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MINIMAL STATE SPACE REALIZATION OF MIMO SYSTEMS IN THE MAX ALGEBRA*

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Abstract

The topic of this paper is the (partial) minimal realization problem in the max algebra, which is one of the modeling frameworks that can be used to model discrete event systems. We use the fact that a system of multivariate max-algebraic polynomial equalities can be transformed into an Extended Linear Complementarity Problem to find all equivalent minimal state space realizations of a multiple input multiple output (MIMO) max-linear discrete event system starting from its impulse response matrices. We also give a geometrical description of the set of all minimal state space realizations.

1 Introduction

1.1 Overview

In this paper we consider discrete event systems, examples of which are flexible manufacturing systems, subway traffic networks, parallel processing systems, telecommunication networks, ... There exists a wide range of frameworks to model and analyze discrete event systems: Petri nets, generalized semi-Markov processes, formal languages, perturbation analysis, computer simulation and so on. We concentrate on a subclass of discrete event systems that can be described with the max algebra [1, 2, 3]. Although the description of these systems is non-linear in linear algebra, the model becomes 'linear' when we formulate it in the max algebra. In this paper we only consider systems that can be described by a time-invariant state space model. Therefore, we limit ourselves to deterministic systems, i.e. systems in which the sequence and the durations of the activities are fixed or can be determined in advance.

In order to analyze systems it is advantageous to have a compact description, i.e. a description with as few parameters as possible. For a system that can be described by a max-linear state space model this gives rise to the minimal state space realization problem. In this paper we address the (partial) minimal state space realization problem for max-algebraic multiple input multiple output (MIMO) systems. First we discuss the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities and then we use the results to solve the minimal state space realization problem for MIMO max-linear discrete event systems. We also give a geometric characterization of the set of all minimal state space realizations and illustrate the procedure with an example.

1.2 The max algebra

In this section we give a short introduction to the max algebra and state the definitions, theorems and properties that we need in the remainder of this paper. A more complete overview of the max algebra can be found in [1, 3]. The basic max-algebraic operations are defined as follows:

$$
\begin{aligned}
a \oplus b &= \max(a, b) \\
a \otimes b &= a + b
\end{aligned}
$$

where $a, b \in \mathbb{R} \cup \{-\infty\}$. The resulting structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the max algebra. The zero element for $\oplus$ is $\varepsilon \stackrel{\text{def}}{=} -\infty : \forall a \in \mathbb{R} \cup \{\varepsilon\} : a \oplus \varepsilon = a = \varepsilon \oplus a$. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$.

Let $r \in \mathbb{R}$. The rth max-algebraic power of $a \in \mathbb{R}$ is represented by $a^{\otimes^r}$ and corresponds to ra in linear algebra. Hence, $a^{\otimes^0} = 0$ and $a^{\otimes^{-1}} = -a$ is the inverse element of a w.r.t. $\otimes$ in $\mathbb{R}_\varepsilon$. If $r > 0$ then $\varepsilon^{\otimes^r} = \varepsilon$; if $r \leqslant 0$ then $\varepsilon^{\otimes^r}$ is not defined.

The max-algebraic operations are extended to matrices

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in the usual way. If $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ then $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$. If $A \in \mathbb{R}_\varepsilon^{m\times p}$ and $B \in \mathbb{R}_\varepsilon^{p\times n}$ then $(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$. The matrix E_n is the n by n identity matrix in the max algebra: $(E_n)_{ij} = 0$ if $i = j$ and $(E_n)_{ij} = \varepsilon$ if $i \neq j$. The m by n max-algebraic zero matrix is represented by $\mathcal{E}_{m\times n}$: $(\mathcal{E}_{m\times n})_{ij} = \varepsilon$ for all i, j.

