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The Singular Value Decomposition and the QR Decomposition in the Extended Max Algebra¹

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Abstract. In this paper we present an alternative proof for the existence theorem of the singular value decomposition in the extended max algebra and we propose some possible extensions of the max-algebraic singular value decomposition. We also prove the existence of a kind of QR decomposition in the extended max algebra.

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1 Introduction

1.1 Overview

One of the possible frameworks to describe and analyze discrete event systems (such as flexible manufacturing processes, railroad traffic networks, telecommunication networks, ...) is the max algebra [1, 5, 6]. A class of discrete event systems, the timed event graphs, can be described by a model that is linear in the max algebra. There exists a remarkable analogy between max-algebraic system theory and system theory for linear systems. However, in contrast to linear system theory the mathematical foundations of the max-algebraic system theory are not as fully developed as those of the classic linear system theory, although some of the properties and concepts of linear algebra, such as Cramer's rule, the Cayley-Hamilton theorem, eigenvalues, eigenvectors, ... also have a max-algebraic equivalent. In [14] Olsder and Roos have used a kind of link between the field of the real numbers and the max algebra based on asymptotic equivalences to show that every matrix has at least one max-algebraic eigenvalue and to prove a max-algebraic version of Cramer's rule and of the Cayley-Hamilton theorem. In [8] we have extended this link and used it to define the singular value decomposition (SVD) in the extended max algebra, which is a kind of symmetrization of the max algebra [9, 13].

In this paper we present an alternative proof for the existence theorem of the max-algebraic SVD based on Kogbetliantz's SVD algorithm [4, 11]. Furthermore, we prove the existence of a kind of QR decomposition (QRD) in the extended max algebra. We also propose possible extensions of the max-algebraic SVD.

This paper is organized as follows: In Section 1 we recapitulate the most important concepts, definitions and properties of [8] and give some additional definitions and properties. In Section 2 we present an alternative proof for the existence theorem of the max-algebraic SVD and we prove the existence of the max-algebraic QRD. In Section 3 we propose possible extensions of the max-algebraic SVD and the max-algebraic QRD. We conclude with some examples.

1.2 Notations and definitions

We use f or $f(\cdot)$ to represent a function. The value of f at x is denoted by $f(x)$. If $a \in \mathbb{R}^n$, then a_i is the i th component of a . If A is a matrix, then a_{ij} or $(A)_{ij}$ is the entry on the i th row and the j th column. The i th row of A is represented by A_i . The n by n identity matrix is denoted by I_n and the m by n zero matrix is denoted by $O_{m \times n}$. We use \mathbb{R}_0 to represent the set of all the real numbers except for 0 ($\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$). The set of the nonnegative real numbers is denoted by \mathbb{R}^+ .

Definition 1.1 (Analytic function) *A real function f is analytic in a point $\alpha \in \mathbb{R}$ if the Taylor series of f with center α exists and if there is a neighborhood of α where the Taylor series converges to f .*

A real function f is analytic in an interval $[\alpha, \beta]$ if it is analytic in every point of that interval. A real matrix-valued function is analytic in $[\alpha, \beta]$ if all its entries are analytic in $[\alpha, \beta]$.

Definition 1.2 (Asymptotic equivalence) *Let $\alpha \in \mathbb{R} \cup \{\infty\}$ and let f and g be real functions. The function f is asymptotically equivalent to g in the neighborhood of α , denoted by $f(x) \sim g(x), x \rightarrow \alpha$, if $\lim_{x \rightarrow \alpha} \frac{f(x)}{g(x)} = 1$.*

If $\beta \in \mathbb{R}$ and if $\exists \delta > 0, \forall x \in (\beta - \delta, \beta + \delta) \setminus \{\beta\} : f(x) = 0$ then $f(x) \sim 0, x \rightarrow \beta$.

We say that $f(x) \sim 0, x \rightarrow \infty$ if $\exists K \in \mathbb{R}, \forall x > K : f(x) = 0$.

If $F(\cdot)$ and $G(\cdot)$ are real m by n matrix-valued functions then $F(x) \sim G(x), x \rightarrow \alpha$ if $f_{ij}(x) \sim g_{ij}(x), x \rightarrow \alpha$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Note that the main difference with the classic definition of asymptotic equivalence is that Definition 1.2 also allows us to say that a function is asymptotically equivalent to 0.

1.3 The max algebra and the extended max algebra

In this section we give a short introduction to the max algebra and the extended max algebra. A complete overview of the max algebra can be found in [1, 6]. The basic max-algebraic operations are defined as follows:

$$x \oplus y = \max(x, y) \tag{1}$$

$$x \otimes y = x + y \tag{2}$$

where $x, y \in \mathbb{R} \cup \{-\infty\}$. The resulting structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the max algebra. The zero element for \oplus in \mathbb{R}_{\max} is represented by $\varepsilon \stackrel{\text{def}}{=} -\infty$. Define $\mathbb{R}_{\varepsilon} = \mathbb{R} \cup \{-\infty\}$. Let $r \in \mathbb{R}$. The r th max-algebraic power of $x \in \mathbb{R}$ is denoted by $x^{\otimes r}$ and corresponds to rx in linear algebra. If $x \in \mathbb{R}$ then $x^{\otimes 0} = 0$ and the inverse element of x w.r.t. \otimes is $x^{\otimes -1} = -x$. If $r > 0$ then $\varepsilon^{\otimes r} = \varepsilon$. If $r \leq 0$ then $\varepsilon^{\otimes r}$ is not defined.

The max-algebraic operations are extended to matrices in the usual way. If $\alpha \in \mathbb{R}_{\varepsilon}$ and if $X, Y \in \mathbb{R}_{\varepsilon}^{m \times n}$ then $(\alpha \otimes X)_{ij} = \alpha \otimes x_{ij}$ and $(X \oplus Y)_{ij} = x_{ij} \oplus y_{ij}$ for all i, j . If $X \in \mathbb{R}_{\varepsilon}^{m \times p}$ and $Y \in \mathbb{R}_{\varepsilon}^{p \times n}$ then $(X \otimes Y)_{ij} = \bigoplus_{k=1}^p x_{ik} \otimes y_{kj}$ for all i, j .

The matrix E_n is the n by n max-algebraic identity matrix: $(E_n)_{ii} = 0$ for $i = 1, 2, \dots, n$ and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The m by n max-algebraic zero matrix is represented by $\mathcal{E}_{m \times n}$: $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j . The off-diagonal entries of a max-algebraic diagonal matrix $D \in \mathbb{R}_{\varepsilon}^{m \times n}$ are equal to ε : $d_{ij} = \varepsilon$ for all i, j with $i \neq j$. A matrix $R \in \mathbb{R}_{\varepsilon}^{m \times n}$ is a max-algebraic upper triangular matrix if $r_{ij} = \varepsilon$ for all i, j with $i > j$.

In contrast to linear algebra, there exist no inverse elements w.r.t. \oplus in \mathbb{R}_{ε} . To overcome this problem we need the extended max algebra \mathbb{S}_{\max} [1, 9, 13], which is a kind of symmetrization of the max algebra. We shall restrict ourselves to a short introduction to the most important features of \mathbb{S}_{\max} . For a more formal derivation the interested reader is referred to [1, 8, 9, 13].

First we define two new elements for every $x \in \mathbb{R}_{\varepsilon}$: $\ominus x$ and x^{\bullet} . This gives rise to an extension \mathbb{S} of \mathbb{R}_{ε} that contains three classes of elements:

- $\mathbb{S}^{\oplus} \equiv \mathbb{R}_{\varepsilon}$, the set of the max-positive or zero elements,
- $\mathbb{S}^{\ominus} = \{\ominus x \mid x \in \mathbb{R}_{\varepsilon}\}$, the set of max-negative or zero elements,
- $\mathbb{S}^{\bullet} = \{x^{\bullet} \mid x \in \mathbb{R}_{\varepsilon}\}$, the set of the balanced elements.

We have $\mathbb{S} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \cup \mathbb{S}^{\bullet}$ and $\mathbb{S}^{\oplus} \cap \mathbb{S}^{\ominus} \cap \mathbb{S}^{\bullet} = \{\varepsilon\}$ since $\varepsilon = \ominus \varepsilon = \varepsilon^{\bullet}$. The max-positive and max-negative elements and the zero element ε are called signed ($\mathbb{S}^{\vee} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$).

The \oplus and the \otimes operation can be extended to \mathbb{S} . The resulting structure $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$

is called the extended max algebra. The \oplus law is associative, commutative and idempotent in \mathbb{S} and its zero element is ε ; the \otimes law is associative and commutative in \mathbb{S} and its unit element is 0. Furthermore, \otimes is distributive w.r.t. \oplus in \mathbb{S} . If $x, y \in \mathbb{R}_\varepsilon$ then

$$\begin{aligned} x \oplus (\ominus y) &= x && \text{if } x > y, \\ x \oplus (\ominus y) &= \ominus y && \text{if } x < y, \\ x \oplus (\ominus x) &= x^\bullet . \end{aligned}$$

We have $\forall a, b \in \mathbb{S}$:

$$\begin{aligned} a^\bullet &= (\ominus a)^\bullet = (a^\bullet)^\bullet \\ a \otimes b^\bullet &= (a \otimes b)^\bullet \\ \ominus(\ominus a) &= a \\ \ominus(a \oplus b) &= (\ominus a) \oplus (\ominus b) \\ \ominus(a \otimes b) &= (\ominus a) \otimes b . \end{aligned}$$

The last three properties allow us to write $a \ominus b$ instead of $a \oplus (\ominus b)$. So the \ominus operator in \mathbb{S}_{\max} could be considered as the equivalent of the - operator in linear algebra.

Let $a \in \mathbb{S}$. The max-positive part a^\oplus and the max-negative part a^\ominus of a are defined as follows:

- if $a \in \mathbb{S}^\oplus$ then $a^\oplus = a$ and $a^\ominus = \varepsilon$,
- if $a \in \mathbb{S}^\ominus$ then $a^\oplus = \varepsilon$ and $a^\ominus = \ominus a$,
- if $a \in \mathbb{S}^\bullet$ then $\exists b \in \mathbb{R}_\varepsilon$ such that $a = b^\bullet$ and then $a^\oplus = a^\ominus = b$.

So $a = a^\oplus \ominus a^\ominus$ and $a^\oplus, a^\ominus \in \mathbb{R}_\varepsilon$. We define the max-absolute value of $a \in \mathbb{S}$ as $|a|_\oplus = a^\oplus \oplus a^\ominus$. In linear algebra we have $\forall x \in \mathbb{R} : x - x = 0$, but in \mathbb{S}_{\max} we have $\forall a \in \mathbb{S} : a \ominus a = a^\bullet \neq \varepsilon$ unless $a = \varepsilon$, the zero element for \oplus . Therefore, we introduce a new relation, the balance relation, represented by ∇ .

Definition 1.3 (Balance relation) Consider $a, b \in \mathbb{S}$. We say that a balances b , denoted by $a \nabla b$, if $a^\oplus \oplus b^\ominus = a^\ominus \oplus b^\oplus$.

