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The Singular Value Decomposition in the Extended Max Algebra is an Extended Linear Complementarity Problem[1]

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Abstract. We show that the problem of finding a singular value decomposition of a matrix in the extended max algebra can be reformulated as an Extended Linear Complementarity Problem. This allows us to compute all the max-algebraic singular value decompositions of a matrix. This technique can also be used to calculate many other max-algebraic matrix decompositions.

1 Introduction

1.1 Overview

One of the possible frameworks to describe and analyze discrete event systems (such as flexible manufacturing processes, railroad traffic networks, telecommunication networks, ...) is the max algebra [1, 2, 3]. The elements of the max algebra are the real numbers and $-\infty$, and the admissible operations are the maximum and the addition. A class of discrete event systems, the timed event graphs, can be described by a model that is linear in the max algebra. There exists a remarkable analogy between linear algebra and the max algebra: many properties and concepts of linear algebra such as Cramer's rule, the Cayley-Hamilton theorem, eigenvalues, eigenvectors, ... also have a max-algebraic equivalent. However, there are also some major differences that make that the mathematical foundations of the max algebra are not as fully developed as those of linear algebra.

In [8, 6] we have introduced a link between the field of the real numbers and the extended max algebra, which is a kind of symmetrization of the max algebra. We have used this link to prove the existence of a singular value decomposition (SVD) and a QR decomposition (QRD) of a matrix in the extended max algebra and to calculate these decompositions.

In this paper we present an alternative method to calculate the max-algebraic SVD and other max-algebraic matrix decompositions. This method is based on the fact that a system of multivariate max-algebraic polynomial equalities and inequalities can be transformed into

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an Extended Linear Complementarity Problem (ELCP) [4]. The ELCP is an extension of the well-known Linear Complementarity Problem, which is one of the fundamental problems of mathematical programming. In [4] we have developed an algorithm to find all the solutions of an ELCP. We shall use this algorithm to calculate the max-algebraic SVD and the max-algebraic QRD of a matrix. This method also gives us a geometrical insight in the set of all max-algebraic SVDs or QRDs of a given matrix.

In Section 1 we explain the notations we use in this paper and we give some definitions and properties. We also include a short introduction to the max algebra and the extended max algebra and we discuss the link between the field of the real numbers and the extended max algebra. In Section 2 we briefly treat the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities. In Section 3 we recapitulate the main results on the max-algebraic SVD and the max-algebraic QRD of [8, 6] and we prove that a matrix with finite entries always has a max-algebraic SVD with finite singular values and finite singular vectors. In Section 4 we show that the problem of finding a max-algebraic SVD or a max-algebraic QRD of a matrix can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities and thus also as an ELCP. Next we show that the extended definitions of the max-algebraic SVD and the max-algebraic QRD that were proposed in [6] also lead to a system of multivariate max-algebraic polynomial equalities and inequalities or an ELCP. We conclude with some worked examples.

1.2 Notations and definitions

If $a \in \mathbb{R}^n$, then a_i is the ith component of a. If A is a matrix, then a_{ij} or $(A)_{ij}$ is the entry on the ith row and the jth column. The ith row of A is represented by $A_{i.}$. The n by n identity matrix is denoted by I_n and the m by n zero matrix is denoted by $O_{m \times n}$.

If $f : A \to C$ is a function then the value of f at $x \in A$ is denoted by $f(x)$. The number of elements of the domain of definition D_f of the function f is denoted by $\# D_f$. If $f : A \to C$ and $g : B \to D$ are functions and if $A \cap B = \emptyset$, then $f \cup g$ is a function that is defined as follows: $f \cup g : A \cup B \to C \cup D$ with

$$
\begin{aligned}
(f \cup g)(x) &= f(x) \qquad \text{if } x \in A\,, \\
&= g(x) \qquad \text{if } x \in B\ .
\end{aligned}
$$

Definition 1.1 (Asymptotic equivalence) *Let $\alpha \in \mathbb{R} \cup \{\infty\}$ and let f and g be real functions. The function f is asymptotically equivalent to g in the neighborhood of α, denoted by $f(x) \sim g(x)$, $x \to \alpha$, if $\displaystyle\lim_{x \to \alpha} \frac{f(x)}{g(x)} = 1$.*
If $\beta \in \mathbb{R}$ and if $\exists \delta > 0, \forall x \in (\beta - \delta, \beta + \delta) \setminus \{\beta\} : f(x) = 0$ then $f(x) \sim 0$, $x \to \beta$.
We say that $f(x) \sim 0$, $x \to \infty$ if $\exists K \in \mathbb{R}, \forall x > K : f(x) = 0$.
If $F(\cdot)$ and $G(\cdot)$ are real m by n matrix-valued functions then $F(x) \sim G(x)$, $x \to \alpha$ if $f_{ij}(x) \sim g_{ij}(x)$, $x \to \alpha$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The main difference with the classic definition of asymptotic equivalence is that Definition 1.1 also allows us to say that a function is asymptotically equivalent to 0.

1.3 The max algebra and the extended max algebra

Now we give a short introduction to the max algebra and the extended max algebra. A complete overview of the max algebra can be found in [1, 3]. The basic max-algebraic operations are defined as follows:

$$x \oplus y = \max(x, y) \tag{1}$$

$$x \otimes y = x + y \tag{2}$$

where $x, y \in \mathbb{R} \cup \{-\infty\}$. The resulting structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the max algebra. Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$. The zero element for $\oplus$ in $\mathbb{R}_\varepsilon$ is ε. Let $r \in \mathbb{R}$. The rth max-algebraic power of $x \in \mathbb{R}$ is denoted by $x^{\otimes^r}$ and corresponds to rx in linear algebra. If $x \in \mathbb{R}$ then $x^{\otimes^0} = 0$ and the inverse element of x w.r.t. $\otimes$ is $x^{\otimes^{-1}} = -x$. If $r > 0$ then $\varepsilon^{\otimes^r} = \varepsilon$. If $r \leqslant 0$ then $\varepsilon^{\otimes^r}$ is not defined.