In contrast to linear algebra, there exist no inverse elements w.r.t. $\oplus$ in $\mathbb{R}_\varepsilon$. To overcome this problem we shall use the extended max algebra $\mathbb{S}_{\max}$ [8, 10], which is a kind of symmetrization of the max algebra. We shall restrict ourselves to an intuitive introduction to the most important features of $\mathbb{S}_{\max}$. For a formal derivation and for the proofs of the properties and theorems of this section the interested reader is referred to [1, 8, 10].
We introduce two new elements for each element $x \in \mathbb{R}_\varepsilon$: $\ominus x$ and $x^\bullet$. This gives rise to an extension $\mathbb{S}$ of $\mathbb{R}_\varepsilon$ that contains three classes of elements:

- the max-positive or zero elements: $\mathbb{S}^\oplus \equiv \mathbb{R}_\varepsilon$
- the max-negative or zero elements: $\mathbb{S}^\ominus = \{ \ominus a \mid a \in \mathbb{R}_\varepsilon \}$
- the balanced elements: $\mathbb{S}^\bullet = \{ a^\bullet \mid a \in \mathbb{R}_\varepsilon \}$

where $\mathbb{S} = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$. By definition we have $\varepsilon = \ominus\varepsilon = \varepsilon^\bullet$. The elements of $\mathbb{S}^\oplus$ and $\mathbb{S}^\ominus$ are called signed.
The $\oplus$ operation between an element of $\mathbb{S}^\oplus$ and an element of $\mathbb{S}^\ominus$ is defined as follows:

$$
\begin{aligned}
a \oplus (\ominus b) &= a && \text{if } a > b\,, \\
a \oplus (\ominus b) &= \ominus b && \text{if } a < b\,, \\
a \oplus (\ominus a) &= a^\bullet
\end{aligned}
$$

where $a,\, b \in \mathbb{R}_\varepsilon$. The $\ominus$ sign corresponds to the $-$ sign in linear algebra. By analogy we write $a \ominus b$ instead of $a \oplus (\ominus b)$. We have

$$
\begin{aligned}
\ominus(\ominus a) &= a \\
(\ominus a) \oplus (\ominus b) &= \ominus(a \oplus b) \\
(\ominus a) \otimes b = a \otimes (\ominus b) &= \ominus(a \otimes b)
\end{aligned}
$$

for $a,\, b \in \mathbb{S}$. If one of the operands is balanced, we evaluate the expression as follows:

$$
\begin{aligned}
a \oplus b^\bullet &= a \oplus (b \oplus (\ominus b)) \\
a \otimes b^\bullet &= a \otimes (b \oplus (\ominus b))
\end{aligned}
$$

and we use the fact that both $\oplus$ and $\otimes$ are associative and commutative in $\mathbb{S}$ and that $\otimes$ is distributive w.r.t. $\oplus$ in $\mathbb{S}$. The resulting structure $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$ is called the extended max algebra.

Let $a \in \mathbb{S}$. The max-positive part a^+ and the max-negative part a^- of a are defined as follows:

- if $a \in \mathbb{R}_\varepsilon$ then $a^+ = a$ and $a^- = \varepsilon$
- if $a \in \mathbb{S}^\ominus$ then $a^+ = \varepsilon$ and $a^- = \ominus a$
- if $a \in \mathbb{S}^\bullet$ then there exists an element $b \in \mathbb{R}_\varepsilon$ such that $a = b^\bullet$ and then $a^+ = a^- = b$.

So $a^+, a^- \in \mathbb{R}_\varepsilon$ and $a = a^+ \ominus a^-$.
In linear algebra we have $a - a = 0$ for all $a \in \mathbb{R}$ but in $\mathbb{S}_{\max}$ we have $a \ominus a = a^\bullet \neq \varepsilon$ (unless $a = \varepsilon$). Therefore, we introduce a new relation, the balance relation, represented by ∇.

Definition 1.1 (Balance) *Consider $a, b \in \mathbb{S}$. We say that a balances b, denoted by $a \,\nabla\, b$, if and only if $a^+ \oplus b^- = b^+ \oplus a^-$.*

Property 1.2 $\forall a^\bullet \in \mathbb{S}^\bullet : a^\bullet = a \ominus a \,\nabla\, \varepsilon$.

We could say that the balance relation is the $\mathbb{S}_{\max}$ counterpart of the equality relation. However, the balance relation is not an equivalence relation, since it is not transitive.
A $\ominus$ sign in a balance means that the element should be at the other side:

Property 1.3 $\forall a, b, c \in \mathbb{S} : a \ominus b \,\nabla\, c$ *if and only if* $a \,\nabla\, b \oplus c$.