Since $\forall a \in \mathbb{S} : a \ominus a = a^\bullet = |a|_\oplus \ominus |a|_\oplus \nabla \varepsilon$, we could say that the balance relation in \mathbb{S} is the counterpart of the equality relation in linear algebra. The balance relation is reflexive and symmetric but it is not transitive. The balance relation is extended to matrices in the usual way: if $A, B \in \mathbb{S}^{m \times n}$ then $A \nabla B$ if $a_{ij} \nabla b_{ij}$ for all i, j .

An element with a \ominus sign can be transferred to the other side of a balance as follows:

Proposition 1.4 $\forall a, b, c \in \mathbb{S} : a \ominus c \nabla b$ if and only if $a \nabla b \oplus c$.

If both sides of a balance are signed, we can replace the balance by an equality:

Proposition 1.5 $\forall a, b \in \mathbb{S}^\vee : a \nabla b \Rightarrow a = b$.

The above properties can be extended to the matrix case as follows:

Proposition 1.6 $\forall A, B, C \in \mathbb{S}^{m \times n} : A \ominus C \nabla B$ if and only if $A \nabla B \oplus C$.

Proposition 1.7 $\forall A, B \in (\mathbb{S}^\vee)^{m \times n} : A \nabla B \Rightarrow A = B$.

Definition 1.8 (Max-algebraic norm) The max-algebraic norm of a vector $a \in \mathbb{S}^n$ is defined as

$$\|a\|_{\oplus} = \bigoplus_{i=1}^n |a_i|_{\oplus} = \bigoplus_{i=1}^n (a_i^{\oplus} \oplus a_i^{\ominus}) .$$

The max-algebraic norm of a matrix $A \in \mathbb{S}^{m \times n}$ is defined as

$$\|A\|_{\oplus} = \bigoplus_{i=1}^m \bigoplus_{j=1}^n |a_{ij}|_{\oplus} .$$

Definition 1.9 (Max-algebraic determinant) Consider a matrix $A \in \mathbb{S}^{n \times n}$. The max-algebraic determinant of A is defined as

$$\det_{\oplus} A = \bigoplus_{\sigma \in \mathcal{P}_n} \text{sgn}_{\oplus}(\sigma) \otimes \bigotimes_{i=1}^n a_{i\sigma(i)}$$

where \mathcal{P}_n is the set of all the permutations of $\{1, \dots, n\}$, and $\text{sgn}_{\oplus}(\sigma) = 0$ if the permutation σ is even and $\text{sgn}_{\oplus}(\sigma) = \ominus 0$ if the permutation is odd.

Theorem 1.10 Let $A \in \mathbb{S}^{n \times n}$. The homogeneous linear balance $A \otimes x \nabla \mathcal{E}_{n \times 1}$ has a non-trivial signed solution if and only if $\det_{\oplus} A \nabla \varepsilon$.

Proof: See [9]. □

Definition 1.11 (Max-linear independence)

A set of vectors $\{x_i \in \mathbb{S}^n \mid i = 1, 2, \dots, m\}$ is max-linearly independent if the only solution of

$$\bigoplus_{i=1}^m \alpha_i \otimes x_i \nabla \mathcal{E}_{n \times 1}$$

with $\alpha_i \in \mathbb{S}^{\vee}$ is $\alpha_1 = \alpha_2 = \dots = \alpha_m = \varepsilon$. Otherwise we say that the vectors x_i are max-linearly dependent.

1.4 A link between the field of the real numbers and the extended max algebra

In [8] we have introduced the following mapping for $x \in \mathbb{R}_{\varepsilon}$:

$$\begin{aligned} \mathcal{F}(x, s) &= \mu e^{xs} \\ \mathcal{F}(\ominus x, s) &= -\mu e^{xs} \\ \mathcal{F}(x^{\bullet}, s) &= \nu e^{xs} \end{aligned}$$

where μ is an arbitrary positive real number or parameter and ν is an arbitrary real number or parameter different from 0 and s is a real parameter. Note that $\mathcal{F}(\varepsilon, s) = 0$.

To reverse the mapping \mathcal{F} we have to take $\lim_{s \rightarrow \infty} \frac{\log(|\mathcal{F}(x, s)|)}{s}$ and adapt the max-sign depending on the sign of the coefficient of the exponential. So if f is a real function, if $x \in \mathbb{R}_{\varepsilon}$

and if μ is a positive real number or if μ is a parameter that can only take on positive real values then

$$\begin{aligned} f(s) \sim \mu e^{xs}, s \rightarrow \infty &\Rightarrow \mathcal{R}(f) = x \\ f(s) \sim -\mu e^{xs}, s \rightarrow \infty &\Rightarrow \mathcal{R}(f) = \ominus x \end{aligned}$$

where \mathcal{R} is the reverse mapping of \mathcal{F} . If ν is a parameter that can take on both positive and negative real values then

$$f(s) \sim \nu e^{xs}, s \rightarrow \infty \Rightarrow \mathcal{R}(f) = x^\bullet .$$

Note that if the coefficient of e^{xs} is a number then the reverse mapping always yields a signed result.

Now we have for $a, b, c \in \mathbb{S}$:

$$a \oplus b = c \rightarrow \mathcal{F}(a, s) + \mathcal{F}(b, s) \sim \mathcal{F}(c, s), s \rightarrow \infty \quad (3)$$

$$\mathcal{F}(a, s) + \mathcal{F}(b, s) \sim \mathcal{F}(c, s), s \rightarrow \infty \rightarrow a \oplus b \nabla c \quad (4)$$

$$a \otimes b = c \leftrightarrow \mathcal{F}(a, s) \cdot \mathcal{F}(b, s) = \mathcal{F}(c, s) \quad \text{for all } s \in \mathbb{R} \quad (5)$$

for an appropriate choice of the μ 's and ν 's in $\mathcal{F}(c, s)$ in (3) and in (5) from the left to the right. The balance in (4) results from the fact that we can have cancellation of equal terms with opposite sign in $(\mathbb{R}, +, \times)$ whereas this is in general not possible in \mathbb{S}_{\max} since for all $a \in \mathbb{S} \setminus \{\varepsilon\} : a \ominus a \neq \varepsilon$. So we have the following correspondences:

$$(\mathbb{R}^+, +, \times) \leftrightarrow (\mathbb{R}_\varepsilon, \oplus, \otimes) = \mathbb{R}_{\max}$$

$$(\mathbb{R}, +, \times) \leftrightarrow (\mathbb{S}, \oplus, \otimes) = \mathbb{S}_{\max} .$$

We can extend this mapping to matrices such that if $A \in \mathbb{S}^{m \times n}$ then $\tilde{A}(\cdot) = \mathcal{F}(A, \cdot)$ is a real m by n matrix-valued function with $\tilde{a}_{ij}(s) = \mathcal{F}(a_{ij}, s)$ for some choice of the μ 's and ν 's. Note that the mapping is performed entrywise. If A, B and C are matrices with entries in \mathbb{S} , we have

$$A \oplus B = C \rightarrow \mathcal{F}(A, s) + \mathcal{F}(B, s) \sim \mathcal{F}(C, s), s \rightarrow \infty \quad (6)$$

$$\mathcal{F}(A, s) + \mathcal{F}(B, s) \sim \mathcal{F}(C, s), s \rightarrow \infty \rightarrow A \oplus B \nabla C \quad (7)$$

$$A \otimes B = C \rightarrow \mathcal{F}(A, s) \cdot \mathcal{F}(B, s) \sim \mathcal{F}(C, s), s \rightarrow \infty \quad (8)$$

$$\mathcal{F}(A, s) \cdot \mathcal{F}(B, s) \sim \mathcal{F}(C, s), s \rightarrow \infty \rightarrow A \otimes B \nabla C \quad (9)$$

for an appropriate choice of the μ 's and ν 's in $\mathcal{F}(C, s)$ in (6) and (8).

2 The singular value decomposition and the QR decomposition in the extended max algebra

In [8] we have used the mapping from $(\mathbb{R}, +, \times)$ to \mathbb{S}_{\max} and the reverse mapping to prove the existence of a kind of singular value decomposition in \mathbb{S}_{\max} . Now we give an alternative proof of the existence theorem based on Kogbetliantz's SVD algorithm. The entries of the matrices that are used in this proof are sums or series of exponentials. Therefore, we first study some properties of this kind of functions.

Definition 2.1 (Sum or series of exponentials) *Let \mathcal{S}_e be the set of real analytic functions f that can be written as a (possibly infinite, but absolutely convergent) sum of exponentials for x large enough:*

$$\mathcal{S}_e = \left\{ f \mid \exists K \in \mathbb{R} \text{ such that } f \text{ is analytic in } [K, \infty) \text{ and either} \right.$$

$$\forall x \geq K : f(x) = \sum_{i=0}^n \alpha_i e^{a_i x} \quad (10)$$

with $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}_0$, $a_i \in \mathbb{R}_\varepsilon$ and $a_0 > a_1 > \dots > a_n$; or

$$\forall x \geq K : f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} \quad (11)$$

with $\alpha_i \in \mathbb{R}_0$, $a_i \in \mathbb{R}$, $a_i > a_{i+1}$, $\lim_{i \rightarrow \infty} a_i = \varepsilon$ and

where the series converges absolutely for $x \geq K$ } .

Since we allow exponents that are equal to $\varepsilon = -\infty$, the zero function can also be considered as an exponential: $0 = 1 \cdot e^{\varepsilon x}$. Because the sequence of exponents is decreasing and the coefficients cannot be equal to 0, the sum of exponentials that corresponds to the zero function consists of exactly one term.

If $f \in \mathcal{S}_e$ is a series of the form (11) then the set $\{a_i \mid i = 0, 1, \dots, \infty\}$ has no finite accumulation point since the sequence $\{a_i\}_{i=0}^{\infty}$ is decreasing and unbounded from below. Note that series of the form (11) are related to (generalized) Dirichlet series [12].

Proposition 2.2 (Uniform convergence) *If $f \in \mathcal{S}_e$ is a series:*

$$\exists K \in \mathbb{R} \text{ such that } \forall x \geq K : f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x}$$

with $\alpha_i \in \mathbb{R}_0$, $a_i \in \mathbb{R}_\varepsilon$, $a_i > a_{i+1}$ and where the series converges absolutely for $x \geq K$, then the series $\sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ converges uniformly in $[K, \infty)$.

Proof: Since $f(x)$ can be written as a series, we know that $a_0 \neq \varepsilon$. Hence,

$$\sum_{i=0}^{\infty} \alpha_i e^{a_i x} = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{\infty} \frac{\alpha_i}{\alpha_0} e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{\infty} \gamma_i e^{c_i x} \right)$$

with $\gamma_i = \frac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0 < 0$. Since $\sum_{i=0}^{\infty} \alpha_i e^{a_i K}$ converges absolutely $\sum_{i=1}^{\infty} \gamma_i e^{c_i K}$ also converges absolutely.

If $x \geq K$ then $e^{c_i x} \leq e^{c_i K}$ since $c_i < 0$. So $\forall x \geq K : |\gamma_i e^{c_i x}| < |\gamma_i e^{c_i K}|$. We already know that $\sum_{i=1}^{\infty} |\gamma_i e^{c_i K}|$ converges. Now we can apply the Weierstrass M -test. Therefore, the series $\sum_{i=1}^{\infty} \gamma_i e^{c_i x}$ converges uniformly in $[K, \infty)$ and as a consequence, $\sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ also converges uniformly in $[K, \infty)$. \square

The behavior of the functions in \mathcal{S}_e in the neighborhood of ∞ is given by the following property:

Proposition 2.3 *Every function $f \in \mathcal{S}_e$ is asymptotically equivalent to an exponential in the neighborhood of ∞ :*

$$f \in \mathcal{S}_e \Rightarrow f(x) \sim \alpha_0 e^{a_0 x}, x \rightarrow \infty$$

with $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}_\varepsilon$.