The max-algebraic operations are extended to matrices as follows: if $\alpha \in \mathbb{R}_\varepsilon$ and if X, $Y \in \mathbb{R}_\varepsilon^{m\times n}$ then $(\alpha \otimes X)_{ij} = \alpha \otimes x_{ij}$ and $(X \oplus Y)_{ij} = x_{ij} \oplus y_{ij}$ for all i, j. If $X \in \mathbb{R}_\varepsilon^{m\times p}$ and $Y \in \mathbb{R}_\varepsilon^{p\times n}$ then $(X \otimes Y)_{ij} = \bigoplus_{k=1}^{p} x_{ik} \otimes y_{kj}$ for all i, j. The matrix E_n is the n by n max-algebraic identity matrix: $(E_n)_{ii} = 0$ for $i = 1, 2, \ldots, n$ and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The m by n max-algebraic zero matrix is represented by $\mathcal{E}_{m\times n}$: $(\mathcal{E}_{m\times n})_{ij} = \varepsilon$ for all i, j. The off-diagonal entries of a max-algebraic diagonal matrix $D \in \mathbb{R}_\varepsilon^{m\times n}$ are equal to ε: $d_{ij} = \varepsilon$ for all i, j with $i \neq j$. A matrix $R \in \mathbb{R}_\varepsilon^{m\times n}$ is a max-algebraic upper triangular matrix if $r_{ij} = \varepsilon$ for all i, j with $i > j$. A max-algebraic permutation matrix is square matrix P with exactly one 0 entry in each row and in each column and where the other entries are equal to ε.

In contrast to linear algebra, there exist no inverse elements w.r.t. $\oplus$ in $\mathbb{R}_\varepsilon$. To overcome this problem we need the extended max algebra $\mathbb{S}_{\max}$ [1, 9, 10], which is a kind of symmetrization of the max algebra. We shall restrict ourselves to a short introduction to the most important features of $\mathbb{S}_{\max}$. For a more formal derivation the interested reader is referred to [1, 8, 9, 10].
First we define two new elements for every $x \in \mathbb{R}_\varepsilon$: $\ominus x$ and $x^\bullet$. This gives rise to an extension $\mathbb{S}$ of $\mathbb{R}_\varepsilon$ that contains three classes of elements:

- $\mathbb{S}^\oplus \equiv \mathbb{R}_\varepsilon$, the set of the max-positive or zero elements,
- $\mathbb{S}^\ominus = \{\, \ominus x \,|\, x \in \mathbb{R}_\varepsilon \,\}$, the set of max-negative or zero elements,
- $\mathbb{S}^\bullet = \{\, x^\bullet \,|\, x \in \mathbb{R}_\varepsilon \,\}$, the set of the balanced elements.

We have $\mathbb{S} = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$ and $\mathbb{S}^\oplus \cap \mathbb{S}^\ominus \cap \mathbb{S}^\bullet = \{\varepsilon\}$ since $\varepsilon = \ominus\varepsilon = \varepsilon^\bullet$. The max-positive elements, the max-negative elements and the zero element ε are called signed ($\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$). The $\oplus$ and the $\otimes$ operation can be extended to $\mathbb{S}$. The resulting structure $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$ is called the extended max algebra. The $\oplus$ law is associative, commutative and idempotent in $\mathbb{S}$ and its zero element is ε; the $\otimes$ law is associative and commutative in $\mathbb{S}$ and its unit element is 0. The $\otimes$ law is distributive w.r.t. the $\oplus$ law in $\mathbb{S}$. If $x, y \in \mathbb{R}_\varepsilon$ then

$$\begin{aligned} x \oplus (\ominus y) &= x && \text{if } x > y\,, \\ x \oplus (\ominus y) &= \ominus y && \text{if } x < y\,, \\ x \oplus (\ominus x) &= x^\bullet && . \end{aligned}$$

Furthermore, we have

$$\begin{aligned}
a^\bullet &= (\ominus a)^\bullet = (a^\bullet)^\bullet \\
a \otimes b^\bullet &= (a \otimes b)^\bullet \\
\ominus(\ominus a) &= a \\
\ominus(a \oplus b) &= (\ominus a) \oplus (\ominus b) \\
\ominus(a \otimes b) &= (\ominus a) \otimes b
\end{aligned}$$

for all $a, b \in \mathbb{S}$. The last three properties allow us to write $a \ominus b$ instead of $a \oplus (\ominus b)$. So the $\ominus$ operator in $\mathbb{S}_{\max}$ could be considered as the equivalent of the - operator in linear algebra.
Let $a \in \mathbb{S}$. The max-positive part $a^\oplus$ and the max-negative part $a^\ominus$ of a are defined as follows:

- if $a \in \mathbb{S}^\oplus$ then $a^\oplus = a$ and $a^\ominus = \varepsilon$,
- if $a \in \mathbb{S}^\ominus$ then $a^\oplus = \varepsilon$ and $a^\ominus = \ominus a$,
- if $a \in \mathbb{S}^\bullet$ then $\exists b \in \mathbb{R}_\varepsilon$ such that $a = b^\bullet$ and then $a^\oplus = a^\ominus = b$.

Note that $a = a^\oplus \ominus a^\ominus$ and $a^\oplus, a^\ominus \in \mathbb{R}_\varepsilon$. We define the max-absolute value of $a \in \mathbb{S}$ as $|a|_\oplus = a^\oplus \oplus a^\ominus$.
We say that $a \in \mathbb{S}$ is finite if $|a|_\oplus \in \mathbb{R}$. If $|a|_\oplus = \varepsilon$ then we say that a is infinite.

Proposition 1.2 $\forall a, b \in \mathbb{S} : |a \oplus b|_\oplus = |a|_\oplus \oplus |b|_\oplus$.

Proposition 1.3 $\forall a, b \in \mathbb{S} : |a \otimes b|_\oplus = |a|_\oplus \otimes |b|_\oplus$.

In linear algebra we have $\forall x \in \mathbb{R} : x - x = 0$, but in $\mathbb{S}_{\max}$ we have $\forall a \in \mathbb{S} : a \ominus a = a^\bullet \neq \varepsilon$ unless $a = \varepsilon$, the zero element for $\oplus$. Therefore, we introduce a new relation, the balance relation, represented by ∇.

Definition 1.4 (Balance relation) *Consider $a, b \in \mathbb{S}$. We say that a balances b, denoted by $a \nabla b$, if $a^\oplus \oplus b^\ominus = a^\ominus \oplus b^\oplus$.*

Since $\forall a \in \mathbb{S} : a \ominus a = a^\bullet = |a|_\oplus \ominus |a|_\oplus \nabla \varepsilon$, we could say that the balance relation in $\mathbb{S}$ is the counterpart of the equality relation in linear algebra. The balance relation is reflexive and symmetric but it is not transitive. The balance relation is extended to matrices in the usual way: if $A, B \in \mathbb{S}^{m \times n}$ then $A \nabla B$ if $a_{ij} \nabla b_{ij}$ for all i, j.
An element with a $\ominus$ sign can be transferred to the other side of a balance as follows:

Proposition 1.5 $\forall a, b, c \in \mathbb{S} : a \ominus c \nabla b$ *if and only if* $a \nabla b \oplus c$.

If both sides of a balance are signed, we can replace the balance by an equality:

Proposition 1.6 $\forall a, b \in \mathbb{S}^\vee : a \nabla b \Rightarrow a = b$.