If both sides of a balance are signed, we can replace the balance by an equality:

Property 1.4 $\forall a, b \in \mathbb{S}^\oplus \cup \mathbb{S}^\ominus : a \,\nabla\, b \Rightarrow a = b$.

Definition 1.5 (Determinant) *Let $A \in \mathbb{S}^{n\times n}$. The max-algebraic determinant of A is defined as*

$$
\det A = \bigoplus_{\sigma\in\mathcal{P}_n} \operatorname{sgn}(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)}
$$

where $\mathcal{P}_n$ is the set of all permutations of $\{1, \ldots, n\}$, and $\operatorname{sgn}(\sigma) = 0$ *if the permutation σ is even and* $\operatorname{sgn}(\sigma) = \ominus 0$ *if the permutation is odd.*

Definition 1.6 (Minor rank) *Consider $A \in \mathbb{S}^{m\times n}$. The max-algebraic minor rank of A is the dimension of the largest square submatrix A_{sub} of A such that* $\det A_{\text{sub}} \not\nabla\, \varepsilon$.

Definition 1.7 (Characteristic equation) *Let $A \in \mathbb{S}^{n\times n}$. The max-algebraic characteristic equation of A is defined as* $\det(A \ominus \lambda \otimes E_n) \,\nabla\, \varepsilon$.

If we work this out, we get

$$
\lambda^{\otimes^n} \oplus \bigoplus_{p=1}^{n} a_p \otimes \lambda^{\otimes^{n-p}} \,\nabla\, \varepsilon \ .
$$

Theorem 1.8 (Cayley-Hamilton)
In $\mathbb{S}_{\max}$ every square matrix satisfies its characteristic equation.

2 Systems of multivariate max-algebraic polynomial equalities and inequalities

In this section we consider systems of multivariate max-algebraic polynomial equalities and inequalities, which can be seen as a generalized framework for many important max-algebraic problems such as matrix decompositions, transformation of state space models, state space realization of impulse responses, construction of matrices with a given characteristic polynomial and so on [6, 7]. Consider the following problem:

> Given a set of integers $\{m_k\}$ and three sets of real numbers $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $i = 1, \ldots, m_k$, $j = 1, \ldots, n$ and $k = 1, \ldots, p_1 + p_2$, find $x \in \mathbb{R}^n$ such that
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1, \ldots, p_1 \ , \quad (1)$$
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} \leqslant b_k \qquad \text{for } k = p_1 + 1, \ldots, p_1 + p_2 \ , \quad (2)$$
>
> or show that no such x exists.

We call (1) – (2) a system of multivariate max-algebraic polynomial equalities and inequalities. Note that the exponents can be negative or real.

In [7] we have shown that this problem is equivalent to an Extended Linear Complementarity Problem (ELCP) [5]. This leads to an algorithm that yields the entire solution set of problem (1) – (2). In general this solution set consists of the union of faces of a polyhedron $\mathcal{P}$ and is defined by three sets of vectors $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and a set Λ. These sets can be characterized as follows:

- $\mathcal{X}^{\text{cen}}$ is a set of *central rays* of $\mathcal{P}$. It is a basis for the largest linear subspace of $\mathcal{P}$. Let us call $\mathcal{P}_{\text{red}}$ the polyhedron obtained by subtracting this largest linear subspace from $\mathcal{P}$.
- $\mathcal{X}^{\text{inf}}$ is a set of *extreme rays* or *vertices at infinity* of the polyhedron $\mathcal{P}_{\text{red}}$.
- $\mathcal{X}^{\text{fin}}$ is the set of the *finite vertices* of the polyhedron $\mathcal{P}_{\text{red}}$.
- Λ is a set of pairs $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\}$ with $\mathcal{X}_s^{\text{inf}} \subset \mathcal{X}^{\text{inf}}$, $\mathcal{X}_s^{\text{fin}} \subset \mathcal{X}^{\text{fin}}$ and $\mathcal{X}_s^{\text{fin}} \neq \emptyset$. Each pair determines a face $\mathcal{F}_s$ of the polyhedron $\mathcal{P}$ that belongs to the solution set: $\mathcal{X}_s^{\text{inf}}$ contains the extreme rays of $\mathcal{F}_s$ and $\mathcal{X}_s^{\text{fin}}$ contains the finite vertices of $\mathcal{F}_s$.

