_p}(K)\}_{p=0}^{\infty}$ converge to an orthogonal matrix for some real number $K \geq P$. Consider two arbitrary indices l_p and l_q. If K is large enough then the difference

between two corresponding entries of $U_{l_p}(\cdot)$ and $U_{l_q}(\cdot)$ either grows or diminishes monotonically on $[K, \infty)$ since these entries are asymptotically equivalent to an exponential in the neighborhood of ∞. This also holds for the entries of $V_{l_p}(\cdot)$ and $V_{l_q}(\cdot)$.
Now we select a new subsequence of $\{U_{l_p}(\cdot)\}_{p=0}^{\infty}$ and $\{V_{l_p}(\cdot)\}_{p=0}^{\infty}$ such that the absolute values of the differences between corresponding entries diminish monotonically on $[K, \infty)$. This can be done by applying the selection principle again, first on the sequence $\{U_{l_p}(Q)\}_{p=0}^{\infty}$ and then on the corresponding subsequence of $\{V_{l_p}(Q)\}_{p=0}^{\infty}$, with $Q \gg K$. Let the resulting new subsequences be given by $\{U_{m_p}(\cdot)\}_{p=0}^{\infty}$ and $\{V_{m_p}(\cdot)\}_{p=0}^{\infty}$. Then we have

$$\begin{aligned} &\forall s \geq K, \forall p, q \in \mathbb{N}, \forall i, j : \\ &\quad \left| \big(U_{m_p}(s)\big)_{ij} - \big(U_{m_q}(s)\big)_{ij} \right| \leq \left| \big(U_{m_p}(K)\big)_{ij} - \big(U_{m_q}(K)\big)_{ij} \right| \end{aligned}$$

and

$$\forall p, q \in \mathbb{N}, \forall i, j : \lim_{s \to \infty} \big(U_{m_p}(s)\big)_{ij} = \lim_{s \to \infty} \big(U_{m_q}(s)\big)_{ij} \quad .$$

Analogous expressions hold for the entries of $V_{m_p}(\cdot)$ and $V_{m_q}(\cdot)$.
So the sequence $\{U_{m_p}(\cdot)\}_{p=0}^{\infty}$ converges uniformly to a matrix-valued function $\tilde{U}(\cdot)$ in $[K, \infty)$. Therefore, $\tilde{U}(\cdot)$ is continuous in $[K, \infty)$ and its entries are also asymptotically equivalent to an exponential in the neighborhood of ∞. Furthermore, $\tilde{U}(s)$ is orthogonal for each $s \in [K, \infty)$. This also holds for $\tilde{V}(\cdot) = \lim\limits_{p \to \infty} V_{m_p}(\cdot)$.
Hence, $\tilde{U}(\cdot)\, \tilde{\Psi}(\cdot)\, \tilde{V}^T(\cdot)$ is a "*continuous* SVD" of $\tilde{D}(\cdot)$ on $[K, \infty)$ for which the singular values and the entries of the singular vectors are asymptotically equivalent to an exponential in the neighborhood of ∞. Note that we have not proved that $\tilde{U}(\cdot)\, \tilde{\Psi}(\cdot)\, \tilde{V}^T(\cdot)$ is an *analytic* SVD of $\tilde{D}(\cdot)$ since this is not necessary for the remainder of the proof.

Now we define a matrix-valued function $\tilde{\Sigma}(\cdot)$ such that $\tilde{\Sigma}(s) = e^{cs}\, \tilde{\Psi}(s)$. Then $\tilde{U}(s)\, \tilde{\Sigma}(s)\, \tilde{V}^T(s)$ is a constant SVD of $\tilde{A}(s)$ for each $s \in [K, \infty)$:

$$\tilde{A}(s) = \tilde{U}(s)\, \tilde{\Sigma}(s)\, \tilde{V}^T(s) \tag{31}$$
$$\tilde{U}^T(s)\, \tilde{U}(s) = I_m \tag{32}$$
$$\tilde{V}^T(s)\, \tilde{V}(s) = I_n \tag{33}$$

and the entries of $\tilde{U}(\cdot)$, $\tilde{\Sigma}(\cdot)$ and $\tilde{V}(\cdot)$ are asymptotically equivalent to an exponential in the neighborhood of ∞. Furthermore, the singular values $\tilde{\sigma}_i(\cdot) \stackrel{\text{def}}{=} (\tilde{\Sigma}(\cdot))_{ii}$ are nonnegative and their dominant exponents are ordered. Now we use the reverse mapping $\mathcal{R}$ to obtain a max-algebraic SVD of A.