These properties can be extended to the matrix case as follows:

Proposition 1.7 $\forall A, B, C \in \mathbb{S}^{m \times n} : A \ominus C \nabla B$ *if and only if* $A \nabla B \oplus C$.

Proposition 1.8 $\forall A, B \in (\mathbb{S}^\vee)^{m \times n} : A \nabla B \Rightarrow A = B$.

Definition 1.9 (Max-algebraic norm) *The max-algebraic norm of a vector $a \in \mathbb{S}^n$ is defined as*

$$\|a\|_{\oplus} = \bigoplus_{i=1}^{n} |a_i|_{\oplus} = \bigoplus_{i=1}^{n} (a_i^{\oplus} \oplus a_i^{\ominus}) \ .$$

The max-algebraic norm of a matrix $A \in \mathbb{S}^{m\times n}$ is defined as

$$\|A\|_{\oplus} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_{\oplus} \ .$$

Definition 1.10 (Max-algebraic determinant) *Consider a matrix $A \in \mathbb{S}^{n\times n}$. The max-algebraic determinant of A is defined as*

$$\det_{\oplus} A = \bigoplus_{\sigma \in \mathcal{P}_n} \operatorname{sgn}_{\oplus}(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)}$$

where $\mathcal{P}_n$ is the set of all the permutations of $\{1, 2, \ldots, n\}$ and $\operatorname{sgn}_{\oplus}(\sigma) = 0$ if the permutation σ is even and $\operatorname{sgn}_{\oplus}(\sigma) = \ominus 0$ if the permutation is odd.

Theorem 1.11 *Let $A \in \mathbb{S}^{n\times n}$. The homogeneous linear balance $A \otimes x \ \nabla \ \mathcal{E}_{n\times 1}$ has a non-trivial signed solution if and only if $\det_{\oplus} A \ \nabla \ \varepsilon$.*

Proof: See [9]. □

Definition 1.12 (Max-linear independence)
A set of vectors $\{\, x_i \in \mathbb{S}^n \,|\, i = 1, 2, \ldots, m \,\}$ is max-linearly independent if the only signed solution of

$$\bigoplus_{i=1}^{m} \alpha_i \otimes x_i \ \nabla \ \mathcal{E}_{n\times 1}$$

is $\alpha_1 = \alpha_2 = \ldots = \alpha_m = \varepsilon$. Otherwise we say that the vectors x_i are max-linearly dependent.

So if $A \in \mathbb{S}^{n\times n}$ then the columns of A are max-linearly independent if and only if $\det_{\oplus} A \ \not\nabla \ \varepsilon$.

1.4 A link between the field of the real numbers and the extended max algebra

In [8] we have introduced the following mapping for $x \in \mathbb{R}_{\varepsilon}$:

$$\begin{aligned} \mathcal{F}(x, s) &= \mu e^{xs} \\ \mathcal{F}(\ominus x, s) &= -\mu e^{xs} \\ \mathcal{F}(x^{\bullet}, s) &= \nu e^{xs} \end{aligned}$$

where μ is an arbitrary positive real number or parameter and ν is an arbitrary real number or parameter different from 0 and s is a real parameter. Note that $\mathcal{F}(\varepsilon, s) = 0$.
To reverse the mapping $\mathcal{F}$ we have to take $\lim\limits_{s\to\infty} \dfrac{\log(\,|\,\mathcal{F}(x, s)\,|\,)}{s}$ and adapt the max-sign depending on the sign of the coefficient of the exponential. So if f is a real function, if $x \in \mathbb{R}_{\varepsilon}$

and if μ is a positive real number or if μ is a parameter that can only take on positive real values then

$$
\begin{aligned}
f(s) \ \sim \ \mu e^{xs} \ , \ s \to \infty \quad &\Rightarrow \quad \mathcal{R}(f) = x \\
f(s) \ \sim \ -\mu e^{xs} \ , \ s \to \infty \quad &\Rightarrow \quad \mathcal{R}(f) = \ominus x
\end{aligned}
$$

where $\mathcal{R}$ is the reverse mapping of $\mathcal{F}$. If ν is a parameter that can take on both positive and negative real values then

$$f(s) \ \sim \ \nu e^{xs} \ , \ s \to \infty \quad \Rightarrow \quad \mathcal{R}(f) = x^{\bullet} \ .$$

Note that if the coefficient of e^{xs} is a number then the reverse mapping always yields a signed result.
Now we have for $a, b, c \in \mathbb{S}$:

$$a \oplus b = c \quad \to \quad \mathcal{F}(a,s) + \mathcal{F}(b,s) \ \sim \ \mathcal{F}(c,s) \ , \ s \to \infty \tag{3}$$

$$\mathcal{F}(a,s) + \mathcal{F}(b,s) \ \sim \ \mathcal{F}(c,s) \ , \ s \to \infty \quad \to \quad a \oplus b \ \nabla \ c \tag{4}$$

$$a \otimes b = c \quad \leftrightarrow \quad \mathcal{F}(a,s) \cdot \mathcal{F}(b,s) = \mathcal{F}(c,s) \qquad \text{for all } s \in \mathbb{R} \tag{5}$$

for an appropriate choice of the μ's and ν's in $\mathcal{F}(c,s)$ in (3) and in (5) from the left to the right. This leads to the following correspondences:

$$
\begin{aligned}
(\mathbb{R}^+, +, \times) \quad &\leftrightarrow \quad (\mathbb{R}_\varepsilon, \oplus, \otimes) \ = \ \mathbb{R}_{\max} \\
(\mathbb{R}, +, \times) \quad &\leftrightarrow \quad (\mathbb{S}, \oplus, \otimes) \ = \ \mathbb{S}_{\max} \ .
\end{aligned}
$$

We can extend this mapping to matrices such that if $A \in \mathbb{S}^{m \times n}$ then $\tilde{A}(\cdot) = \mathcal{F}(A, \cdot)$ is a real m by n matrix-valued function with $\tilde{a}_{ij}(s) = \mathcal{F}(a_{ij}, s)$ for some choice of the μ's and ν's. Note that the mapping is performed entrywise — it is not a matrix exponential! If A, B and C are matrices with entries in $\mathbb{S}$, we have

$$A \oplus B = C \quad \to \quad \mathcal{F}(A,s) + \mathcal{F}(B,s) \ \sim \ \mathcal{F}(C,s) \ , \ s \to \infty \tag{6}$$

$$\mathcal{F}(A,s) + \mathcal{F}(B,s) \ \sim \ \mathcal{F}(C,s) \ , \ s \to \infty \quad \to \quad A \oplus B \ \nabla \ C \tag{7}$$

$$A \otimes B = C \quad \to \quad \mathcal{F}(A,s) \cdot \mathcal{F}(B,s) \ \sim \ \mathcal{F}(C,s) \ , \ s \to \infty \tag{8}$$

$$\mathcal{F}(A,s) \cdot \mathcal{F}(B,s) \ \sim \ \mathcal{F}(C,s) \ , \ s \to \infty \quad \to \quad A \otimes B \ \nabla \ C \tag{9}$$

for an appropriate choice of the μ's and ν's in $\mathcal{F}(C,s)$ in (6) and (8).