Proof: If $f \in \mathcal{S}_e$ then there exists a real number K such that

$$\forall x \geq K : f(x) = \sum_{i=0}^n \alpha_i e^{a_i x}$$

with $n \in \mathbb{N} \cup \{\infty\}$, $\alpha_i \in \mathbb{R}_0$ and $a_i \in \mathbb{R}_\varepsilon$. If $n = \infty$ then f is a series that converges absolutely in $[K, \infty)$.

If $a_0 = \varepsilon$ then there exists a real number K such that $\forall x \geq K : f(x) = 0$ and thus

$$f(x) \sim 0 = 1 \cdot e^{\varepsilon x}, x \rightarrow \infty$$

by Definition 1.2.

If $n = 1$ then $f(x) = \alpha_0 e^{a_0 x}$ and thus $f(x) \sim \alpha_0 e^{a_0 x}$, $x \rightarrow \infty$ with $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}_\varepsilon$.

From now on we assume that $n > 1$ and $a_0 \neq \varepsilon$. Then we can rewrite $f(x)$ as

$$f(x) = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^n \frac{\alpha_i}{\alpha_0} e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} (1 + p(x))$$

with

$$p(x) = \sum_{i=1}^n \gamma_i e^{c_i x}$$

where $\gamma_i = \frac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0$. Note that $p \in \mathcal{S}_e$. Since $a_i < a_0$, we have $c_i < 0$ and consequently

$$\lim_{x \rightarrow \infty} p(x) = \lim_{x \rightarrow \infty} \sum_{i=1}^n \gamma_i e^{c_i x} = \sum_{i=1}^n \left(\lim_{x \rightarrow \infty} \gamma_i e^{c_i x} \right) = 0 \quad . \quad (12)$$

We can interchange the summation and the limit in (12) even if $n = \infty$ since in that case the series $\sum_{i=1}^{\infty} \gamma_i e^{c_i x}$ converges uniformly in $[K, \infty)$ by Proposition 2.2.

Now we have

$$\lim_{x \rightarrow \infty} \frac{f(x)}{\alpha_0 e^{a_0 x}} = \lim_{x \rightarrow \infty} \frac{\alpha_0 e^{a_0 x} (1 + p(x))}{\alpha_0 e^{a_0 x}} = \lim_{x \rightarrow \infty} (1 + p(x)) = 1$$

and thus

$$f(x) \sim \alpha_0 e^{a_0 x}, \quad x \rightarrow \infty$$

where $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}$. □

Definition 2.4 (Sign function) *The sign function $\text{sgn}(\cdot)$ is a real function that is defined as follows:*

$$\begin{aligned} \text{sgn}(x) &= 1 && \text{if } x > 0, \\ &= -1 && \text{if } x < 0, \\ &= 0 && \text{if } x = 0. \end{aligned}$$

Now we prove that the set \mathcal{S}_e is closed under some basic operations:

Theorem 2.5 *If f and g belong to \mathcal{S}_e then ρf , $f + g$, $f - g$, $f \cdot g$, f^n , $|f|$ and $\text{sgn}(f)$ also belong to \mathcal{S}_e for every $\rho \in \mathbb{R}$ and $n \in \mathbb{N}$.*

If $\frac{1}{f}$, $\frac{f}{g}$ and \sqrt{f} are defined, they also belong to \mathcal{S}_e .

Proof: If f and g belong to \mathcal{S}_e then there exists a real number K such that

$$\forall x \geq K : f(x) = \sum_{i=0}^n \alpha_i e^{a_i x} \quad \text{and} \quad g(x) = \sum_{j=0}^m \beta_j e^{b_j x}$$

with $m, n \in \mathbb{N} \cup \{\infty\}$, $\alpha_i, \beta_j \in \mathbb{R}_0$ and $a_i, b_j \in \mathbb{R}_\varepsilon$. If m or n is equal to ∞ then the corresponding series converges absolutely in $[K, \infty)$.

If $a_0 = \varepsilon$ then $f(x) = 0$ for $x \geq K$, which means that $|f(x)| = 0$ and $\text{sgn}(f(x)) = 0$ for $x \geq K$ and therefore, $|f|$ and $\text{sgn}(f)$ belong to \mathcal{S}_e .

If $a_0 \neq \varepsilon$ then there exists a real number $L \geq K$ such that either $f(x) > 0$ or $f(x) < 0$ for $x \geq L$, since $f(x) \sim \alpha_0 e^{a_0 x}$ for $x \rightarrow \infty$ with $\alpha_0 \neq 0$ by Proposition 2.3. Hence, either $|f(x)| = f(x)$ and $\text{sgn}(f(x)) = 1$, or $|f(x)| = -f(x)$ and $\text{sgn}(f(x)) = -1$ for $x \geq L$. So in this case $|f|$ and $\text{sgn}(f)$ also belong to \mathcal{S}_e .

Since f and g are analytic in $[K, \infty)$, the functions ρf , $f + g$, $f - g$, $f \cdot g$ and f^n are also analytic in $[K, \infty)$.

If $a_0 = \varepsilon$ then $f(x) = 0$ for $x \geq K$ and then $\frac{1}{f(x)}$ is not defined for $x \geq K$. If $a_0 \neq \varepsilon$ then we already know that there exists a real number $L \geq K$ such that either $f(x) > 0$ or $f(x) < 0$ for $x \geq L$. So $\frac{1}{f}$ is defined and analytic in $[L, \infty)$. An analogous reasoning can be made for $\frac{f}{g}$.

If $a_0 = \varepsilon$ then $\sqrt{f(x)} = 0$ for $x \geq K$. So \sqrt{f} is analytic in $[K, \infty)$. If $a_0 \neq \varepsilon$ and if $\sqrt{f(x)}$ is defined for x large enough then $\alpha_0 > 0$ and there exists a real number $L \geq K$ such that $\forall x \geq L : f(x) > 0$. Therefore, \sqrt{f} is analytic in $[L, \infty)$.

Now we prove that these functions can be written as a sum of exponentials or as an absolutely convergent series of exponentials.

We may assume without loss of generality that both m and n are equal to ∞ . If m or n are

finite then we can always add dummy terms of the form $0 \cdot e^{\varepsilon x}$ to $f(x)$ or $g(x)$. Afterwards we can then remove terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$ to obtain an expression with nonzero coefficients and decreasing exponents. So now we have two absolute convergent series of exponentials f and g .

If $\rho = 0$ then $\rho f(x) = 0$ and thus $\rho f \in \mathcal{S}_e$.

If $\rho \neq 0$ then $\rho f(x) = \sum_{i=0}^{\infty} (\rho \alpha_i) e^{a_i x}$. The series $\sum_{i=0}^{\infty} (\rho \alpha_i) e^{a_i x}$ also converges absolutely and has the same exponents as $f(x)$. Consequently, $\rho f \in \mathcal{S}_e$.

The sum $f(x) + g(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} + \sum_{j=0}^{\infty} \beta_j e^{b_j x}$ is also an absolutely convergent series of exponentials for $x \geq K$. This means that sum of this series does not change if we rearrange the terms. Therefore, $f(x) + g(x)$ can be written in the format of Definition 2.1 by reordering the terms and adding up terms with equal exponents and removing terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$, if there are any. If the result is a series then the sequence of exponents is decreasing and unbounded from below. So $f + g \in \mathcal{S}_e$.

Since $f - g = f + (-1) \cdot g$, the function $f - g$ also belongs to \mathcal{S}_e .

The series corresponding to f and g converge absolutely for $x \geq K$. Therefore, their Cauchy product will also converge absolutely for $x \geq K$ and it will be equal to $f \cdot g$:

$$f(x) \cdot g(x) = \sum_{i=0}^{\infty} \sum_{j=0}^i \alpha_j \beta_{i-j} e^{(a_j + b_{i-j})x} \quad \text{for } x \geq K .$$

Using the same procedure as for $f + g$, we can also write this product in the format (10) or (11). Hence, $f \cdot g \in \mathcal{S}_e$.

We can make repeated use of the fact that $f \cdot g \in \mathcal{S}_e$ if $f, g \in \mathcal{S}_e$ to prove that f^n with $n \in \mathbb{N}$ also belongs to \mathcal{S}_e .

If $\frac{1}{f}$ is defined then there exists a real number $L \geq K$ such that $\forall x \geq L : f(x) \neq 0$. Hence, $a_0 \neq \varepsilon$. We rewrite $f(x)$ as follows:

$$f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{\infty} \frac{\alpha_i}{\alpha_0} e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} (1 + p(x))$$

with

$$p(x) = \sum_{i=1}^{\infty} \gamma_i e^{c_i x}$$

where $\gamma_i = \frac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0 < 0$. So p also belongs to \mathcal{S}_e .

If $c_1 = \varepsilon$ then $p(x) = 0$ and $\frac{1}{f(x)} = \frac{1}{\alpha_0} e^{-a_0 x}$. Hence, $\frac{1}{f} \in \mathcal{S}_e$.

Now assume that $c_1 \neq \varepsilon$. Since $\{c_i\}_{i=1}^{\infty}$ is a decreasing sequence of negative numbers with $\lim_{i \rightarrow \infty} c_i = \varepsilon = -\infty$ and since p converges uniformly in $[L, \infty) \subset [K, \infty)$ by Proposition 2.2, $|p(x)|$ will be smaller than 1 if x is large enough, say, if $x \geq M$. If we use the Taylor series expansion of $\frac{1}{1+x}$, we obtain

$$\frac{1}{1+p(x)} = \sum_{k=0}^{\infty} (-1)^k p^k(x) \quad \text{if } |p(x)| < 1 . \quad (13)$$

We already know that $p \in \mathcal{S}_e$. Hence, $p^n \in \mathcal{S}_e$. Since $|p(x)| < 1$ for $x \geq M$ and since the highest exponent in $p^n(x)$ is equal to nc_1 , which means that the dominant exponent of $p^n(x)$ tends to $-\infty$ as $n \rightarrow \infty$, the coefficients and the exponents of more and more successive terms of the partial sum $s_n(x) = \sum_{k=0}^n (-1)^k p^k(x)$ will not change any more as n becomes larger and larger. Therefore, the series in (13) also is a sum of exponentials:

$$\frac{1}{1+p(x)} = \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=1}^{\infty} \gamma_i e^{c_i x} \right)^k = \sum_{k=0}^{\infty} d_k e^{\delta_k x} \quad \text{for } x \geq M . \quad (14)$$

First we prove that this series also converges absolutely. Define

$$p^*(x) = \sum_{i=1}^{\infty} |\gamma_i| e^{c_i x} .$$

The series p^* converges absolutely and uniformly in $[K, \infty)$ since p converges absolutely in $[K, \infty)$. Furthermore, $\{c_i\}_{i=1}^{\infty}$ is a decreasing and unbounded sequence of negative real numbers. So $|p^*(x)|$ will be smaller than 1 if x is large enough, say, if $x \geq N$. So

$$\frac{1}{1+p^*(x)} = \sum_{k=0}^{\infty} (-1)^k (p^*(x))^k \quad \text{for } x \geq N .$$

This series converges absolutely in $[N, \infty)$ and since

$$\begin{aligned} \sum_{k=0}^{\infty} |d_k| e^{\delta_k x} &\leq \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} |\gamma_i| e^{c_i x} \right)^k \\ &\leq \sum_{k=0}^{\infty} |(p^*(x))^k| , \end{aligned}$$

the series (14) also converges absolutely in $[N, \infty)$.