Since we have used numbers instead of parameters for the coefficients of the exponentials in $\mathcal{F}(A,\cdot)$, the coefficients of the exponentials in the singular values and the entries of the singular vectors are also numbers. Therefore, the reverse mapping will only yield signed results.
If we define

$$\Sigma = \mathcal{R}\Big(\tilde{\Sigma}(\cdot)\Big)\ ,\ U = \mathcal{R}\Big(\tilde{U}(\cdot)\Big)\ ,\ V = \mathcal{R}\Big(\tilde{V}(\cdot)\Big)\ \text{ and } \sigma_i = (\Sigma)_{ii} = \mathcal{R}(\tilde{\sigma}_i(\cdot))$$

then Σ is a max-algebraic diagonal matrix since its off-diagonal entries are equal to ε, and U and V have signed entries. Furthermore, (31) – (33) result in

$$\begin{aligned} A &\ \nabla\ U \otimes \Sigma \otimes V^T \\ U^T \otimes U &\ \nabla\ E_m \\ V^T \otimes V &\ \nabla\ E_n\ . \end{aligned}$$

We have $\|\tilde{A}(s)\|_{\mathrm{F}} \sim \gamma e^{cs}$, $s \to \infty$ with $\gamma > 0$ since $c = \|A\|_\oplus$ is the largest exponent that appears in the entries of $\tilde{A}(\cdot)$. So $\mathcal{R}\Big(\|\tilde{A}(\cdot)\|_{\mathrm{F}}\Big) = c = \|A\|_\oplus$.
By (1) we have

$$\frac{1}{\sqrt{n}}\,\|\tilde{A}(s)\|_{\mathrm{F}} \ \leq\ \|\tilde{A}(s)\|_2 \ \leq\ \|\tilde{A}(s)\|_{\mathrm{F}} \qquad \text{for all } s \in \mathbb{R}\ .$$

Since $\tilde{\sigma}_1(s) = \|\tilde{A}(s)\|_2$ for $s \geq K$ and since the mapping $\mathcal{R}$ preserves the order, this leads to $\|A\|_\oplus \leq \sigma_1 \leq \|A\|_\oplus$ and consequently,

$$\sigma_1 = \|A\|_\oplus\ . \tag{34}$$

The singular values $\tilde{\sigma}_i(\cdot)$ are nonnegative and ordered in $[K,\infty)$. Hence, $\sigma_i \in \mathbb{R}_\varepsilon$ for $i = 1,2,\ldots,r$ and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq \varepsilon$. ■

Proposition 21. *Let $A \in \mathbb{S}^{m\times n}$. There always exists a max-algebraic SVD $U \otimes \Sigma \otimes V^T$ of A for which $\sigma_1 = \|A\|_\oplus$.*

Proof. This was already proved in the proof of Theorem 20 (cf. equation (34)). ■

If $A \in \mathbb{S}^{m\times n}$ and if $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A then U is a signed square m by m matrix that satisfies $U^T \otimes U\ \nabla\ E_m$. We shall now prove some properties of this kind of matrices.

Proposition 22. *Consider $U \in (\mathbb{S}^\vee)^{m\times m}$. If $U^T \otimes U\ \nabla\ E_m$ then we have $\|u_i\|_\oplus = 0$ for $i = 1,2,\ldots,m$.*

Proof. Since $U^T \otimes U \nabla E_m$, we have $(U^T \otimes U)_{ii} \nabla 0$ for $i = 1, 2, \ldots, m$. Hence,

$$\bigoplus_{k=1}^{m} {u_{ki}}^{\otimes^2} \nabla 0 \qquad \text{for } i = 1, 2, \ldots, m \ . \tag{35}$$