2 Systems of multivariate max-algebraic polynomial equalities and inequalities

In this section we consider systems of multivariate max-algebraic polynomial equalities and inequalities, which can be seen as a generalized framework for many important max-algebraic problems such as matrix decompositions, transformation of state space models, state space realization of impulse responses, construction of matrices with a given characteristic polynomial and so on [5, 7].
Consider the following problem:

Given a set of integers $\{m_k\}$ and three sets of real numbers $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $k = 1, 2, \ldots, p_1 + p_2$, $i = 1, 2, \ldots, m_k$ and $j = 1, 2, \ldots, n$, find $x \in \mathbb{R}^n$ such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1, 2, \ldots, p_1 \,, \tag{10}$$

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} \leqslant b_k \qquad \text{for } k = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2 \,, \tag{11}$$

or show that no such x exists.

We call (10) – (11) a system of multivariate max-algebraic polynomial equalities and inequalities. Note that the exponents can be negative or real.

In [7] we have shown that this problem is equivalent to an Extended Linear Complementarity Problem (ELCP) [4]:

Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets ϕ_j, $j = 1, 2, \ldots, m$, of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i \; = \; 0 \tag{12}$$

subject to

$$\begin{aligned} Ax &\geqslant c \\ Bx &= d \;, \end{aligned}$$

or show that no such x exists.

In [4] we have developed an algorithm to compute all the solutions of an ELCP. Consequently, we can also calculate the entire solution set of problem (10) – (11). This solution set can be described in terms of linear algebra concepts as follows: in general it consists of the union of faces of a polyhedron $\mathcal{P}$ and is defined by three sets of vectors $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and a set Λ. These sets can be characterized as follows:

- $\mathcal{X}^{\text{cen}}$ is a set of *central rays* of $\mathcal{P}$. It is a basis for the largest linear subspace of $\mathcal{P}$. Let $\mathcal{P}_{\text{red}}$ be the polyhedron obtained by subtracting this largest linear subspace from $\mathcal{P}$.
- $\mathcal{X}^{\text{inf}}$ is a set of *extreme rays* or *vertices at infinity* of the polyhedron $\mathcal{P}_{\text{red}}$.
- $\mathcal{X}^{\text{fin}}$ is the set of the *finite vertices* of the polyhedron $\mathcal{P}_{\text{red}}$.
- Λ is a set of pairs $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\}$ with $\mathcal{X}_s^{\text{inf}} \subset \mathcal{X}^{\text{inf}}$, $\mathcal{X}_s^{\text{fin}} \subset \mathcal{X}^{\text{fin}}$ and $\mathcal{X}_s^{\text{fin}} \neq \emptyset$. Each pair $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\}$ determines a face $\mathcal{F}_s$ of the polyhedron $\mathcal{P}$ that belongs to the solution set: $\mathcal{X}_s^{\text{inf}}$ contains the extreme rays of $\mathcal{F}_s$ and $\mathcal{X}_s^{\text{fin}}$ contains the finite vertices of $\mathcal{F}_s$. The rays and vertices of $\mathcal{X}_s^{\text{inf}} \cup \mathcal{X}_s^{\text{fin}}$ are called *cross-complementary* since every combination of the form

  $$x \;=\; \sum_{x_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k$$

  with $\kappa_k, \mu_k \geqslant 0$ and $\sum_k \mu_k = 1$ satisfies the *complementarity condition* (12).

The solution set of problem (10) – (11) can be characterized by the following theorems:

Theorem 2.1 *When $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and Λ are given, then x is a solution of the system of multivariate max-algebraic polynomial equalities and inequalities if and only if there exists a pair $\left\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\right\} \in \Lambda$ such that*

$$x = \sum_{x_k \in \mathcal{X}^{\text{cen}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k \tag{13}$$

with $\lambda_k \in \mathbb{R},\ \kappa_k, \mu_k \geqslant 0$ and $\sum_k \mu_k = 1$.

Theorem 2.2 *In general the set of (finite) solutions of a system of multivariate max-algebraic polynomial equalities and inequalities either is empty or consists of the union of faces of a polyhedron.*

Remark 2.3 In order to apply the ELCP technique we have only considered finite coefficients and solutions with finite components in the formulation of problem (10) – (11).
However, in some cases we can allow b_k's that are equal to ε. Then we have to introduce a positive number ξ that is large enough and transform equations of the form $\bigoplus_i t_i = \varepsilon$ into $\bigoplus_i t_i \leqslant -\xi$. Once we have found a solution x of the system of multivariate max-algebraic polynomial equalities and inequalities we replace every negative component of x that has the same order of magnitude as ξ by ε provided that this does not cause any problems arising from taking negative powers of ε, and that x has no positive components of the same order of magnitude as ξ. Positive components of the same order of magnitude as ξ would have to be replaced by ∞, but ∞ does not belong to $\mathbb{R}_\varepsilon$.
Another way to obtain solutions with components equal to ε is to allow some of the λ_k's or κ_k's in (13) to become infinite, but in a controlled way, since we only allow infinite components that are equal to ε and since negative powers of ε are not defined. Solutions obtained in this way will correspond to points at infinity of the polyhedron $\mathcal{P}$. Since the max operation hides small numbers from larger numbers, it suffices in practice to replace the negative components that are large enough in absolute value by ε provided that there are no positive components of the same order of magnitude.
The solutions of systems of multivariate max-algebraic polynomial equalities and inequalities that arise from max-algebraic SVDs and max-algebraic QRDs will always be bounded from above. This means that in these cases there will be no solutions with positive components of the same order of magnitude as ξ if we take ξ large enough.
We shall illustrate all these techniques in Example 6.1.

3 The singular value decomposition and the QR decomposition in the extended max algebra

In this section we recapitulate the main results of [8, 6] concerning the max-algebraic singular value decomposition (SVD) and the max-algebraic QR decomposition (QRD). We also prove that for matrix with finite entries there always exists a max-algebraic SVD with finite singular values and finite singular vectors.