Note that the set of exponents of the series (13) and (14) will have no finite accumulation point since the highest exponent in p^n is equal to nc_1 . Since the series (14) converges absolutely, we can reorder the terms. After reordering the terms and adding up terms with the same exponents and removing terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$ if necessary, the sequence of exponents will be decreasing and unbounded from below.

This means that $\frac{1}{1+p} \in \mathcal{S}_e$ and thus also $\frac{1}{f} \in \mathcal{S}_e$.

As a consequence, $\frac{f}{g} = f \cdot \frac{1}{g}$ also belongs to \mathcal{S}_e .

If $a_0 = \varepsilon$ then $\sqrt{f(x)} = 0$ for $x \geq K$ and thus $\sqrt{f} \in \mathcal{S}_e$.

If $a_0 \neq \varepsilon$ and if \sqrt{f} is defined in some interval $[P, \infty)$ then $\alpha_0 > 0$ and

$$\sqrt{f(x)} = \sqrt{\alpha_0} e^{\frac{\alpha_0}{2}x} \sqrt{1+p(x)} .$$

Now we can use the Taylor series expansion of $\sqrt{1+x}$. This leads to

$$\sqrt{1+p(x)} = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-k\right) k!} p^k(x) \quad \text{if } |p(x)| < 1 ,$$

where Γ is the gamma function. If we apply the same reasoning as for $\frac{1}{1+p}$, we obtain $\sqrt{1+p} \in \mathcal{S}_e$ and thus also $\sqrt{f} \in \mathcal{S}_e$. \square

Now we give an alternative proof for the existence theorem of the max-algebraic SVD:

Theorem 2.6 (Existence of the singular value decomposition in \mathbb{S}_{\max})

Let $A \in \mathbb{S}^{m \times n}$ and let $r = \min(m, n)$. Then there exist a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m \times n}$ and matrices $U \in (\mathbb{S}^\vee)^{m \times m}$ and $V \in (\mathbb{S}^\vee)^{n \times n}$ such that

$$A \nabla U \otimes \Sigma \otimes V^T \tag{15}$$

with

$$\begin{aligned} U^T \otimes U &\nabla E_m \\ V^T \otimes V &\nabla E_n \end{aligned}$$

and $\|A\|_{\oplus} \geq \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq \varepsilon$ where $\sigma_i = (\Sigma)_{ii}$.

Every decomposition of the form (15) that satisfies the above conditions is called a max-algebraic singular value decomposition of A .

Proof: If $A \in \mathbb{S}^{m \times n}$ has entries that are not signed we can always define a signed m by n matrix \hat{A} such that

$$\begin{aligned} \hat{a}_{ij} &= a_{ij} && \text{if } a_{ij} \text{ is signed,} \\ &= |a_{ij}|_{\oplus} && \text{if } a_{ij} \text{ is not signed.} \end{aligned}$$

Since $|\hat{a}_{ij}|_{\oplus} = |a_{ij}|_{\oplus}$ for all i, j , we have $\|\hat{A}\|_{\oplus} = \|A\|_{\oplus}$. Furthermore, $\forall a, b \in \mathbb{S} : a \nabla b \Rightarrow a \bullet \nabla b$, which means that $\hat{A} \nabla U \otimes \Sigma \otimes V^T$ would imply $A \nabla U \otimes \Sigma \otimes V^T$. Therefore, it is sufficient to prove this theorem for signed matrices A .

So from now on we assume that A is signed. First we define $c = \|A\|_{\oplus}$.

If $c = \varepsilon$ then $A = \mathcal{E}_{m \times n}$. If we take $U = E_m$, $\Sigma = \mathcal{E}_{m \times n}$ and $V = E_n$, we have $A = U \otimes \Sigma \otimes V^T$, $U^T \otimes U = E_m$, $V^T \otimes V = E_n$ and $\sigma_1 = \sigma_2 = \dots = \sigma_r = \varepsilon = \|A\|_{\oplus}$. So $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A .

From now on we assume that $c \neq \varepsilon$. We may also assume without loss of generality that $m \leq n$. If $m > n$ then we can apply the subsequent reasoning on A^T since $A \nabla U \otimes \Sigma \otimes V^T$ if and only if $A^T \nabla V \otimes \Sigma^T \otimes U^T$. So $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A if and only if $V \otimes \Sigma^T \otimes U^T$ is a max-algebraic SVD of A^T .

Now we have to distinguish between two different situations depending on whether or not all the a_{ij} 's have a finite max-absolute value.

Case 1: all the a_{ij} 's have a finite max-absolute value.

We construct a matrix-valued function $\tilde{A}(\cdot) = \mathcal{F}(A, \cdot)$. Hence, we have $\tilde{a}_{ij}(s) = \gamma_{ij} e^{c_{ij}s}$ for all $s \in \mathbb{R}$ with $\gamma_{ij} \in \mathbb{R}_0$ and $c_{ij} = |a_{ij}|_{\oplus} \in \mathbb{R}_\varepsilon$.

We select the coefficients γ_{ij} such that the generic rank of $\tilde{A}(\cdot)$ is m . This can be effectuated by choosing the γ_{ij} 's such that

$$\text{rank } \tilde{A}(0) = \text{rank } \Gamma = m \tag{16}$$

where $(\Gamma)_{ij} = \gamma_{ij}$, since the rank of $\tilde{A}(\cdot)$ is constant except in some non-generic points where the rank drops. Therefore, condition (16) ensures that the generic rank of $\tilde{A}(\cdot)$ is m and that $\tilde{A}(\cdot)$ has no singular values that are identically zero.

Furthermore, if $m > 1$ then we select the γ_{ij} 's such that $\tilde{A}(s)$ has no multiple singular values except maybe in some non-generic points. This condition will guarantee the asymptotic quadratic convergence of Kogbetliantz's SVD algorithm. Since we are only interested in the asymptotic behavior of the entries of $\tilde{A}(s)$, we can always add an extra exponential of the form $\delta_{ij}e^{d_{ij}s}$ with $d_{ij} < c_{ij}$ to $\tilde{a}_{ij}(s)$ such that $\text{rank } \tilde{A}(0) = \text{rank } (\Gamma + \Delta) = m$ if this should be necessary to obtain distinct singular values for almost all values of s .

Now we define a matrix-valued function $\tilde{B}(\cdot)$ such that $\tilde{B}(s) = e^{-(c+1)s}\tilde{A}(s)$. So

$$\tilde{b}_{ij}(s) = \gamma_{ij}e^{-b_{ij}s} + \delta_{ij}e^{-f_{ij}s} \quad \text{for all } s \in \mathbb{R}$$

with $b_{ij} = c + 1 - c_{ij} > 0$ and $f_{ij} = c + 1 - d_{ij} > b_{ij}$. The entries of $\tilde{B}(\cdot)$ are in \mathcal{S}_e .

If $I \subset \mathbb{R}$ then $\tilde{U}(s)\tilde{\Psi}(s)\tilde{V}^T(s)$ is a (constant) SVD of $\tilde{B}(s)$ for each $s \in I$ if and only if $\tilde{U}(s)\tilde{\Sigma}(s)\tilde{V}^T(s)$ with $\tilde{\Sigma}(s) = e^{(c+1)s}\tilde{\Psi}(s)$ is a (constant) SVD of $\tilde{A}(s)$ for each $s \in I$.

We shall apply Kogbetliantz's SVD algorithm [4, 11] on $\tilde{B}(\cdot)$. This algorithm can be considered as an extension of Jacobi's method for the calculation of the eigenvalue decomposition of a symmetric matrix. For a matrix $B \in \mathbb{R}^{m \times n}$ (with $m \leq n$) a sequence of matrices is generated as follows:

$$\begin{aligned} X_0 &= I_m, & Y_0 &= I_n, & S_0 &= B \\ X_k &= G_k X_{k-1}, & Y_k &= H_k Y_{k-1}, & S_k &= G_k S_{k-1} H_k^T \end{aligned} \quad \text{for } k = 1, 2, 3, \dots$$

such that

$$\|S_k\|_{\text{off}} \stackrel{\text{def}}{=} \sqrt{\sum_{i \neq j} (S_k)_{ij}^2}$$

decreases monotonously. So S_k tends more and more to a diagonal matrix as the iteration process progresses. If $m = n$ then the orthogonal updating transformations G_k and H_k are elementary rotations that are chosen such that $(S_k)_{i_k j_k} = (S_k)_{j_k i_k} = 0$ for some pair of indices (i_k, j_k) . As a result we have

$$\|S_k\|_{\text{off}}^2 = \|S_{k-1}\|_{\text{off}}^2 - (S_{k-1})_{i_k j_k}^2 - (S_{k-1})_{j_k i_k}^2 .$$

If $m < n$ and if $m < j_k \leq n$ then only $S_{i_k j_k}$ is zeroed and only one transformation is applied (the identity matrix is taken for G_k). We shall use the cyclic version of Kogbetliantz's SVD algorithm: the indices i_k and j_k are chosen such that the entries in the upper triangular part of the S_k 's are selected row by row. If $m = n$, this yields the following sequence for the pairs of indices (i_k, j_k) :

$$(1, 2) \rightarrow (1, 3) \rightarrow (1, 4) \rightarrow \dots \rightarrow (1, n) \rightarrow (2, 3) \rightarrow (2, 4) \rightarrow \dots \rightarrow (n-1, n) .$$

If $m < n$ then the last pair of indices is (m, n) . A complete cycle is called a sweep and corresponds to $N = \frac{(2n - m - 1)m}{2}$ iterations.

Note that

$$\forall k \in \mathbb{N} : B = X_k^T S_k Y_k .$$

Since G_k and H_k are orthogonal matrices, we have

$$\forall k \in \mathbb{N} : \|S_k\|_{\text{F}} = \|B\|_{\text{F}} \quad (17)$$

and

$$\forall k \in \mathbb{N} : X_k^T X_k = I_m \text{ and } Y_k^T Y_k = I_n .$$

If we define $S = \lim_{k \rightarrow \infty} S_k$, $X = \lim_{k \rightarrow \infty} X_k$ and $Y = \lim_{k \rightarrow \infty} Y_k$ then S is a diagonal matrix and X and Y are orthogonal matrices. After applying a permutation such that the diagonal entries of S are ordered we obtain an SVD of B :

$$B = (X^T P^T) \cdot (P S P^T) \cdot (P Y) = U \Psi V^T$$

with $U = X^T P^T$, $\Psi = P S P^T$ and $V = Y^T P^T$ and where P is a permutation matrix. The convergence of the cyclic Kogbetliantz algorithm is quadratic for k large enough [15]:

$$\exists K \in \mathbb{N} \text{ such that } \forall k \geq K : \|S_{k+N}\|_{\text{off}} \leq c \|S_k\|_{\text{off}}^2 . \quad (18)$$

The operations used in Kogbetliantz's SVD algorithm are additions, multiplications, subtractions, divisions, square roots, absolute values and sign functions. So if we apply this algorithm to a matrix with entries in \mathcal{S}_e then the entries of all the matrices generated during the iteration process also belong to \mathcal{S}_e by Theorem 2.5.