The solution set of problem (1) – (2) is characterized by the following theorems:

Theorem 2.1 *When $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and Λ are given, then x is a finite solution of the system of multivariate max-algebraic polynomial equalities and inequalities if and only if there exists a pair $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\} \in \Lambda$ such that*

$$x = \sum_{x_k \in \mathcal{X}^{\text{cen}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k, \mu_k \geqslant 0$ and $\sum_k \mu_k = 1$.

Theorem 2.2 *In general the set of the (finite) solutions of a system of multivariate max-algebraic polynomial equalities and inequalities consists of the union of faces of a polyhedron.*

Remark: Solutions for which some of the components are equal to ε can be obtained by a limit or a threshold procedure. These solutions would correspond to points at infinity of the polyhedron $\mathcal{P}$. See [7] for more information on this subject.

3 Minimal state space realization

Consider a discrete event system that can be described by the following nth order state space model with m inputs and l outputs:

$$x(k+1) = A \otimes x(k) \ \oplus \ B \otimes u(k) \qquad (3)$$

$$y(k) = C \otimes x(k) \qquad (4)$$

where $A \in \mathbb{R}_\varepsilon^{n \times n}$, $B \in \mathbb{R}_\varepsilon^{n \times m}$ and $C \in \mathbb{R}_\varepsilon^{l \times n}$. The vector x represents the state, u is the input vector and y is the output vector of the system.

If we apply a unit impulse: $e(k) = 0$ if $k = 0$ and $e(k) = \varepsilon$ if $k \neq 0$, to the ith input of the system and if we assume that the initial state $x(0)$ satisfies $x(0) = \mathcal{E}_{n \times 1}$, we get

$$y(k) = C \otimes A^{\otimes^{k-1}} \otimes B_{.i} \qquad \text{for } k = 1, 2, \ldots$$

as the output of the system, where $B_{.i}$ is the ith column of B. We repeat this experiment for all inputs $i = 1, 2, \ldots, m$ and store the outputs in l by m matrices $G_k = C \otimes A^{\otimes^k} \otimes B$ for $k = 0, 1, \ldots$. The G_k's are called the *impulse response matrices* or *Markov parameters*.

Suppose that A, B and C are unknown, and that we only know the Markov parameters (e.g. from experiments – where we assume that the system is max-linear and time-invariant and that there is no noise present). How can we construct A, B and C from the G_k's? This problem is called state space realization. If we make the dimension of A minimal, we have a minimal state space realization problem.

In order to solve this problem we first need a lower bound r for the minimal system order. As a direct consequence of the Cayley-Hamilton theorem the Markov parameters satisfy the max-algebraic characteristic equation of A. So we could try to find a stable relationship of the form

$$\bigoplus_{p=0}^{r} a_p \otimes G_{k+r-p} \ \nabla\ \mathcal{E}_{l\times m} \qquad \text{for } k=0,1,\ldots \tag{5}$$

with as few terms as possible, where the a_p's should correspond to the coefficients of the characteristic equation of a matrix with entries in $\mathbb{R}_\varepsilon$. Expression (5) is a system of linear balances with the a_p's as unknowns. In [6, 4] we have applied this procedure to obtain a lower bound r for the minimal system order of a single input single output system. If we decompose the a_p's as $a_p = a_p^+ \ominus a_p^-$ and if we use Properties 1.3 and 1.4, we can transform (5) into a system of multivariate max-algebraic polynomial equalities with the max-positive and the max-negative parts of the a_p's as variables.

We could also use the following theorem [8, 9] to obtain a lower bound for the minimal system order:

Theorem 3.1 *Let $G_k = C \otimes A^{\otimes^k} \otimes B$ for $k=0,1,\ldots$ be the Markov parameters of a time-invariant max-linear system with system matrices A, B and C. Then the max-algebraic minor rank of the block Hankel matrix*

$$H = \begin{bmatrix} G_0 & G_1 & G_2 & \ldots \\ G_1 & G_2 & G_3 & \ldots \\ G_2 & G_3 & G_4 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is a lower bound for the minimal system order.