We have

$$\begin{aligned} {u_{ki}}^{\otimes^2} &= (u_{ki}^{\oplus} \ominus u_{ki}^{\ominus})^{\otimes^2} \\ &= (u_{ki}^{\oplus})^{\otimes^2} \ominus u_{ki}^{\oplus} \otimes u_{ki}^{\ominus} \ominus u_{ki}^{\ominus} \otimes u_{ki}^{\oplus} \oplus (u_{ki}^{\ominus})^{\otimes^2} \\ &= (u_{ki}^{\oplus})^{\otimes^2} \oplus (u_{ki}^{\ominus})^{\otimes^2} \end{aligned}$$

since the entries of U are signed and thus $u_{ki}^{\oplus} = \varepsilon$ or $u_{ki}^{\ominus} = \varepsilon$. So ${u_{ki}}^{\otimes^2}$ is also signed, which means that both sides of the balance (35) are signed. By Proposition 8 this leads to

$$\bigoplus_{k=1}^{m} \left((u_{ki}^{\oplus})^{\otimes^2} \oplus (u_{ki}^{\ominus})^{\otimes^2} \right) = 0 \qquad \text{for } i = 1, 2, \ldots, m \ .$$

Since $\forall x, y \in \mathbb{R}_\varepsilon : (x \oplus y)^{\otimes^2} = x^{\otimes^2} \oplus y^{\otimes^2}$, this results in

$$\bigoplus_{k=1}^{m} (u_{ki}^{\oplus} \oplus u_{ki}^{\ominus})^{\otimes^2} = 0 \qquad \text{for } i = 1, 2, \ldots, m \ . \tag{36}$$

If $x \in \mathbb{R}_\varepsilon$ then $x^{\otimes^2}$ is equal to $2 \cdot x$ in linear algebra. Therefore, (36) is equivalent to

$$\bigoplus_{k=1}^{m} (u_{ki}^{\oplus} \oplus u_{ki}^{\ominus}) = 0 \qquad \text{for } i = 1, 2, \ldots, m$$

and this results in $\|u_i\|_{\oplus} = 0$ for $i = 1, 2, \ldots, m$. ∎

Corollary 23. *Consider $U \in (\mathbb{S}^\vee)^{m \times m}$. If $U \otimes U^T \nabla E_m$ then we have $|u_{ij}|_{\oplus} \leq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$.*

Now we can show why we really need the extended max algebra $\mathbb{S}_{\max}$ to define the max-algebraic singular value decomposition: the class of matrices (with entries in $\mathbb{R}_\varepsilon$) that have max-algebraic SVD in which U and V have only entries in $\mathbb{R}_\varepsilon$ is rather limited. The matrix $U \in \mathbb{R}_\varepsilon^{m \times m}$ then has to satisfy $U^T \otimes U \nabla E_m$ or equivalently $U^T \otimes U = E_m$ since the entries of $U^T \otimes U$ are signed. In other words, U should be invertible in $\mathbb{R}_{\max}$. It can be shown [4] that the only matrices that are invertible in $\mathbb{R}_{\max}$ are matrices

of the form $D \otimes P$ where D is a square max-algebraic diagonal matrix with non-ε diagonal entries and P is a max-algebraic permutation matrix (i.e. a square matrix with exactly one 0 entry in each row and in each column and where the other entries are equal to ε). So $U = D_1 \otimes P_1$ and $V = D_2 \otimes P_2$ where D_1 and D_2 are square max-algebraic diagonal matrices with non-ε diagonal entries and where P_1 and P_2 are max-algebraic permutation matrices. Since the max-algebraic norm of the columns of U and V is equal to 0 by Proposition 22 the diagonal entries of D_1 and D_2 have to be equal to 0, which means that $D_1 = E_m$ and $D_2 = E_n$. As a consequence we have $A = U \otimes \Sigma \otimes V^T = P_1 \otimes \Sigma \otimes P_2^T$. Hence, A has to be a permuted max-algebraic diagonal matrix.
So only permuted max-algebraic diagonal matrices with entries in $\mathbb{R}_\varepsilon$ have a max-algebraic SVD with entries in $\mathbb{R}_\varepsilon$. This could be compared with the class of real matrices in linear algebra that have an SVD with only nonnegative entries: using analogous reasoning one can prove that this class is the set of real permuted diagonal matrices. Furthermore, it is obvious that each SVD in $\mathbb{R}_{\max}$ is also an SVD in $\mathbb{S}_{\max}$.