If f , g and h belong to \mathcal{S}_e , they are asymptotically equivalent to an exponential in the neighborhood of ∞ by Proposition 2.3. So if L is large enough, then $f(L) \geq 0$ and $g(L) \geq h(L)$ imply that $\forall s \geq L : f(s) \geq 0$ and $g(s) \geq h(s)$. This is one of the reasons that Kogbetliantz's SVD algorithm also works for matrices with entries in \mathcal{S}_e instead of in \mathbb{R} .

Now we apply Kogbetliantz's SVD algorithm on $\tilde{B}(\cdot)$. Let $\tilde{S}_k(\cdot)$, $\tilde{X}_k(\cdot)$ and $\tilde{Y}_k(\cdot)$ be the matrix-valued functions obtained in the k th step of the algorithm. Let $\tilde{\Psi}_k(\cdot)$, $\tilde{U}_k(\cdot)$ and $\tilde{V}_k(\cdot)$ be the permuted versions of $\tilde{S}_k(\cdot)$, $\tilde{X}_k(\cdot)$ and $\tilde{Y}_k(\cdot)$ respectively.

The exponents of the entries of $\tilde{B}(\cdot)$ are negative and the same holds for the exponents of the entries of $\tilde{S}_k(\cdot)$ since $\|\tilde{S}_k(\cdot)\|_{\text{F}} = \|\tilde{B}(\cdot)\|_{\text{F}}$ by (17). Hence, (18) means that the largest off-diagonal exponent approximately doubles each N steps. Since the Frobenius norm of $\tilde{S}_k(\cdot)$ stays constant during the iteration, the exponents of the updates of the diagonal entries also approximately double each N steps. Therefore, more and more successive terms of the series of the diagonal entries of $\tilde{S}_k(\cdot)$ stay constant as the iteration process progresses. This also holds for the series of the entries of $\tilde{X}_k(\cdot)$ and $\tilde{Y}_k(\cdot)$.

In theory we should run the iteration process forever. However, since we are only interested in the asymptotic behavior of the singular values and the entries of the singular vectors we can stop the iteration as soon as the dominant exponents do not change anymore, i.e. after a *finite* number of iteration steps.

If $\tilde{U}(\cdot) \tilde{\Psi}(\cdot) \tilde{V}^T(\cdot)$ is a path of (approximate) SVDs of $\tilde{B}(\cdot)$ on some interval $[L, \infty)$ that was obtained by the above procedure, then the entries of $\tilde{U}(\cdot)$, $\tilde{\Psi}(\cdot)$ and $\tilde{V}(\cdot)$ are in \mathcal{S}_e and $\tilde{U}(s)$ and $\tilde{V}(s)$ are orthogonal for each $s \in [L, \infty)$.

Furthermore, it should be pointed out that we are not really interested in a path of *exact* SVDs of $\tilde{B}(\cdot)$. Let $\tilde{\Delta}_k(\cdot)$ be the diagonal matrix-valued function obtained by removing the

off-diagonal entries from $\tilde{\Psi}_k(\cdot)$ after the k th iteration step. If we define the matrix-valued function $\tilde{C}_k(\cdot) = \tilde{U}_k(\cdot) \tilde{\Delta}_k(\cdot) \tilde{V}_k^T(\cdot)$, then we have a path of exact SVDs of $\tilde{C}(\cdot)$ on some interval $[L, \infty)$. This means that we could also stop the iteration process as soon as

$$\tilde{b}_{ij}(s) \sim \tilde{c}_{ij}(s), \quad s \rightarrow \infty \quad \text{for all } i, j .$$

Define a matrix-valued function $\tilde{\Sigma}(\cdot)$ such that $\tilde{\Sigma}(s) = e^{(c+1)s} \tilde{\Psi}(s)$. Then $\tilde{U}(\cdot) \tilde{\Sigma}(\cdot) \tilde{V}^T(\cdot)$ is a path of (approximate) SVDs of $\tilde{A}(\cdot)$ on $[L, \infty)$:

$$\tilde{A}(s) \sim \tilde{U}(s) \tilde{\Sigma}(s) \tilde{V}^T(s), \quad s \rightarrow \infty \quad (19)$$

$$\tilde{U}^T(s) \tilde{U}(s) = I_m \quad \text{if } s \geq L \quad (20)$$

$$\tilde{V}^T(s) \tilde{V}(s) = I_n \quad \text{if } s \geq L . \quad (21)$$

So now we have proved that there exists a path of (approximate) SVDs of $\tilde{A}(\cdot)$ for which the singular values and the entries of the singular vectors belong to \mathcal{S}_e and are asymptotically equivalent to an exponential in the neighborhood of ∞ by Proposition 2.3.

If we apply the reverse mapping \mathcal{R} , we obtain a max-algebraic SVD of A . Since we have used numbers instead of parameters for the coefficients of the exponentials in $\mathcal{F}(A, \cdot)$, the coefficients of the exponentials in the singular values and the entries of the singular vectors are also numbers. Therefore, the reverse mapping only yields signed results. If we define

$$\Sigma = \mathcal{R}(\tilde{\Sigma}(\cdot)), \quad U = \mathcal{R}(\tilde{U}(\cdot)), \quad V = \mathcal{R}(\tilde{V}(\cdot)) \quad \text{and } \sigma_i = (\Sigma)_{ii} = \mathcal{R}(\tilde{\sigma}_i(\cdot)) ,$$

then Σ is a max-algebraic diagonal matrix and U and V have signed entries. If we apply the reverse mapping \mathcal{R} to (19)–(21), we get

$$A \nabla U \otimes \Sigma \otimes V^T$$

$$U^T \otimes U \nabla E_m$$

$$V^T \otimes V \nabla E_n .$$

The $\tilde{\sigma}_i(\cdot)$'s are positive in $[L, \infty)$ and therefore, $\sigma_i \in \mathbb{R}_\varepsilon$. Since the $\tilde{\sigma}_i(\cdot)$'s are ordered in $[L, \infty)$, their dominant exponentials are also ordered. Hence, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq \varepsilon$.

We have $\|\tilde{A}(s)\|_F \sim \gamma e^{cs}$, $s \rightarrow \infty$ with $\gamma > 0$ since $c = \|A\|_\oplus$ is the largest exponent that appears in the entries of $\tilde{A}(\cdot)$. So $\mathcal{R}(\|\tilde{A}(\cdot)\|_F) = c = \|A\|_\oplus$. If $M \in \mathbb{R}^{m \times n}$ then

$$\frac{1}{\sqrt{n}} \|M\|_F \leq \|M\|_2 \leq \|M\|_F .$$

Hence,

$$\frac{1}{\sqrt{n}} \|\tilde{A}(s)\|_F \leq \|\tilde{A}(s)\|_2 \leq \|\tilde{A}(s)\|_F \quad \text{if } s \geq L .$$

Since $\tilde{\sigma}_1(s) \sim \|\tilde{A}(s)\|_2$, $s \rightarrow \infty$ and since the mapping \mathcal{R} preserves the order, this leads to $\|A\|_\oplus \leq \sigma_1 \leq \|A\|_\oplus$ and consequently, $\sigma_1 = \|A\|_\oplus$.

Case 2: not all the a_{ij} 's have a finite max-absolute value.

Now some of the entries of A are equal to ε and it is possible that there are singular values that are identically zero. Therefore, we cannot use the technique that was used in Case 1 to ensure that there are no multiple singular values. Hence, we cannot guarantee the quadratic convergence of Kogbetliantz's SVD algorithm anymore without making some extra assumptions (see [2, 3]).

Therefore, we construct a sequence $\{A_k\}_{k=0}^{\infty}$ of m by n matrices such that

$$\begin{aligned} (A_k)_{ij} &= a_{ij} && \text{if } |a_{ij}|_{\oplus} \text{ is finite,} \\ &= P - k && \text{if } |a_{ij}|_{\oplus} = \varepsilon, \end{aligned}$$

where $P = \|A\|_{\oplus} - 1$. So the entries of the matrices A_k are finite and $\|A\|_{\oplus} = \|A_k\|_{\oplus}$ for all $k \in \mathbb{N}$.

Now we construct the corresponding sequence of $\mathcal{F}(A_k, \cdot)$'s where we always take the same coefficients γ_{ij} , δ_{ij} and d_{ij} . We calculate a path of (approximate) SVDs $\tilde{U}_k(\cdot) \tilde{\Sigma}_k(\cdot) \tilde{V}_k^T(\cdot)$ for each $\mathcal{F}(A_k, \cdot)$ using the method discussed above. In general, it is possible that some sequences of the dominant exponents and the corresponding coefficients of the entries of $\tilde{U}_k(\cdot)$ and $\tilde{V}_k(\cdot)$ have more than one accumulation point. However, since we use a fixed calculation scheme (the cyclic Kogbetliantz algorithm), all the sequences will have exactly one accumulation point. So some of the dominant exponents will reach a finite limit as k goes to ∞ , while the other dominant exponents will tend to $-\infty$. If we take the reverse mapping \mathcal{R} , we get a sequence of max-algebraic SVDs $\{U_k \otimes \Sigma_k \otimes V_k^T\}_{k=0}^{\infty}$ where some of the entries, viz. those that correspond to dominant exponents that tend to $-\infty$, tend to ε as $k \rightarrow \infty$. Note that $(\Sigma_k)_{ii} \leq (\Sigma_k)_{11} \leq \|A\|_{\oplus}$ for all i .

If we define

$$U = \lim_{k \rightarrow \infty} U_k, \quad \Sigma = \lim_{k \rightarrow \infty} \Sigma_k \quad \text{and} \quad V = \lim_{k \rightarrow \infty} V_k$$

then

$$\begin{aligned} A &\nabla U \otimes \Sigma \otimes V^T \\ U^T \otimes U &\nabla E_m \\ V^T \otimes V &\nabla E_n . \end{aligned}$$

Since the diagonal entries of the Σ_k 's are max-positive or zero, ordered and less than or equal to $\|A\|_{\oplus}$, the diagonal entries of Σ are also max-positive or zero, ordered and less than or equal to $\|A\|_{\oplus}$. So $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A . \square

Although this alternative proof technique leads to a proof that is longer than that of [8], it has the advantage that it can also be used to prove the existence of other max-algebraic matrix decompositions fairly easily. We shall demonstrate this by proving the existence of the max-algebraic equivalent of the QR decomposition (QRD).

Definition 2.7 (QR decomposition) *The QR decomposition of a real m by n matrix A is given by*

$$A = QR ,$$

where $Q \in \mathbb{R}^{m \times m}$ is orthogonal and $R \in \mathbb{R}^{m \times n}$ is upper triangular.

Note that $\|A\|_F = \|R\|_F$ since Q is an orthogonal matrix.