In practice we only consider a truncated version of the semi-infinite block Hankel matrix H:

$$H_{p,q} = \begin{bmatrix} G_0 & G_1 & \ldots & G_p \\ G_1 & G_2 & \ldots & G_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_q & G_{q+1} & \ldots & G_{p+q} \end{bmatrix} .$$

The max-algebraic minor rank of $H_{p,q}$ is a lower bound for the minimal system order.

We assume that the entries of all the G_k's are finite and that the system exhibits a periodic steady state behavior of the following kind:

$$\exists n_0, d \in \mathbb{N} \text{ and } \exists c \in \mathbb{R} \text{ such that}$$
$$\forall n \geqslant n_0 \ :\ G_{n+d} = c^{\otimes^d} \otimes G_n \ . \tag{6}$$

It can be shown [1, 8] that a sufficient condition for (6) to hold is that the system matrix A is irreducible, i.e. $(A \oplus A^{\otimes^2} \oplus \ldots \oplus A^{\otimes^n})_{ij} \neq \varepsilon$ for all i,j. This will for example be the case for a discrete event system without separate independent subsystems and with a cyclic behavior or with feedback from the output to the input like flexible production systems in which the parts are carried around on a limited number of pallets that circulate in the system.

We start with r equal to the lower bound. Now we try to find an rth order partial state space realization of the given impulse response: we have to find $A \in \mathbb{R}_\varepsilon^{r\times r}$, $B \in \mathbb{R}_\varepsilon^{r\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times r}$ such that

$$C \otimes A^{\otimes^k} \otimes B \ =\ G_k \qquad \text{for } k=0,1,\ldots,N-1 \tag{7}$$

for N large enough. If we work out the equations of the form (7), we get for $k=0$:

$$\bigoplus_{p=1}^{r} c_{ip} \otimes b_{pj} \ =\ (G_0)_{ij} \tag{8}$$

for $i=1,2,\ldots,l$ and $j=1,2,\ldots,m$. For $k>0$ we obtain

$$\bigoplus_{p=1}^{r}\bigoplus_{q=1}^{r} c_{ip} \otimes (A^{\otimes^k})_{pq} \otimes b_{qj} \ =\ (G_k)_{ij} \tag{9}$$

for $i=1,2,\ldots,l$ and $j=1,2,\ldots,m$. Since

$$(A^{\otimes^k})_{pq} \ =\ \bigoplus_{i_1=1}^{r} \ldots \bigoplus_{i_{k-1}=1}^{r} a_{pi_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1}q} \ ,$$

equation (9) can be rewritten as

$$\bigoplus_{p=1}^{r}\bigoplus_{q=1}^{r}\bigoplus_{s=1}^{r^{k-1}} c_{ip} \otimes \bigotimes_{u=1}^{r}\bigotimes_{v=1}^{r} a_{uv}{}^{\otimes^{\gamma_{kpqsuv}}} \otimes b_{qj} \ =\ (G_k)_{ij} \tag{10}$$

where γ_{kpqsuv} is the number of times that a_{uv} appears in the sth term of $(A^{\otimes^k})_{pq}$. If a_{uv} does not appear in that term, we take $\gamma_{kpqsuv}=0$ since $a^{\otimes^0} = 0 \cdot a = 0$, the identity element for $\otimes$. If we use the fact that $\forall x \in \mathbb{R}_\varepsilon : x \oplus x = x$ and $\forall x,y \in \mathbb{R}_\varepsilon : x \otimes y \leqslant x\otimes x \oplus y \otimes y$, we can remove many redundant terms. There are then w_{kij} terms in (10) where $w_{kij} \leqslant r^{k+1}$.

If we put all unknowns in one large vector x of length $r(r+m+l)$, we have to solve a system of multivariate max-algebraic polynomial equations of the following form:

$$\bigoplus_{p=1}^{r}\bigotimes_{q=1}^{r(r+m+l)} x_q{}^{\otimes^{\delta_{0ijpq}}} \ =\ (G_0)_{ij} \tag{11}$$

$$\bigoplus_{p=1}^{w_{kij}}\bigotimes_{q=1}^{r(r+m+l)} x_q{}^{\otimes^{\delta_{kijpq}}} \ =\ (G_k)_{ij} \tag{12}$$

for $i=1,2,\ldots,l$; $j=1,2,\ldots,m$ and $k=1,2,\ldots,N-1$.