From Theorem 20 we know that the max-algebraic singular values of a matrix A are bounded from above since the largest max-algebraic singular value σ_1 is less than or equal to $\|A\|_\oplus$. Furthermore, by Proposition 21 there always exists a max-algebraic SVD for which σ_1 is equal to this upper bound. The following proposition tells us when the upper bound for σ_1 is tight for all the max-algebraic SVDs of A:

PROPOSITION 24. *Consider $A \in \mathbb{S}^{m \times n}$. If there is at least one signed entry in A that is equal to $\|A\|_\oplus$ in max-absolute value then $\sigma_1 = \|A\|_\oplus$ for every max-algebraic SVD of A.*

Proof. Consider an arbitrary max-algebraic SVD of A: $A \mathrel{\nabla} U \otimes \Sigma \otimes V^T$. If we extract the max-positive and the max-negative part of each matrix, we get

$$A^\oplus \ominus A^\ominus \ \nabla \ (U^\oplus \ominus U^\ominus) \otimes \Sigma \otimes (V^\oplus \ominus V^\ominus)^T \ .$$

Using Proposition 10 this balance can be rewritten as

$$\begin{aligned} A^\oplus \ \oplus \ U^\oplus \otimes \Sigma \otimes (V^\ominus)^T \ \oplus \ U^\ominus \otimes \Sigma \otimes (V^\oplus)^T \quad \nabla \\ A^\ominus \ \oplus \ U^\oplus \otimes \Sigma \otimes (V^\oplus)^T \ \oplus \ U^\ominus \otimes \Sigma \otimes (V^\ominus)^T \ . \end{aligned} \tag{37}$$

Both sides of this balance are signed and by Proposition 11 we can replace the balance by an equality. Let $r = \min(m, n)$ and let a_{pq} be the signed entry of A for which $|a_{pq}|_\oplus = \|A\|_\oplus$. If we select the equality that corresponds to the pth row and the qth column of (37), we get

$$a_{pq}^\oplus \ \oplus \ \bigoplus_{k=1}^{r} u_{pk}^\oplus \otimes \sigma_k \otimes v_{qk}^\ominus \ \oplus \ \bigoplus_{k=1}^{r} u_{pk}^\ominus \otimes \sigma_k \otimes v_{qk}^\oplus \ =$$

$$a_{pq}^{\ominus} \;\oplus\; \bigoplus_{k=1}^{r} u_{pk}^{\oplus} \otimes \sigma_k \otimes v_{qk}^{\oplus} \;\oplus\; \bigoplus_{k=1}^{r} u_{pk}^{\ominus} \otimes \sigma_k \otimes v_{qk}^{\ominus} \;. \tag{38}$$

First we assume that $a_{pq} \in \mathbb{S}^{\oplus}$ and consequently $a_{pq}^{\ominus} = \varepsilon$. The entries of U and V are less than or equal to 0 in max-absolute value by Corollary 23. Hence,

$$u_{pk}^{\oplus},\, u_{pk}^{\ominus},\, v_{qk}^{\oplus},\, v_{qk}^{\ominus} \;\leq\; 0 \qquad \text{for } k = 1, 2, \ldots, m \tag{39}$$

and thus,

$$u_{pk}^{\oplus} \otimes \sigma_k \otimes v_{qk}^{\ominus} \;\leq \sigma_k \leq \|A\|_{\oplus} \quad \text{and} \quad u_{pk}^{\ominus} \otimes \sigma_k \otimes v_{qk}^{\oplus} \;\leq \sigma_k \leq \|A\|_{\oplus}$$

for $k = 1, 2, \ldots, m$. So the left hand side of (38) is equal to $a_{pq}^{\oplus} = \|A\|_{\oplus}$, which means that there has to exist an index l such that

$$u_{pl}^{\oplus} \otimes \sigma_l \otimes v_{ql}^{\oplus} \;=\; a_{pq}^{\oplus} \quad \text{or} \quad u_{pl}^{\ominus} \otimes \sigma_l \otimes v_{ql}^{\ominus} \;=\; a_{pq}^{\oplus} \;.$$

Because of (39) this is only possible if $\sigma_l \geq a_{pq}^{\oplus} = \|A\|_{\oplus}$. Since $\|A\|_{\oplus} \geq \sigma_1 \geq \sigma_l$, this means that $\sigma_1 = \sigma_l = \|A\|_{\oplus}$.
If $a_{pq} \in \mathbb{S}^{\ominus}$, analogous reasoning also leads to $\sigma_1 = \|A\|_{\oplus}$. ■