Theorem 2.8 *If $\tilde{A}(\cdot) \in (\mathcal{S}_e)^{m \times n}$ then there exists a path of QR factorizations $\tilde{Q}(\cdot)\tilde{R}(\cdot)$ of $\tilde{A}(\cdot)$ for which the entries of $\tilde{Q}(\cdot)$ and $\tilde{R}(\cdot)$ also belong to \mathcal{S}_e .*

Proof: We can use Householder or Givens transformations to calculate the QR decomposition of a matrix [10]. If we apply the algorithms in their most elementary form (i.e. without the refinements necessary to avoid overflow and to guarantee numerical stability), we only have to use additions, multiplications, subtractions, divisions and square roots. Hence, the entries of resulting matrices $\tilde{Q}(\cdot)$ and $\tilde{R}(\cdot)$ belong to \mathcal{S}_e by Theorem 2.5. \square

As a direct consequence we have

Theorem 2.9 (Max-algebraic QR decomposition) *If $A \in \mathbb{S}^{m \times n}$ then there exist a matrix $Q \in (\mathbb{S}^\vee)^{m \times m}$ and a max-algebraic upper triangular matrix $R \in (\mathbb{S}^\vee)^{m \times n}$ such that*

$$A \nabla Q \otimes R \tag{22}$$

with

$$Q^T \otimes Q \nabla E_m$$

and $\|R\|_{\oplus} \leq \|A\|_{\oplus}$.

Every decomposition of the form (22) that satisfies the above conditions is called a max-algebraic QR decomposition of A .

The condition $\sigma_1 \leq \|A\|_{\oplus}$ in the definition of the max-algebraic SVD is necessary in order to obtain singular values that are bounded from above as is shown by the following example:

Example 2.10 Consider

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and define

$$U = \begin{bmatrix} 0 & 0 & \ominus 0 & 0 \\ 0 & \ominus 0 & \ominus 0 & \ominus 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ominus 0 & \ominus 0 \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \sigma \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & \ominus 0 & 0 & \ominus 0 \\ 0 & 0 & 0 & \ominus 0 \\ \ominus 0 & 0 & 0 & \ominus 0 \\ 0 & \ominus 0 & 0 & 0 \end{bmatrix}.$$

Then we have

$$U^T \otimes U = V^T \otimes V = \begin{bmatrix} 0 & 0^\bullet & 0^\bullet & 0^\bullet \\ 0^\bullet & 0 & 0^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet & 0 & 0^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet & 0 \end{bmatrix} \nabla E_4$$

and

$$U \otimes \Sigma \otimes V^T = \begin{bmatrix} \sigma^\bullet & \sigma^\bullet & \sigma^\bullet & \sigma^\bullet \\ \sigma^\bullet & \sigma^\bullet & \sigma^\bullet & \sigma^\bullet \\ \sigma^\bullet & \sigma^\bullet & \sigma^\bullet & \sigma^\bullet \\ \sigma^\bullet & \sigma^\bullet & \sigma^\bullet & \sigma^\bullet \end{bmatrix}, \tag{23}$$

which means that $U \otimes \Sigma \otimes V^T \nabla A$ for every $\sigma \geq 0$.

So if the condition $\sigma_1 \leq \|A\|_{\oplus}$ would not be included in the definition of the max-algebraic SVD, (23) would be a max-algebraic SVD of A for every $\sigma \geq 0$. \square

Likewise, the condition $\|R\|_{\oplus} \leq \|A\|_{\oplus}$ in Theorem 2.9 is necessary to bound the components of R from above:

Example 2.11 Consider

$$A = \begin{bmatrix} 0 & \ominus 0 & 0 \\ 0 & 0 & 0 \\ \ominus 0 & 0 & 0 \end{bmatrix}.$$

Without the condition $\|R\|_{\oplus} \leq \|A\|_{\oplus}$ every max-algebraic product of the form

$$Q \otimes R = \begin{bmatrix} 0 & \ominus 0 & 0 \\ 0 & 0 & \ominus 0 \\ \ominus 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & \varepsilon & \rho \\ \varepsilon & 0 & \rho \\ \varepsilon & \varepsilon & \rho \end{bmatrix} = \begin{bmatrix} 0 & \ominus 0 & \rho \bullet \\ 0 & 0 & \rho \bullet \\ \ominus 0 & 0 & \rho \bullet \end{bmatrix}$$

with $\rho \geq 0$ would have been a max-algebraic QRD of A . \square

In [8] we have defined a rank based on the minimal max-algebraic SVD of a matrix. The same can be done with the max-algebraic QRD: we could define the max-algebraic QR rank of a matrix A as the minimal possible number of non- ε rows of R over the set of all possible max-algebraic QRDs of A :

Definition 2.12 (Max-algebraic QR rank) Let $A \in \mathbb{S}^{m \times n}$. The max-algebraic QR rank of A is defined as

$$\text{rank}_{\oplus, \text{QR}}(A) = \min \left\{ \rho \mid A \nabla \bigoplus_{i=1}^{\rho} q_i \otimes R_i, \begin{array}{l} Q \otimes R \text{ is a max-algebraic} \\ \text{QR decomposition of } A \end{array} \right\}$$

where q_i is the i th column of Q , R_i is the i th row of R and $\bigoplus_{i=1}^{\rho} q_i \otimes R_i$ is equal to $\mathcal{E}_{m \times n}$ by definition.

3 Extensions of the max-algebraic SVD

In this section we propose possible extensions of the definition of the max-algebraic SVD. If U is a (real) m by m matrix then $U^T U = I_m$ if and only if $U U^T = I_m$. However, in the extended max algebra $U^T \otimes U \nabla E_m$ does not always imply $U \otimes U^T \nabla E_m$ as is shown by the following example:

Example 3.1 Consider

$$U = \begin{bmatrix} 0 & \ominus(-1) & 0 & -1 \\ -1 & 0 & -1 & \ominus 0 \\ \ominus 0 & \varepsilon & 0 & -1 \\ \varepsilon & 0 & \varepsilon & 0 \end{bmatrix}.$$

We have

$$U^T \otimes U = \begin{bmatrix} 0 & (-1)^\bullet & 0^\bullet & (-1)^\bullet \\ (-1)^\bullet & 0 & (-1)^\bullet & 0^\bullet \\ 0^\bullet & (-1)^\bullet & 0 & (-1)^\bullet \\ (-1)^\bullet & 0^\bullet & (-1)^\bullet & 0 \end{bmatrix} \nabla E_4$$

but

$$U \otimes U^T = \begin{bmatrix} 0 & (-1)^\bullet & 0^\bullet & (-1)^\bullet \\ (-1)^\bullet & 0 & (-1)^\bullet & 0^\bullet \\ 0^\bullet & (-1)^\bullet & 0 & -1 \\ (-1)^\bullet & 0^\bullet & -1 & 0 \end{bmatrix} \nabla E_4$$

since $(U \otimes U^T)_{34} = (U \otimes U^T)_{43} = -1 \nabla \varepsilon$. \square

In the proof of the existence theorem of the max-algebraic SVD we have seen that for every matrix $A \in \mathbb{S}^{m \times n}$ (with finite entries) there is at least one max-algebraic SVD that corresponds to a path of (approximate) SVDs $\tilde{U}(\cdot) \tilde{\Sigma}(\cdot) \tilde{V}^T(\cdot)$ of $\tilde{A}(\cdot) = \mathcal{F}(A, \cdot)$ on some interval $[L, \infty)$. So if $s \geq L$ then $\tilde{U}(s)$ satisfies both $\tilde{U}^T(s) \tilde{U}(s) = I_m$ and $\tilde{U}(s) \tilde{U}^T(s) = I_m$, and $\tilde{V}(s)$ satisfies both $\tilde{V}^T(s) \tilde{V}(s) = I_n$ and $\tilde{V}(s) \tilde{V}^T(s) = I_n$. Therefore, we could add two extra conditions to the definition of the max-algebraic SVD: $U \otimes U^T \nabla E_m$ and $V \otimes V^T \nabla E_n$. This yields:

Theorem 3.2 *Let $A \in \mathbb{S}^{m \times n}$ and let $r = \min(m, n)$. Then there exist a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m \times n}$ and matrices $U \in (\mathbb{S}^\vee)^{m \times m}$ and $V \in (\mathbb{S}^\vee)^{n \times n}$ such that*

$$A \nabla U \otimes \Sigma \otimes V^T$$

with

$$U^T \otimes U \nabla E_m$$

$$U \otimes U^T \nabla E_m$$

$$V^T \otimes V \nabla E_n$$

$$V \otimes V^T \nabla E_n$$

and $\|A\|_{\oplus} \geq \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq \varepsilon$ where $\sigma_i = (\Sigma)_{ii}$.

Furthermore, the left singular vectors of the path of (approximate) SVDs of $\tilde{A}(\cdot)$ will be linearly independent on $[L, \infty)$ since $\tilde{U}^T(s) \tilde{U}(s) = I_m$ for each value of $s \geq L$. The right singular vectors will also be linearly independent. However, in the extended max algebra the condition $U^T \otimes U \nabla E_m$ is not sufficient for max-linear independence of the columns of U even if the entries U are signed as is shown by the following example:

Example 3.3 Consider

$$U = \begin{bmatrix} 0 & \ominus 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \ominus 0 \end{bmatrix}.$$

We have

$$U^T \otimes U = U \otimes U^T = \begin{bmatrix} 0 & 0^\bullet & 0^\bullet \\ 0^\bullet & 0 & 0^\bullet \\ 0^\bullet & 0^\bullet & 0 \end{bmatrix} \nabla E_3.$$

Furthermore, $\det_{\oplus} U = 0^{\bullet}$. So by Theorem 1.10 there exists a signed solution of $\alpha_1 \otimes u_1 \oplus \alpha_2 \otimes u_2 \oplus \alpha_3 \otimes u_3 \nabla \mathcal{E}_{3 \times 1}$, viz. $\alpha_1 = 0$, $\alpha_2 = \ominus 0$ and $\alpha_3 = \ominus 0$. Therefore, the vectors u_1 , u_2 and u_3 are max-linearly dependent by Definition 1.11. \square

If we want the left singular vectors to be max-linearly independent and if we also want the right singular vectors to be max-linearly independent, we should have $\det_{\oplus} U \nabla \varepsilon$ and $\det_{\oplus} V \nabla \varepsilon$ by Theorem 1.10. So we could also add these conditions to the definition of the max-algebraic SVD. Note that these conditions also imply that the rows of U and V are max-linearly independent since $\det_{\oplus} U = \det_{\oplus} U^T$ also holds in \mathbb{S}_{\max} . This leads to:

Theorem 3.4 *Let $A \in \mathbb{S}^{m \times n}$ and let $r = \min(m, n)$. Then there exist a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_{\varepsilon}^{m \times n}$ and matrices $U \in (\mathbb{S}^{\vee})^{m \times m}$ and $V \in (\mathbb{S}^{\vee})^{n \times n}$ such that*

$$A \nabla U \otimes \Sigma \otimes V^T$$

with

$$\begin{aligned} U^T \otimes U &\nabla E_m \\ U \otimes U^T &\nabla E_m \\ V^T \otimes V &\nabla E_n \\ V \otimes V^T &\nabla E_n \end{aligned} ,$$

where the rows and the columns of U and V are max-linearly independent or equivalently

$$\begin{aligned} \det_{\oplus} U &\nabla \varepsilon \\ \det_{\oplus} V &\nabla \varepsilon \end{aligned}$$

and with $\|A\|_{\oplus} \geq \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq \varepsilon$, where $\sigma_i = (\Sigma)_{ii}$.