Theorem 3.1 (The singular value decomposition in $\mathbb{S}_{\max}$)
Let $A \in \mathbb{S}^{m\times n}$ and let $r = \min(m,n)$. Then there exist a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m\times n}$ and matrices $U \in (\mathbb{S}^\vee)^{m\times m}$ and $V \in (\mathbb{S}^\vee)^{n\times n}$ such that

$$A \ \nabla \ U \otimes \Sigma \otimes V^T \tag{14}$$

with

$$\begin{aligned} U^T \otimes U \quad &\nabla \quad E_m \\ V^T \otimes V \quad &\nabla \quad E_n \end{aligned}$$

and $\|A\|_\oplus \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r \geqslant \varepsilon$ where $\sigma_i = (\Sigma)_{ii}$.
Every decomposition of the form (14) that satisfies the above conditions is called a max-algebraic singular value decomposition of A.

Note that (14) can also be written as

$$A \ \nabla \ \bigoplus_{i=1}^{r} \sigma_i \otimes u_i \otimes v_i^T \tag{15}$$

where u_i is the ith column of U and v_i is the ith column of V.

Proposition 3.2 *Consider $A \in \mathbb{S}^{m\times n}$. If there is at least one signed entry in A that is equal to $\|A\|_\oplus$ in max-absolute value then $\sigma_1 = \|A\|_\oplus$ for every max-algebraic SVD of A.*

In contrast to the singular values in linear algebra the max-algebraic singular values are not always unique. This leads to the definition of a *maximal* max-algebraic SVD – where we take all the singular values as large as possible – and a *minimal* max-algebraic SVD – where we take all the singular values as small as possible. Now we can define a rank based on the max-algebraic SVD:

Definition 3.3 (Max-algebraic SVD rank) *Let $A \in \mathbb{S}^{m\times n}$. The max-algebraic SVD rank of A is defined as*

$$\operatorname{rank}_{\oplus,\mathrm{SVD}}(A) \;=\; \min \left\{ \rho \;\middle|\; A \ \nabla \ \bigoplus_{i=1}^{\rho} \sigma_i \otimes u_i \otimes v_i^T \,,\ U \otimes \Sigma \otimes V^T \text{ is a max-algebraic SVD of } A \right\}$$

where u_i is the ith column of U, v_i is the ith column of V and $\bigoplus_{i=1}^{0} \sigma_i \otimes u_i \otimes v_i^T$ is equal to $\mathcal{E}_{m\times n}$ by definition.

So the max-algebraic SVD rank of a matrix A is equal to the minimal number of non-ε singular values over the set of all the max-algebraic SVDs of A.

Proposition 3.4 *Consider $U \in (\mathbb{S}^\vee)^{m\times m}$. If $U \otimes U^T \ \nabla \ E_m$ then*

$$|u_{ij}|_\oplus \leqslant 0 \qquad \text{for } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,m \ .$$

Theorem 3.5 *Consider a matrix* $A \in \mathbb{S}^{m\times n}$ *with finite entries:* $|a_{ij}|_{\oplus} \neq \varepsilon$ *for all* i, j*. Then there exists a max-algebraic SVD of* A *for which all the singular values and all the components of the singular vectors are finite.*

Proof: First we define a matrix-valued function $\tilde{A}(\cdot) = \mathcal{F}(A, \cdot)$. In [8] we have shown that there exists a path of SVDs $\tilde{U}(\cdot)\,\tilde{\Sigma}(\cdot)\,\tilde{V}^T(\cdot)$ of $\tilde{A}(\cdot)$ on some interval $[L, \infty)$, i.e. for all $s \geqslant L$:

$$\begin{aligned}
\tilde{A}(s) &= \tilde{U}(s)\,\tilde{\Sigma}(s)\,\tilde{V}^T(s) \\
\tilde{U}^T(s)\,\tilde{U}(s) &= I_m \\
\tilde{V}^T(s)\,\tilde{V}(s) &= I_n \ ,
\end{aligned}$$

where the entries of $\tilde{U}(\cdot)$, $\tilde{\Sigma}(\cdot)$ and $\tilde{V}^T(\cdot)$ are asymptotically equivalent to an exponential in the neighborhood of ∞. If we apply the reverse mapping $\mathcal{R}$, we obtain a max-algebraic SVD of A:

$$A \ \nabla \ U \otimes \Sigma \otimes V^T \ . \tag{16}$$

If all the singular values and all the components of the singular vectors of this max-algebraic SVD are finite then the theorem is proved.
Now we shall show how a max-algebraic SVD that contains infinite singular values or singular vectors with infinite components can be transformed into a max-algebraic SVD $\hat{U} \otimes \hat{\Sigma} \otimes \hat{V}^T$ with finite singular values and vectors. This will be done in three steps: first we make all the singular values finite; next we make the components of the left singular vectors finite and finally we make the components of the right singular vectors finite.

Step 1: We make all the singular values finite.
If we extract the max-positive and the max-negative parts of each matrix of (16) and if we use Proposition 1.7, we obtain

$$\begin{aligned}
&A^{\oplus} \ \oplus \ U^{\oplus} \otimes \Sigma \otimes (V^{\ominus})^T \ \oplus \ U^{\ominus} \otimes \Sigma \otimes (V^{\oplus})^T \ \nabla \\
&\qquad\qquad\qquad\qquad A^{\ominus} \ \oplus \ U^{\oplus} \otimes \Sigma \otimes (V^{\oplus})^T \ \oplus \ U^{\ominus} \otimes \Sigma \otimes (V^{\ominus})^T \ .
\end{aligned}$$

Both sides of this balance are signed. So by Proposition 1.8 we can replace the balance by an equality. If we extract the entry on the ith row and the jth column, we obtain

$$\begin{aligned}
&a_{ij}^{\oplus} \ \oplus \ \bigoplus_{k=1}^{r} u_{ik}^{\oplus} \otimes \sigma_k \otimes v_{jk}^{\ominus} \ \oplus \ \bigoplus_{k=1}^{r} u_{ik}^{\ominus} \otimes \sigma_k \otimes v_{jk}^{\oplus} \\
&\qquad = \ a_{ij}^{\ominus} \ \oplus \ \bigoplus_{k=1}^{r} u_{ik}^{\oplus} \otimes \sigma_k \otimes v_{jk}^{\oplus} \ \oplus \ \bigoplus_{k=1}^{r} u_{ik}^{\ominus} \otimes \sigma_k \otimes v_{jk}^{\ominus}
\end{aligned} \tag{17}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Since $|a_{ij}|_{\oplus}$ is finite for all i, j, we can augment the terms of (17) that contain σ_k as long these terms stay less than or equal to $|a_{ij}|_{\oplus}$. Since $|u_{jk}|_{\oplus} \leqslant 0$ and $|v_{jk}|_{\oplus} \leqslant 0$ for all j, k by Proposition 3.4, this condition will be satisfied as long as the $\hat{\sigma}_k$'s stay less than or equal to $|a_{ij}|_{\oplus}$. If we define $f = \min\limits_{i,j} \{ |a_{ij}|_{\oplus} \}$ then f is finite. Assume that $\sigma_l = \sigma_{l+1} = \ldots = \sigma_r = \varepsilon$. If we set

$$\begin{aligned}
\hat{\sigma}_i &= \sigma_j && \text{for } i = 1, 2, \ldots, l-1 \,, \\
&= f && \text{for } i = l, l+1, \ldots, r \,,
\end{aligned}$$

then

$$A \;\nabla\; U \otimes \hat{\Sigma} \otimes V$$

where $\hat{\Sigma}$ is a max-algebraic diagonal matrix with $\hat{\Sigma}_{ii} = \hat{\sigma}_i$. Since the $\hat{\sigma}_i$'s are ordered and since we did not change the other equations, we now have a max-algebraic SVD in which all the singular values are finite.