Note that the condition of Proposition 24 is always satisfied if all the entries of the matrix A are signed. For a matrix A that does not satisfy the condition of Proposition 24 it is indeed possible that there exists a max-algebraic SVD for which the largest singular value is less than $\|A\|_{\oplus}$ as is shown by the following example:

EXAMPLE 25. Consider $A = \left[\; 0^{\bullet} \;\right]$. Then $0 \otimes \sigma \otimes 0$ is a max-algebraic SVD of A for every $\sigma \in \mathbb{R}_{\varepsilon}$ with $\sigma \leq 0 = \|A\|_{\oplus}$ since $0 \otimes \sigma \otimes 0 = \sigma \;\nabla\; 0^{\bullet}$ if $\sigma \leq 0$.

So in contrast to the singular values in linear algebra the max-algebraic singular values are not always unique. This leads to the definition of a *maximal* max-algebraic SVD – where we take all the singular values as large as possible – and a *minimal* max-algebraic SVD – where we take all the singular values as small as possible. The maximal max-algebraic SVD of the matrix A of Example 25 is given by $0 \otimes 0 \otimes 0$ and the minimal max-algebraic SVD is given by $0 \otimes \varepsilon \otimes 0$.

PROPOSITION 26. *Let $A \in \mathbb{S}^{m \times n}$. If $U \otimes \Sigma_{\max} \otimes V^T$ is a maximal max-algebraic SVD of A, then $\sigma_{\max,1} \overset{\text{def}}{=} (\Sigma_{\max})_{11} = \|A\|_{\oplus}$.*

Proof. The definition of the max-algebraic SVD yields an upper bound for $\sigma_{\max,1}$: $\sigma_{\max,1} \leq \|A\|_{\oplus}$ and Proposition 21 tells us that this upper bound is tight. ■

For more information on the max-algebraic SVD, extra properties and possible extensions the interested reader is referred to [6, 7].

The decomposition $A \nabla U \otimes \Sigma \otimes V^T$ can also be written as

$$A \;\nabla\; \bigoplus_{i=1}^{r} \sigma_i \otimes u_i \otimes v_i^T \tag{40}$$

where u_i is the ith column of U and v_i is the ith column of V.
It could be possible that some terms of the right hand side of (40) can be neglected because they are smaller than the other terms. This allows us to define a rank based on the max-algebraic SVD:

Definition 27. Let $A \in \mathbb{S}^{m\times n}$. The max-algebraic SVD rank of A is defined as

$$\operatorname{rank}_{\oplus,\mathrm{SVD}}(A) \;=\; \min \left\{ \rho \;\middle|\; A \;\nabla\; \bigoplus_{i=1}^{\rho} \sigma_i \otimes u_i \otimes v_i^T \,,\; U \otimes \Sigma \otimes V^T \text{ is a max-algebraic SVD of } A \right\}$$

where u_i is the ith column of U, v_i is the ith column of V and $\bigoplus_{i=1}^{0} \sigma_i \otimes u_i \otimes v_i^T$ is equal to $\mathcal{E}_{m\times n}$ by definition.

Let $A \in \mathbb{S}^{m\times n}$ and let $\rho_A = \operatorname{rank}_{\oplus,\mathrm{SVD}}(A)$. If $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A for which $A \nabla \bigoplus_{i=1}^{\rho_A} \sigma_i \otimes u_i \otimes v_i^T$, we can set σ_i with $i > \rho_A$ equal to ε since the corresponding terms can be neglected. So $\operatorname{rank}_{\oplus,\mathrm{SVD}}(A)$ is equal to the minimal number of non-ε singular values in the minimal max-algebraic SVDs of A. This also explains why we have used the condition $\sigma_1 \leq \|A\|_{\oplus}$ instead of $\sigma_1 = \|A\|_{\oplus}$ in Theorem 20: the latter condition would imply that the matrix A of Example 25 would have only one max-algebraic SVD: $0^\bullet \nabla 0 \otimes 0 \otimes 0$ with $\sigma_1 = 0 \neq \varepsilon$. So its minimal max-algebraic SVD would have one non-ε singular value. However, $0^\bullet \nabla \varepsilon$ and thus $\operatorname{rank}_{\oplus,\mathrm{SVD}}(A) = 0$ by Definition 27, which indeed corresponds to the minimal number of non-ε singular values in the minimal max-algebraic SVD of A if we use the condition $\sigma_1 \leq \|A\|_{\oplus}$ in the definition of the max-algebraic SVD.