It is obvious that we can also add the above conditions to the definition of the max-algebraic QR decomposition. This leads to:

Theorem 3.5 *If $A \in \mathbb{S}^{m \times n}$ then there exist a matrix $Q \in (\mathbb{S}^{\vee})^{m \times m}$ and a max-algebraic upper triangular matrix $R \in (\mathbb{S}^{\vee})^{m \times n}$ such that*

$$A \nabla Q \otimes R$$

with

$$\begin{aligned} Q^T \otimes Q &\nabla E_m \\ Q \otimes Q^T &\nabla E_m \\ \det_{\oplus} Q &\nabla \varepsilon \end{aligned}$$

and $\|R\|_{\oplus} \leq \|A\|_{\oplus}$.

4 Examples

In this section we give an example of the calculation of a max-algebraic QRD and a max-algebraic SVD of a matrix using the mapping \mathcal{F} . Other examples can be found in [8, 7].

Example 4.1 We calculate a max-algebraic QRD of

$$A = \begin{bmatrix} 2^\bullet & 0 & \ominus(-1) \\ \varepsilon & \ominus 3 & (-2)^\bullet \\ \ominus 2 & -1 & \ominus 4 \end{bmatrix}.$$

We define $\tilde{A}(\cdot) = \mathcal{F}(A, \cdot)$ where we take all the coefficients μ and ν equal to 1:

$$\tilde{A}(s) = \begin{bmatrix} e^{2s} & 1 & -e^{-s} \\ 0 & -e^{3s} & e^{-2s} \\ -e^{2s} & e^{-s} & -e^{4s} \end{bmatrix}.$$

If we use the Givens QR algorithm [10], we get the following path of QR factorizations of $\tilde{A}(\cdot)$:

$$\tilde{Q}(s) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{e^{-3s} + e^{-4s}}{\sqrt{4 + 2e^{-6s} + 4e^{-7s} + 2e^{-8s}}} & -\frac{1}{\sqrt{2 + e^{-6s} + 2e^{-7s} + e^{-8s}}} \\ 0 & -\frac{\sqrt{2}}{\sqrt{2 + e^{-6s} + 2e^{-7s} + e^{-8s}}} & -\frac{e^{-3s} + e^{-4s}}{\sqrt{2 + e^{-6s} + 2e^{-7s} + e^{-8s}}} \\ -\frac{1}{\sqrt{2}} & \frac{e^{-3s} + e^{-4s}}{\sqrt{4 + 2e^{-6s} + 4e^{-7s} + 2e^{-8s}}} & -\frac{1}{\sqrt{2 + e^{-6s} + 2e^{-7s} + e^{-8s}}} \end{bmatrix}$$

$$\tilde{R}(s) = \begin{bmatrix} \sqrt{2}e^{2s} & \frac{1 - e^{-s}}{\sqrt{2}} & \frac{e^{4s} - e^{-s}}{\sqrt{2}} \\ 0 & \frac{\sqrt{2 + e^{-6s} + 2e^{-7s} + e^{-8s}}}{\sqrt{2}e^{-3s}} & -\frac{e^s + 1 + 2e^{-2s} + e^{-4s} + e^{-5s}}{\sqrt{4 + 2e^{-6s} + 4e^{-7s} + 2e^{-8s}}} \\ 0 & 0 & \frac{e^{4s} + e^{-s} - e^{-5s} - e^{-6s}}{\sqrt{2 + e^{-6s} + 2e^{-7s} + e^{-8s}}} \end{bmatrix}$$

for each $s \in \mathbb{R}$. Hence,

$$\tilde{Q}(s) \sim \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{e^{-3s}}{2} & -\frac{1}{\sqrt{2}} \\ 0 & -1 & -\frac{e^{-3s}}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{e^{-3s}}{2} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad s \rightarrow \infty.$$

$$\tilde{R}(s) \sim \begin{bmatrix} \sqrt{2}e^{2s} & \frac{1}{\sqrt{2}} & \frac{e^{4s}}{\sqrt{2}} \\ 0 & e^{3s} & -\frac{e^s}{2} \\ 0 & 0 & \frac{e^{4s}}{\sqrt{2}} \end{bmatrix}, \quad s \rightarrow \infty$$

If we define $Q = \mathcal{R}(\tilde{Q}(\cdot))$ and $R = \mathcal{R}(\tilde{R}(\cdot))$, we obtain

$$Q = \begin{bmatrix} 0 & -3 & \ominus 0 \\ \varepsilon & \ominus 0 & \ominus(-3) \\ \ominus 0 & -3 & \ominus 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 2 & 0 & 4 \\ \varepsilon & 3 & \ominus 1 \\ \varepsilon & \varepsilon & 4 \end{bmatrix}.$$

We have

$$Q \otimes R = \begin{bmatrix} 2 & 0 & 4^\bullet \\ \varepsilon & \ominus 3 & 1^\bullet \\ \ominus 2 & 0^\bullet & \ominus 4 \end{bmatrix} \nabla A$$

$$Q^T \otimes Q = \begin{bmatrix} 0 & (-3)^\bullet & 0^\bullet \\ (-3)^\bullet & 0 & (-3)^\bullet \\ 0^\bullet & (-3)^\bullet & 0 \end{bmatrix} \nabla E_3$$

and $\|R\|_{\oplus} = 4 = \|A\|_{\oplus}$. Furthermore, $Q \otimes Q^T = Q^T \otimes Q \nabla E_3$ and $\det_{\oplus} Q = 0 \nabla \varepsilon$. \square

Example 4.2 Now we calculate a max-algebraic SVD of

$$B = \begin{bmatrix} -4 & \ominus 2 & 3 \\ \ominus(-3) & \ominus(-5) & 1^\bullet \\ \ominus 0 & 4^\bullet & \ominus 5 \\ 4 & \ominus 5 & 5 \end{bmatrix}.$$

We define $\tilde{B}(\cdot) = \mathcal{F}(B, \cdot)$ where we take all the coefficients μ and ν equal to 1:

$$\tilde{B}(s) = \begin{bmatrix} e^{-4s} & -e^{2s} & e^{3s} \\ -e^{-3s} & -e^{-5s} & e^s \\ -1 & e^{4s} & -e^{5s} \\ e^{4s} & -e^{5s} & e^{5s} \end{bmatrix}.$$

We have calculated the constant SVD of $\tilde{B}(s)$ in a set of discrete points and used interpolation to obtain a path of SVDs $\tilde{U}(\cdot) \tilde{\Sigma}(\cdot) \tilde{V}^T(\cdot)$ of $\tilde{B}(\cdot)$. In Figures 1 and 2 we have plotted the singular values $\tilde{\sigma}_i(\cdot)$ and the components of the first left singular vector $\tilde{u}_1(\cdot)$ of $\tilde{B}(\cdot)$. In Figures 3 and 4 we have plotted the functions $\hat{\sigma}_i(\cdot)$ and $\hat{u}_{i1}(\cdot)$ defined by $\frac{\log \tilde{\sigma}_i(s)}{s}$ and $\frac{\log |\tilde{u}_{i1}(s)|}{s}$ respectively. From these plots we can determine the dominant exponents of the $\tilde{\sigma}_i(\cdot)$'s and the components of $\tilde{u}_1(\cdot)$. If we take the limit of the $\hat{\sigma}_i(\cdot)$'s and the $\hat{u}_{ij}(\cdot)$'s for s going to ∞ and if we take the signs into account – in other words, if we apply the reverse mapping \mathcal{R} – we get the following max-algebraic SVD of B :

$$B \nabla \begin{bmatrix} -2 & -2 & \ominus(-1) & \ominus 0 \\ -4 & -4 & 0 & \ominus(-1) \\ \ominus 0 & \ominus 0 & \ominus(-3) & \ominus(-2) \\ 0 & \ominus 0 & -5 & \ominus(-6) \end{bmatrix} \otimes \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 5 & \varepsilon \\ \varepsilon & \varepsilon & -1 \\ \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} -1 & \ominus(-1) & 0 \\ \ominus 0 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2^\bullet & 3^\bullet & 3 \\ 0^\bullet & 1^\bullet & 1 \\ 4^\bullet & 5^\bullet & \ominus 5 \\ 4 & \ominus 5 & 5^\bullet \end{bmatrix}.$$

\square

5 Conclusions and future research

First we have proved the existence of a kind of singular value decomposition (SVD) in the extended max algebra. Next we have used the same proof technique to prove the existence of a QR factorization of a matrix in the extended max algebra. It is obvious that this proof technique can also be used to prove the existence of max-algebraic equivalents of many other matrix decompositions from linear algebra: it can easily be adapted to prove the existence of an LU decomposition of a matrix in the extended max algebra and to prove the existence of an eigenvalue decomposition for symmetric matrices in the extended max algebra (by using the Jacobi algorithm for the calculation of the eigenvalue decomposition of a symmetric matrix). We have also discussed some possible extensions of the definition of the max-algebraic SVD.

Topics for future research are: further investigation of the properties of the max-algebraic SVD and the max-algebraic QR decomposition, extension of the proof technique of this paper to prove the existence of other max-algebraic matrix decompositions and investigation of possible applications of these matrix decompositions.

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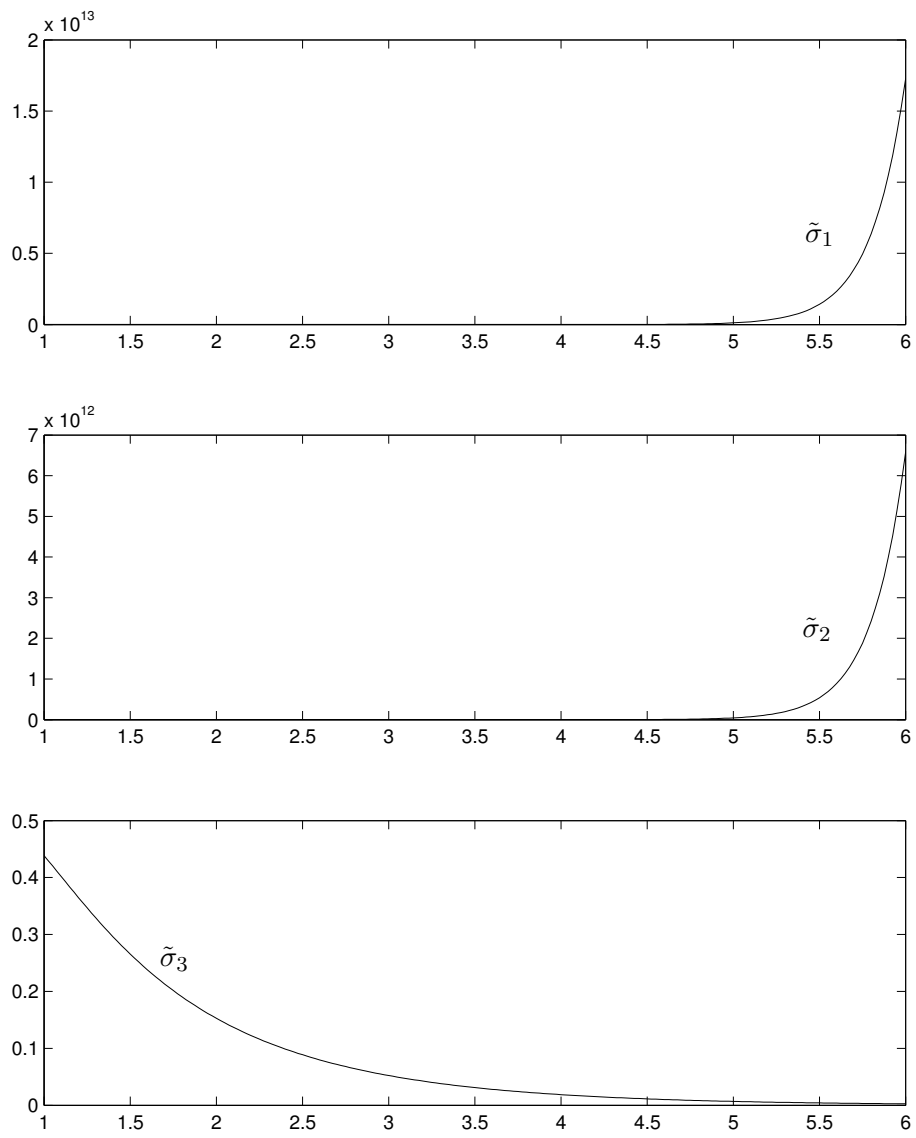


Figure 1: The singular values $\tilde{\sigma}_i(\cdot)$ of $\tilde{B}(\cdot)$.