Step 2: We make the components of the left singular vectors finite.
We shall replace the left singular vectors u_i by new left singular vectors $\hat{u}_i$. If we consider (17) with the σ_k's replaced by the $\hat{\sigma}_k$'s and if we take into account that $|v_{jk}|_{\oplus} \leqslant 0$ for all j, k, then we see that this equation will still hold if the entries of the vectors $\hat{u}_i$ satisfy

$$|\hat{u}_{ik}|_{\oplus} = \hat{u}_{ik}^{\oplus} \oplus \hat{u}_{ik}^{\ominus} \;\leqslant\; \min_j \{ |a_{ij}|_{\oplus} \} - \hat{\sigma}_k \qquad \text{for } i = 1, 2, \ldots, m \text{ and } k = 1, 2, \ldots, r \; . \tag{18}$$

If we define $g = \min_{i,j} \{ |a_{ij}|_{\oplus} \} - \hat{\sigma}_1$, then g is finite. Since $\hat{\sigma}_k \leqslant \hat{\sigma}_1$ for $k = 1, 2, \ldots, r$, condition (18) will be fulfilled if the entries of the vectors $\hat{u}_i$ satisfy

$$|\hat{u}_{ik}|_{\oplus} = \hat{u}_{ik}^{\oplus} \oplus \hat{u}_{ik}^{\ominus} \;\leqslant\; g \qquad \text{for } i = 1, 2, \ldots, m \text{ and } k = 1, 2, \ldots, r \; . \tag{19}$$

Since $\tilde{U}(s)$ is an orthogonal matrix for $s \geqslant L$, either $\det \tilde{U}(s) = 1$ or $\det \tilde{U}(s) = -1$ for $s \geqslant L$. The entries of an orthogonal matrix always lie in the interval $[-1, 1]$. Therefore, all the (dominant) exponents of the entries of $\tilde{U}(s)$ are less than or equal to 0. So $|\det \tilde{U}(s)|$ can only be equal to 1 for $s \geqslant L$ if there exists a permutation φ of the set $\{1, 2, \ldots, m\}$ such that

$$\prod_{i=1}^{m} \tilde{u}_{i\varphi(i)}(s) \;\sim\; c \, , \; s \to \infty$$

with $c \in \mathbb{R}_0$ or equivalently

$$\tilde{u}_{i\varphi(i)}(s) \;\sim\; c_i \, , \; s \to \infty \qquad \text{for } i = 1, 2, \ldots, m \tag{20}$$

with $c_i \in \mathbb{R}_0$. If we apply the reverse mapping $\mathcal{R}$ to (20), we get

$$u_{i\varphi(i)} = 0 \quad \text{or} \quad u_{i\varphi(i)} = \ominus 0 \qquad \text{for } i = 1, 2, \ldots, m \tag{21}$$

since $\mathcal{R}(c_i) = 0$ if $c_i > 0$ and $\mathcal{R}(c_i) = \ominus 0$ if $c_i < 0$.
We shall permute the columns of U such that the entries that are equal to 0 in max-absolute value will be on the main diagonal. This can be done as follows: We define an m by m max-algebraic permutation matrix P such that

$$\begin{aligned} p_{ij} &= 0 && \text{if } i = \varphi(j) \, , \\ &= \varepsilon && \text{otherwise} \, . \end{aligned}$$

If we define $W = U \otimes P$ then W contains the same columns as U but in a (possibly) different order. Furthermore,

$$w_{ii} \;=\; 0 \quad \text{or} \quad w_{ii} \;=\; \ominus 0 \qquad \text{for } i = 1, 2, \ldots, m \; .$$

We have $U = W \otimes P^T$ and $U^T \otimes U = P \otimes W^T \otimes W \otimes P^T$. So $U^T \otimes U \;\nabla\; E_m$ if and only if $W^T \otimes W \;\nabla\; E_m$. Let w_i be the ith column of W. Since $w_i^T \otimes w_i \;\nabla\; 0$ and since both sides of this balance are signed, we have $w_i^T \otimes w_i = 0$ for all i by Proposition 1.6.

Now we copy all the entries of W to $\hat{W}$. We shall update the columns of $\hat{W}$ in two steps. First we make all max-algebraic inner products of two different columns of $\hat{W}$ finite. Next we make the entries of $\hat{W}$ that are still infinite finite.

Step 2a: We make all max-algebraic inner products of two different columns of $\hat{W}$ finite. Define

$$h = \min_{i,j} \left\{ \left| w_i^T \otimes w_j \right|_\oplus \;\middle|\; w_i^T \otimes w_j \neq \varepsilon \right\}$$

and $M = \min(g, h-1)$. Note that h and M are finite. Since $|w_{ij}|_\oplus \leqslant 0$ for all i, j by Proposition 3.4, we have $|w_i^T \otimes w_j|_\oplus \leqslant 0$ for all i, j. Hence, $h \leqslant 0$ and $M < 0$. Furthermore, if $w_i^T \otimes w_j \neq \varepsilon$ then $|w_i^T \otimes w_j|_\oplus > M$.
Consider the following algorithm in which some of the infinite entries of $\hat{W}$ will be replaced by M or $\ominus M$:

for $i = 1, 2, \ldots, m-1$
  for $j = i+1, 2, \ldots, m$
    if $w_i^T \otimes w_j = \varepsilon$
    then
      $\hat{w}_{ij} \leftarrow M$
      $\hat{w}_{ji} \leftarrow (\ominus M) \otimes \hat{w}_{ii} \otimes \hat{w}_{jj}$
    endif
  endfor
endfor

Now we prove that this algorithm will result in

$$\hat{w}_i^T \otimes \hat{w}_j = w_i^T \otimes w_j \qquad \text{if } |w_i^T \otimes w_j|_\oplus \neq \varepsilon\,, \tag{22}$$

$$\hat{w}_i^T \otimes \hat{w}_j = M^\bullet \qquad \text{if } |w_i^T \otimes w_j|_\oplus = \varepsilon\ . \tag{23}$$