We could use the max-algebraic SVD rank in the identification of a max-linear discrete event system from its impulse response:

Suppose that we have a single input single output discrete event system that can be described by an nth order max-algebraic state space model:

$$x(k+1) = A \otimes x(k) \oplus b \otimes u(k) \quad (41)$$
$$y(k) = c^T \otimes x(k) \quad (42)$$

with $A \in \mathbb{R}_\varepsilon^{n \times n}$ and $b, c \in \mathbb{R}_\varepsilon^n$ and where u is the input, y is the output and x is the state vector.
If we apply a unit impulse to the system and if we assume that the initial state $x(0)$ satisfies $x(0) = \mathcal{E}_{n \times 1}$, we get the impulse response as the output of the system. Since $x(0) = \mathcal{E}_{n \times 1}$ leads to

$$x(1) = b\,,\; x(2) = A \otimes b\,,\; \ldots\,,\; x(k) = A^{\otimes^{k-1}} \otimes b\,,\; \ldots\,,$$

the impulse response of the system is given by

$$y(k) = c^T \otimes A^{\otimes^{k-1}} \otimes b \qquad \text{for } k = 1, 2, \ldots\;.$$

Let $g_k = c^T \otimes A^{\otimes^k} \otimes b$ for $k = 0, 1, \ldots$. The g_k's are called the *Markov parameters*.
Suppose that A, b and c are unknown, and that we only know the Markov parameters (e.g. from experiments, where we assume that the system is time-invariant and max-linear – i.e. that it can be described by a state space model of the form (41) – (42) – and that there is no noise present). How can we construct A, b and c from the g_k's? This process is called realization. If we make the dimension of A minimal, we have a minimal realization.
The max-algebraic rank of the Hankel matrix

$$H = \begin{bmatrix} g_0 & g_1 & \cdots & g_q \\ g_1 & g_2 & \cdots & g_{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ g_p & g_{p+1} & \cdots & g_{p+q} \end{bmatrix}$$

with p and q large enough yields a lower bound for the minimal system order [8, 9, 10]. But in the presence of noise this Hankel matrix will almost always be of full rank. However, if we adapt Definition 27 so that we stop adding terms as soon as the matrix A is approximated accurately enough, we could use the max-algebraic SVD rank to get an estimate of the minimal system order of the discrete event system.

5. EXAMPLE

EXAMPLE 28. Consider $A = \begin{bmatrix} 2 & \ominus 5 \\ \ominus 0 & 3 \end{bmatrix}$. Note that the two columns a_1 and a_2 of this matrix are dependent since $a_2 = \ominus 3 \otimes a_1$.
We shall calculate the max-algebraic SVD of this matrix using the mapping $\mathcal{F}$. We define $\tilde{A}(\cdot) = \mathcal{F}(A, \cdot)$ where we take all the coefficients μ equal to 1:

$$\tilde{A}(s) = \begin{bmatrix} e^{2s} & -e^{5s} \\ -1 & e^{3s} \end{bmatrix} .$$

Since this is a 2 by 2 matrix, we can calculate the (constant) SVD of $\tilde{A}(s)$ for $s \in \mathbb{R}$ analytically, e.g. via the eigenvalue decomposition of $\tilde{A}^T(s)\tilde{A}(s)$ (cf. [11, 12]). This yields

$$\tilde{U}(s) = \begin{bmatrix} \dfrac{e^{2s}}{\sqrt{e^{4s}+1}} & \dfrac{1}{\sqrt{e^{4s}+1}} \\ \dfrac{-1}{\sqrt{e^{4s}+1}} & \dfrac{e^{2s}}{\sqrt{e^{4s}+1}} \end{bmatrix} \sim \begin{bmatrix} 1 & e^{-2s} \\ -e^{-2s} & 1 \end{bmatrix}, \ s \to \infty$$

$$\tilde{\Sigma}(s) = \begin{bmatrix} \sqrt{e^{10s}+e^{6s}+e^{4s}+1} & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} e^{5s} & 0 \\ 0 & 0 \end{bmatrix}, \ s \to \infty$$

$$\tilde{V}(s) = \begin{bmatrix} \dfrac{1}{\sqrt{e^{6s}+1}} & \dfrac{e^{3s}}{\sqrt{e^{6s}+1}} \\ \dfrac{-e^{3s}}{\sqrt{e^{6s}+1}} & \dfrac{1}{\sqrt{e^{6s}+1}} \end{bmatrix} \sim \begin{bmatrix} e^{-3s} & 1 \\ -1 & e^{-3s} \end{bmatrix}, \ s \to \infty \ .$$