If we find a solution x of (11) – (12), we extract the entries of the system matrices A, B and C from x.

If we do not get any solutions, this means that r is less than the minimal system order, i.e. the lower bound is not tight. Then we have to augment our estimate of the minimal system order and repeat the above procedure but with $r+1$ instead of r. We continue until we find a solution of (11) − (12).

This yields a minimal state space realization of the first N impulse response matrices. If N is large enough, we can obtain the set of all realizations of the given impulse response by putting all components that are smaller than some threshold equal to ε if necessary.

If the system does not exhibit the steady state behavior of (6) then the procedure presented in this paper will in general only yield partial state space realizations. However, in some cases the ELCP technique can still be applied if an analogous but more complicated threshold procedure is used.

Now we can characterize the set of all (partial) minimal state space realizations of a given impulse response:

Theorem 3.2 *In general the set of all (partial) minimal state space realizations of the impulse response of a max-linear time-invariant discrete event system consists of the union of faces of a polyhedron in the x-space, where x is the vector obtained by putting the components of the system matrices in one large vector.*

4 Example

We shall now illustrate the preceding procedure with an example.

Example 4.1

We start from a system with system matrices:

$$A = \begin{bmatrix} \varepsilon & 14 & \varepsilon \\ 5 & 6 & \varepsilon \\ 1 & 4 & 8 \end{bmatrix}, \; B = \begin{bmatrix} 10 & 11 \\ \varepsilon & 9 \\ 0 & \varepsilon \end{bmatrix} \text{ and}$$

$$C = \begin{bmatrix} 2 & \varepsilon & 9 \\ 0 & 4 & \varepsilon \end{bmatrix} .$$

Now we are going to construct all equivalent minimal state space realizations of this system starting from its impulse response matrices, which are given by

$$\begin{aligned} \{G_k\}_{k=0}^{\infty} \;=\; & \begin{bmatrix} 12 & 13 \\ 10 & 13 \end{bmatrix}, \begin{bmatrix} 20 & 25 \\ 19 & 23 \end{bmatrix}, \begin{bmatrix} 31 & 33 \\ 29 & 32 \end{bmatrix}, \\ & \begin{bmatrix} 39 & 44 \\ 38 & 42 \end{bmatrix}, \begin{bmatrix} 50 & 52 \\ 48 & 51 \end{bmatrix}, \begin{bmatrix} 58 & 63 \\ 57 & 61 \end{bmatrix}, \\ & \begin{bmatrix} 69 & 71 \\ 67 & 70 \end{bmatrix}, \ldots \; . \end{aligned}$$

Note that the G_k's exhibit the behavior of (6) with $n_0 = 1$, $d = 2$ and $c = 9.5$.

The relation of the form (5) with as few terms as possible is given by

$$G_{k+2} \ominus 8 \otimes G_{k+1} \ominus 19 \otimes G_k \;\nabla\; \mathcal{E}_{2\times 2} \quad \text{for } k = 0, 1, \ldots$$

or equivalently

$$G_{k+2} \;=\; 8 \otimes G_{k+1} \;\oplus\; 19 \otimes G_k \qquad \text{for } k = 0, 1, \ldots$$

by Properties 1.3 and 1.4. So the minimal system order is greater than or equal to $r = 2$.

The max-algebraic minor rank of the truncated Hankel matrix $H_{6,6}$ is also equal to 2.

Now we try to find a second order state space realization of the impulse response matrices. Let us take $N = 3$. The ELCP algorithm of [5] yields the rays and vertices of Tables 1 and 2 and the pairs of subsets of Table 3. If we take $N > 3$ we get the same result, but if we take $N < 3$, some combinations of the rays and the vertices only lead to a partial realization of the given impulse response: they only fit the first N impulse response matrices.