First we prove (22).
Later on we shall prove that only infinite entries of $\hat{W}$ are replaced by M or $\ominus M$. The finite entries of $\hat{W}$ do not change. So if $w_i^T \otimes w_j$ is finite then

$$\begin{aligned}
\hat{w}_i^T \otimes \hat{w}_j &= \bigoplus_{\substack{w_{ki} \text{ is finite and} \\ w_{kj} \text{ is finite}}} \hat{w}_{ki} \otimes \hat{w}_{kj} \;\oplus \bigoplus_{\substack{w_{ki} \text{ is infinite or} \\ w_{kj} \text{ is infinite}}} \hat{w}_{ki} \otimes \hat{w}_{kj} \\
&= \bigoplus_{\substack{w_{ki} \text{ is finite and} \\ w_{kj} \text{ is finite}}} w_{ki} \otimes w_{kj} \;\oplus \bigoplus_{\substack{w_{ki} \text{ is infinite or} \\ w_{kj} \text{ is infinite}}} \hat{w}_{ki} \otimes \hat{w}_{kj} \\
&= w_i^T \otimes w_j \;\oplus\; s_{ij}
\end{aligned} \tag{24}$$

where

$$s_{ij} = \bigoplus_{\substack{w_{ki} \text{ is infinite or} \\ w_{kj} \text{ is infinite}}} \hat{w}_{ki} \otimes \hat{w}_{kj}\ .$$

Since

$$\begin{array}{ll} |\hat{w}_{ki}|_{\oplus} \leqslant M \text{ and } |\hat{w}_{kj}|_{\oplus} \leqslant 0 & \text{if } w_{ki} \text{ is infinite,} \\ |\hat{w}_{ki}|_{\oplus} \leqslant 0 \text{ and } |\hat{w}_{kj}|_{\oplus} \leqslant M & \text{if } w_{kj} \text{ is infinite,} \end{array}$$

we have

$$\begin{aligned} |s_{ij}|_{\oplus} &= \bigoplus_{\substack{w_{ki} \text{ is infinite or} \\ w_{kj} \text{ is infinite}}} |\hat{w}_{ki} \otimes \hat{w}_{kj}|_{\oplus} \\ &= \bigoplus_{\substack{w_{ki} \text{ is infinite or} \\ w_{kj} \text{ is infinite}}} |\hat{w}_{ki}|_{\oplus} \otimes |\hat{w}_{kj}|_{\oplus} \\ &\leqslant M \\ &< |w_i^T \otimes w_j|_{\oplus} \ , \end{aligned}$$

where we have used Proposition 1.2 and 1.3. If we combine this with (24), we obtain $\hat{w}_i^T \otimes \hat{w}_j = w_i^T \otimes w_j$. So the values of the finite inner products do not change.
Now we prove (23).
If $w_i^T \otimes w_j = \varepsilon$ then

$$w_{si} \otimes w_{sj} = \varepsilon \qquad \text{for } s = 1, 2, \ldots, m$$

or equivalently

$$w_{si} = \varepsilon \quad \text{or} \quad w_{sj} = \varepsilon \qquad \text{for } s = 1, 2, \ldots, m \ . \tag{25}$$

Since $|w_{ii}|_{\oplus}$ and $|w_{jj}|_{\oplus}$ are equal to 0, this implies that both w_{ij} and w_{ji} are equal to ε.
It is possible that some of the infinite components of $\hat{w}_i$ and $\hat{w}_j$ have already been replaced by M or $\ominus M$. However, $\hat{w}_{ij}$ and $\hat{w}_{ji}$ are still equal to ε since each pair of indices (i, j) is encountered only once in the above algorithm. Hence, we only replace infinite entries of $\hat{W}$ by M or $\ominus M$ if we execute the algorithm.
If we replace $\hat{w}_{ij}$ by M and $\hat{w}_{ji}$ by $(\ominus M) \otimes \hat{w}_{ii} \otimes \hat{w}_{jj}$, we obtain

$$|\hat{w}_{ji}|_{\oplus} \ = \ |(\ominus M) \otimes \hat{w}_{ii} \otimes \hat{w}_{jj}|_{\oplus} \ = \ M \otimes |\hat{w}_{ii}|_{\oplus} \otimes |\hat{w}_{jj}|_{\oplus} \ = \ M \otimes 0 \otimes 0 \ = \ M$$

where we have used Proposition 1.3.
Since M, $\hat{w}_{ii}$ and $\hat{w}_{jj}$ are signed, $\hat{w}_{ji}$ is also signed. So either $\hat{w}_{ji} = M$ or $\hat{w}_{ji} = \ominus M$.
Now we have

$$\begin{aligned} \hat{w}_i^T \otimes \hat{w}_j &= \hat{w}_{ii} \otimes \hat{w}_{ij} \ \oplus \ \hat{w}_{ji} \otimes \hat{w}_{jj} \ \oplus \ \bigoplus_{s \neq i, s \neq j} \hat{w}_{si} \otimes \hat{w}_{sj} \\ &= \hat{w}_{ii} \otimes M \ \oplus \ (\ominus M) \otimes \hat{w}_{ii} \otimes \hat{w}_{jj} \otimes \hat{w}_{jj} \ \oplus \ t_{ij} \end{aligned} \tag{26}$$

where

$$t_{ij} = \bigoplus_{s \neq i, s \neq j} \hat{w}_{si} \otimes \hat{w}_{sj} \ .$$

By (25) we have

$$w_{si} = \varepsilon \text{ and thus } |\hat{w}_{si}|_{\oplus} \leqslant M \quad \text{or} \quad w_{sj} = \varepsilon \text{ and thus } |\hat{w}_{sj}|_{\oplus} \leqslant M$$

for $s = 1, 2, \ldots, m$. Furthermore, since $|w_{pq}|_{\oplus} \leqslant 0$ for all p, q by Proposition 3.4 and since $M < 0$, we have $|\hat{w}_{pq}|_{\oplus} \leqslant 0$ for all p, q. Hence, $|\hat{w}_{si} \otimes \hat{w}_{sj}|_{\oplus} \leqslant M$ and thus $|t_{ij}|_{\oplus} \leqslant M$.
Since $\hat{w}_{jj}$ is signed and $|\hat{w}_{jj}|_{\oplus} = 0$, either $\hat{w}_{jj} = 0$ or $\hat{w}_{jj} = \ominus 0$. So $\hat{w}_{jj} \otimes \hat{w}_{jj} = 0$. Therefore, (26) results in

$$\hat{w}_i^T \otimes \hat{w}_j \;=\; M \otimes \hat{w}_{ii} \,\oplus\, (\ominus M) \otimes \hat{w}_{ii} \,\oplus\, t_{ij} \;.$$

Since $\hat{w}_{ii}$ is either 0 or $\ominus 0$ and since $|t_{ij}|_{\oplus} \leqslant M$, this leads to

$$\hat{w}_i^T \otimes \hat{w}_j = M^{\bullet} \,\oplus\, t_{ij} = M^{\bullet} \;.$$

So now all max-algebraic inner products of two columns of $\tilde{W}$ are finite.