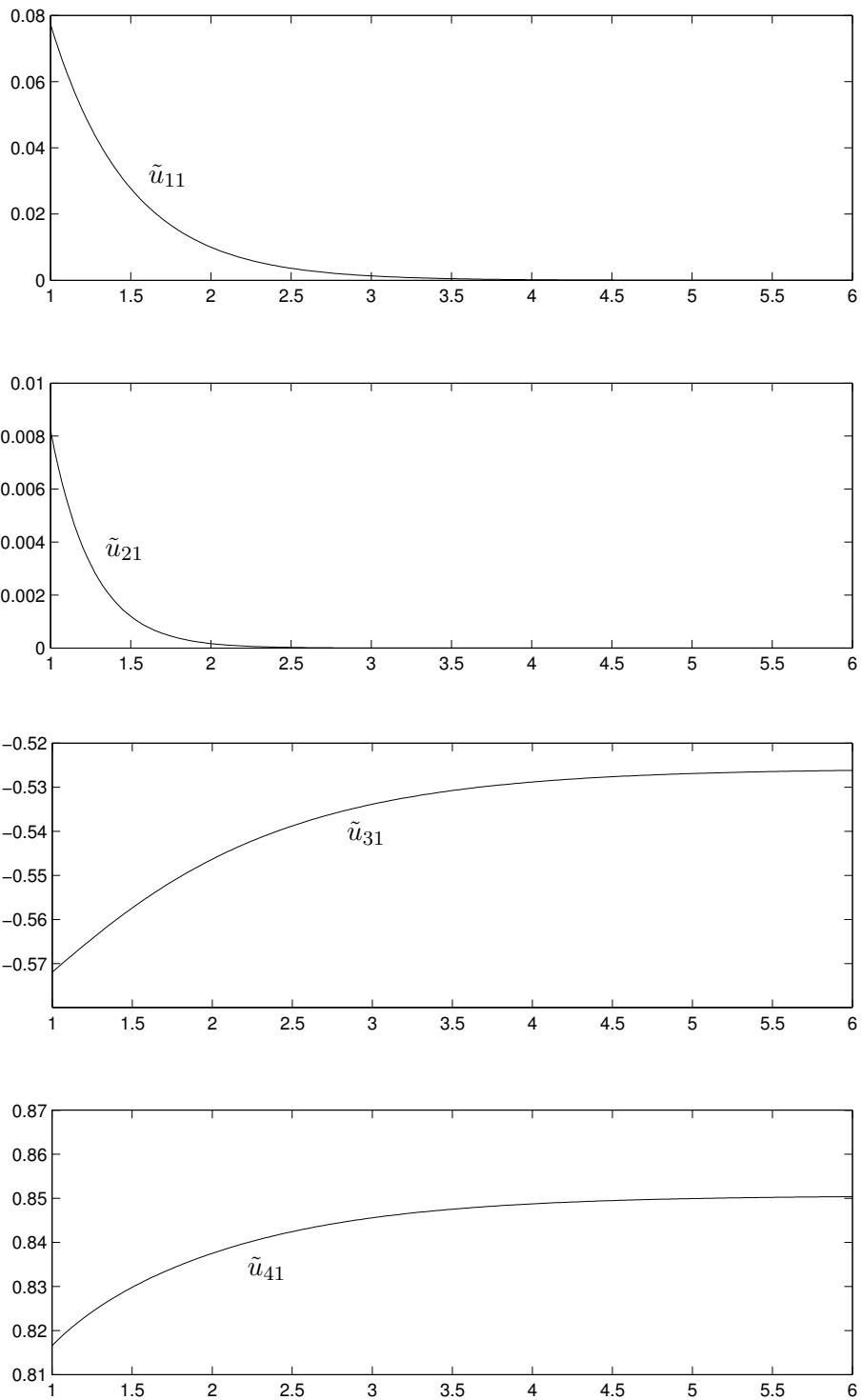


Figure 2: The components $\tilde{u}_{i1}(\cdot)$ of the first left singular vector of $\tilde{B}(\cdot)$.

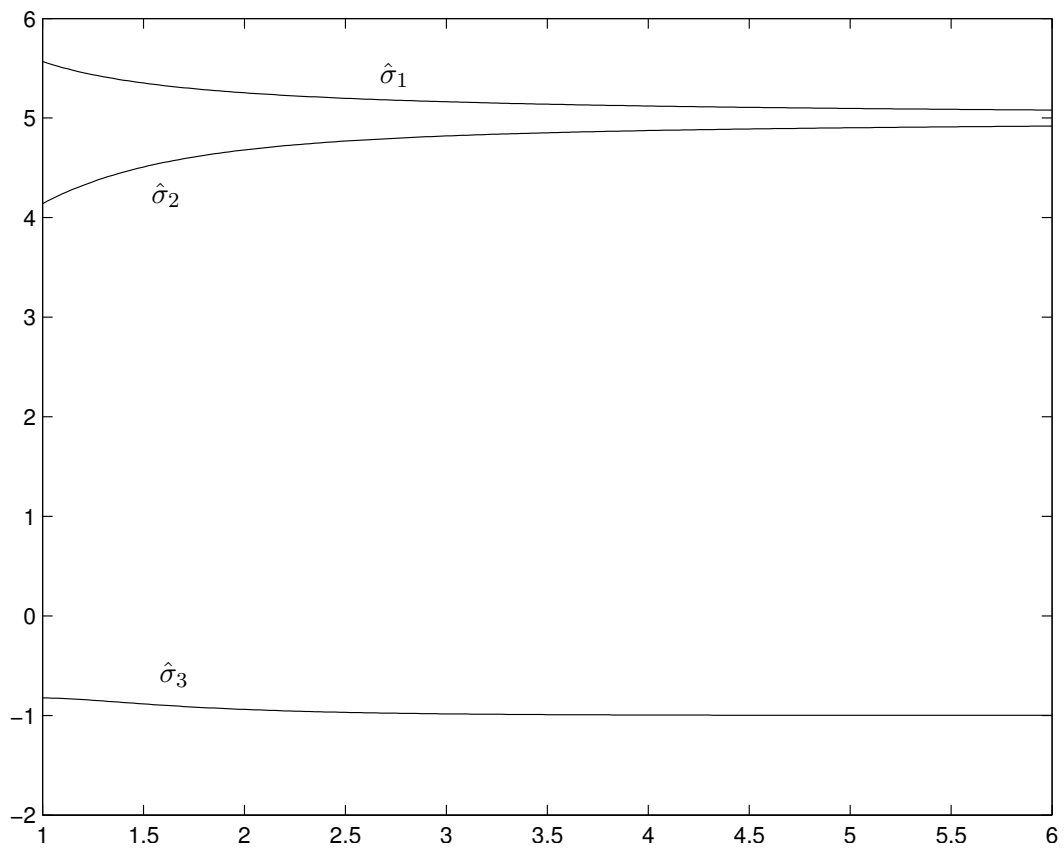


Figure 3: The functions defined by $\hat{\sigma}_i(s) = \frac{\log \tilde{\sigma}_i(s)}{s}$.

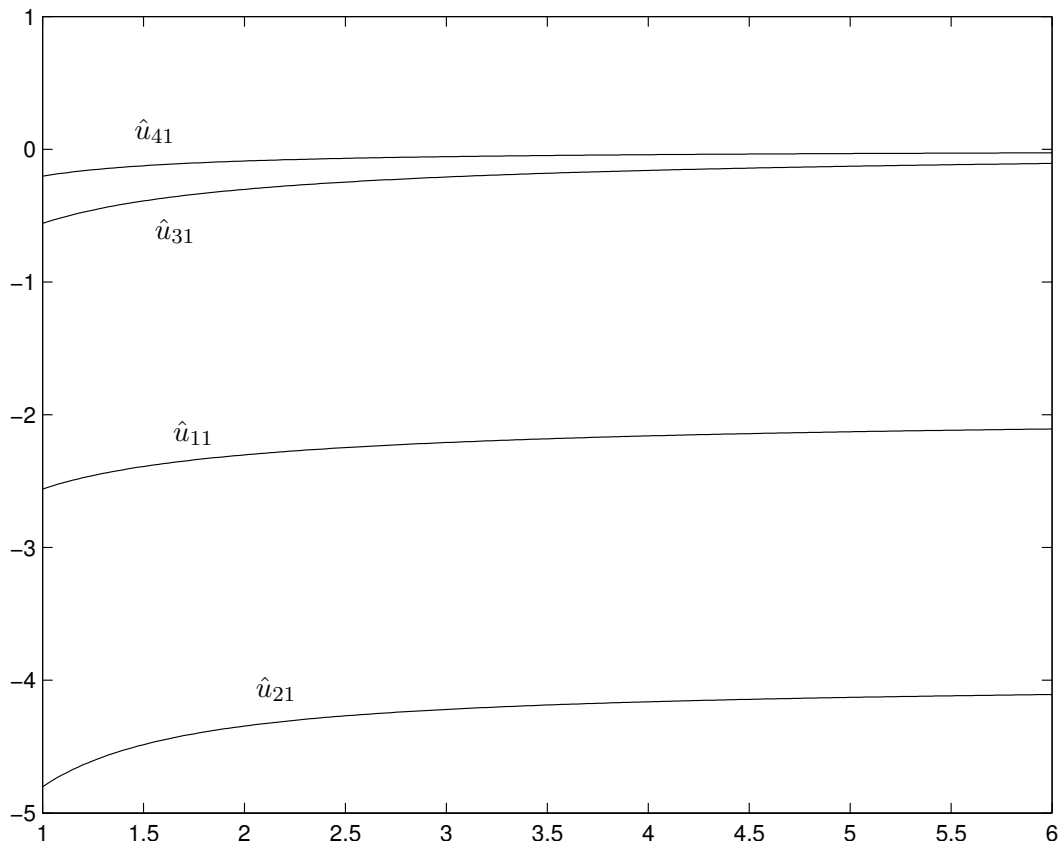


Figure 4: The functions defined by $\hat{u}_{i1}(s) = \frac{\log |\tilde{u}_{i1}(s)|}{s}$.