Note that $\tilde{U}(\cdot)\,\tilde{\Sigma}(\cdot)\,\tilde{V}^T(\cdot)$ is an ASVD of $\tilde{A}(\cdot)$ since all the entries of $\tilde{U}(\cdot)$, $\tilde{\Sigma}(\cdot)$ and $\tilde{V}(\cdot)$ are analytic. If we apply the reverse mapping $\mathcal{R}$, we get the following max-algebraic SVD of A:

$$A \ \nabla \ \begin{bmatrix} 0 & -2 \\ \ominus(-2) & 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} -3 & 0 \\ \ominus 0 & -3 \end{bmatrix}^T = \begin{bmatrix} 2 & \ominus 5 \\ \ominus 0 & 3 \end{bmatrix} .$$

In [7] we have developed another method to calculate all the max-algebraic SVDs of a matrix, without making use of the mapping $\mathcal{F}$. However, in its present form this technique is only suited to calculate the max-algebraic SVD of small-sized matrices. Using this alternative method we find the following max-algebraic SVDs:

$$A \ \nabla \ \begin{bmatrix} 0 & -2 \\ \ominus(-2) & 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & \varepsilon \\ \varepsilon & \sigma_2 \end{bmatrix} \otimes \begin{bmatrix} -3 & 0 \\ \ominus 0 & -3 \end{bmatrix}^T \qquad (43)$$

with $\sigma_2 \leq 0$ or analogous decompositions but with u_2 replaced by $\ominus u_2$, or with v_2 replaced by $\ominus v_2$ or with u_1 and v_1 replaced by $\ominus u_1$ and $\ominus v_1$ respectively.
Note that $\sigma_1 = 5 = \|A\|_{\oplus}$ for all the max-algebraic SVDs (cf. Proposition 24). Taking $\sigma_2 = \varepsilon$ in (43) yields a minimal max-algebraic SVD of A. Since

$$\sigma_1 \otimes u_1 \otimes v_1^T = \begin{bmatrix} 2 & \ominus 5 \\ \ominus 0 & 3 \end{bmatrix} = A \ ,$$

we have $\operatorname{rank}_{\oplus,\mathrm{SVD}}(A) = 1$. If $\sigma_2 = \sigma_{\max,2} = 0$, we have a maximal max-algebraic SVD of A:

$$\sigma_{\max,2} \otimes u_2 \otimes v_2^T = \begin{bmatrix} -2 & -5 \\ 0 & -3 \end{bmatrix}$$

and

$$U \otimes \Sigma_{\max} \otimes V^T = \begin{bmatrix} 2 & \ominus 5 \\ 0^{\bullet} & 3 \end{bmatrix} \nabla A \ .$$

Note that the max-absolute value of every entry of $\sigma_{\max,2} \otimes u_2 \otimes v_2^T$ is smaller than or equal to the max-absolute value of the corresponding entry of $\sigma_1 \otimes u_1 \otimes v_1^T$.

6. CONCLUSIONS AND FUTURE RESEARCH

First we have established a link between the field of the real numbers and the extended max algebra. We have used this link to introduce the max-complex structure $\mathbb{T}_{\max}$, which can be considered as a further extension of the max algebra. We have also defined a kind of singular value decomposition (SVD) in the extended max algebra and proved its existence. Finally, we have defined a rank based on the max-algebraic SVD, which could be used in the identification of max-linear discrete event systems.

Future research topics will include: further investigation of the properties of the SVD in the extended max algebra, development of efficient algorithms to calculate the (minimal) max-algebraic SVD of a matrix and application of the max-algebraic SVD in the system theory for max-linear discrete event systems. Furthermore, it is obvious that many other decompositions and properties of matrices in linear algebra also have a max-algebraic analogue, especially if we make use of the correspondence between $(\mathbb{C}, +, \times)$ and $\mathbb{T}_{\max}$. This will also be a topic for further research.

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