Any arbitrary finite minimal realization can now be expressed as

$$x \;=\; \lambda_1 x_1^{\mathrm{c}} + \lambda_2 x_2^{\mathrm{c}} + \sum_k \kappa_k x_k^{\mathrm{i}} + x_l^{\mathrm{f}} \tag{13}$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\kappa_k \geqslant 0$ and $x_k^{\mathrm{i}} \in \mathcal{X}_s^{\mathrm{inf}}$, $x_l^{\mathrm{f}} \in \mathcal{X}_s^{\mathrm{fin}}$ with $s \in \{1, 2, \ldots, 8\}$ and where x is the column vector obtained by putting the entries of the system matrices in one large column vector. Expression (13) shows that the set of all equivalent minimal state space realizations of the given impulse response is the union of 8 faces of a polyhedron in the x-space. $\Box$

5 Conclusions and future research

We have shown that the problem of finding a minimal state space realization of the impulse response of a multiple input multiple output max-linear time-invariant discrete event system (that exhibits a particular kind of periodic steady state behavior) can be reformulated as a system of multivariate polynomial equations in the max algebra. This means that we can use the ELCP algorithm of [5] to solve such a problem.

One of the main characteristics of the ELCP algorithm of [5] is that it finds all solutions. This provides a geometrical insight in the set of all equivalent minimal state space realizations of an impulse response. On the other hand this also leads to large computation times and storage space requirements if the number of variables and equations is large. Therefore, it might be interesting to develop algorithms that only find one solution as we have done for the minimal realization problem for single input single output systems in [4]. Among the set of all possible realizations we could also try to find certain 'privileged' realizations such as balanced realizations.

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Set	$\mathcal{X}^{\text{cen}}$		$\mathcal{X}^{\text{fin}}$	
Ray	x_1^{c}	x_2^{c}	x_1^{f}	x_2^{f}
a_{11}	0	0	8	8
a_{12}	0	-1	19	19
a_{21}	0	1	0	0
a_{22}	0	0	8	8
b_{11}	1	0	0	0
b_{12}	1	0	5	1
b_{21}	1	1	-8	-11
b_{22}	1	1	-7	-6
c_{11}	-1	0	8	12
c_{12}	-1	-1	20	19
c_{21}	-1	0	8	10
c_{22}	-1	-1	18	19

Table 1: The central rays and the finite vertices for Example 4.1 .

Set	$\mathcal{X}^{\text{inf}}$							
Ray	x_1^{i}	x_2^{i}	x_3^{i}	x_4^{i}	x_5^{i}	x_6^{i}	x_7^{i}	x_8^{i}
a_{11}	0	-1	0	0	0	0	0	0
a_{12}	0	0	0	0	0	0	0	0
a_{21}	0	0	0	0	0	0	0	0
a_{22}	0	0	0	0	0	0	0	-1
b_{11}	0	0	0	0	0	0	0	0
b_{12}	1	0	-1	0	0	0	0	0
b_{21}	1	0	0	0	0	-1	0	0
b_{22}	1	0	0	-1	0	0	0	0
c_{11}	-1	0	0	0	-1	0	0	0
c_{12}	-1	0	0	0	0	0	-1	0
c_{21}	-1	0	0	0	0	0	0	0
c_{22}	-1	0	0	0	0	0	0	0

Table 2: The vertices at infinity for Example 4.1 .

s	$\mathcal{X}_s^{\text{inf}}$	$\mathcal{X}_s^{\text{fin}}$	s	$\mathcal{X}_s^{\text{inf}}$	$\mathcal{X}_s^{\text{fin}}$
1	$\{x_1^{\text{i}}, x_2^{\text{i}}, x_4^{\text{i}}\}$	$\{x_1^{\text{f}}\}$	5	$\{x_3^{\text{i}}, x_6^{\text{i}}, x_8^{\text{i}}\}$	$\{x_2^{\text{f}}\}$
2	$\{x_1^{\text{i}}, x_2^{\text{i}}, x_5^{\text{i}}\}$	$\{x_1^{\text{f}}\}$	6	$\{x_4^{\text{i}}, x_8^{\text{i}}\}$	$\{x_1^{\text{f}}\}$
3	$\{x_2^{\text{i}}, x_3^{\text{i}}\}$	$\{x_2^{\text{f}}\}$	7	$\{x_5^{\text{i}}, x_8^{\text{i}}\}$	$\{x_1^{\text{f}}\}$
4	$\{x_2^{\text{i}}, x_7^{\text{i}}\}$	$\{x_2^{\text{f}}\}$	8	$\{x_6^{\text{i}}, x_7^{\text{i}}, x_8^{\text{i}}\}$	$\{x_2^{\text{f}}\}$

Table 3: The pairs of subsets for Example 4.1 .