Step 2b: We make the remaining infinite entries of $\hat{W}$ finite by replacing them by M.
As already explained above this does not change the value of the finite inner products $\hat{w}_i^T \otimes \hat{w}_j = w_i^T \otimes w_j$. Furthermore, since the other inner products $\hat{w}_i^T \otimes \hat{w}_j$, are already equal to $M^{\bullet}$ their value does not change either. So (22) and (23) still hold.
If we define $\hat{U} = \hat{W} \otimes P^T$, then

$$\begin{aligned}
\hat{u}_i^T \otimes \hat{u}_i &= u_i^T \otimes u_i \;=\; 0\,, \\
\hat{u}_i^T \otimes \hat{u}_j &= u_i^T \otimes u_j \;\nabla\; \varepsilon \qquad \text{if } |u_i^T \otimes u_j|_{\oplus} \neq \varepsilon \text{ and } i \neq j\,, \\
\hat{u}_i^T \otimes \hat{u}_j &= M^{\bullet} \;\nabla\; \varepsilon \qquad\quad\;\; \text{if } |u_i^T \otimes u_j|_{\oplus} = \varepsilon\;.
\end{aligned}$$

So now we have a finite matrix $\hat{U}$ for which $\hat{U}^T \otimes \hat{U} \;\nabla\; E_m$. Furthermore, (17) still holds since condition (19) is satisfied. Therefore, we now have obtained a max-algebraic SVD with finite singular values and finite left singular vectors.

Step 3: Finally we make the components of the right singular vectors finite.
Using a reasoning that is analogous to the one of Step 2 we can transform the right singular vectors v_i into right singular vectors $\hat{v}_i$ with finite entries.

This yields a max-algebraic SVD $\hat{U} \otimes \hat{\Sigma} \otimes \hat{V}$ of A with finite singular values and finite singular vectors. □

Theorem 3.6 (The QR decomposition in $\mathbb{S}_{\max}$) *If $A \in \mathbb{S}^{m\times n}$ then there exist a matrix $Q \in (\mathbb{S}^{\vee})^{m\times m}$ and a max-algebraic upper triangular matrix $R \in (\mathbb{S}^{\vee})^{m\times n}$ such that*

$$A \;\nabla\; Q \otimes R \tag{27}$$

with

$$Q^T \otimes Q \;\nabla\; E_m$$

and $\|R\|_{\oplus} \leqslant \|A\|_{\oplus}$.
Every decomposition of the form (27) that satisfies the above conditions is called a max-algebraic QR decomposition of A.

The decomposition (27) can be rewritten as:

$$A \;\nabla\; \bigoplus_{i=1}^{r} q_i \otimes R_{i.}$$

where q_i is the ith column of Q, $R_{i.}$ is the ith row of R and $r = \min(m, n)$. Now we can also define a max-algebraic QR rank:

Definition 3.7 (Max-algebraic QR rank) *Let $A \in \mathbb{S}^{m \times n}$. The max-algebraic QR rank of A is defined as*

$$\operatorname{rank}_{\oplus,\mathrm{QR}}(A) \;=\; \min \left\{ \rho \;\middle|\; A \;\nabla\; \bigoplus_{i=1}^{\rho} q_i \otimes R_{i.}\,,\;\; Q \otimes R \text{ is a max-algebraic QR decomposition of } A \right\}$$

where q_i is the ith column of Q, $R_{i.}$ is the ith row of R and $\bigoplus_{i=1}^{0} q_i \otimes R_{i.}$ is equal to $\mathcal{E}_{m \times n}$ by definition.

Using a proof that is similar to that of Theorem 3.5 we get:

Theorem 3.8 *Consider a matrix $A \in \mathbb{S}^{m \times n}$ with finite entries: $|a_{ij}|_{\oplus} \neq \varepsilon$ for all i, j. Then there exists a max-algebraic QR decomposition $Q \otimes R$ of A for which all of the entries of Q and all of the entries of the upper triangular part of R are finite.*

4 Calculation of the max-algebraic singular value decomposition and the max-algebraic QR decomposition

We can use the mapping $\mathcal{F}$ to calculate the SVD in $\mathbb{S}_{\max}$. However, max-algebraic singular values and components of the max-algebraic singular vectors that are asymptotically equivalent to an exponential of the form γe^{cs} with $c < 0$ in the neighborhood of ∞ will become almost 0 even for relatively small s. Numerically they are then equal to 0 and they will be mapped to ε instead of c by the reverse mapping $\mathcal{R}$. Therefore, we now present another technique to calculate the SVD in $\mathbb{S}_{\max}$ without making use of the mapping $\mathcal{F}$.
The original problem is:

> Given $A \in \mathbb{S}^{m \times n}$, find a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_{\varepsilon}^{m \times n}$ and matrices $U \in (\mathbb{S}^{\vee})^{m \times m}$ and $V \in (\mathbb{S}^{\vee})^{n \times n}$ such that
>
> $$A \;\nabla\; U \otimes \Sigma \otimes V^T \tag{28}$$
>
> $$U^T \otimes U \;\nabla\; E_m \tag{29}$$
>
> $$V^T \otimes V \;\nabla\; E_n \tag{30}$$
>
> with $\|A\|_{\oplus} \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r \geqslant \varepsilon$, where $\sigma_i = (\Sigma)_{ii}$ and $r = \min(m, n)$.

We shall now transform the above conditions into relations in $\mathbb{R}_{\max}$ and show that we finally get a system of multivariate max-algebraic polynomial equalities and inequalities.
If all the entries of A are finite then there exists a max-algebraic SVD of A with finite singular values and finite singular vectors by Theorem 3.5. If some of the entries of A are not finite, we can use the technique of Remark 2.3 and replace these entries by $-\xi$ where ξ is large enough. This will result in a finite matrix for which there exists a max-algebraic SVD with finite singular values and vectors. Now we shall write down the equations that will yield a

max-algebraic SVD $U \otimes \Sigma \otimes V^T$ of A with finite singular values and finite singular vectors. First of all, we want the entries of U and V to be signed:

$$u_{ij}^{\oplus} \otimes u_{ij}^{\ominus} = \varepsilon \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, m \tag{31}$$

$$v_{ij}^{\oplus} \otimes v_{ij}^{\ominus} = \varepsilon \qquad \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n \ . \tag{32}$$

If we extract the max-positive and the max-negative parts of each matrix, (28) – (30) result in

$$A^{\oplus} \ominus A^{\ominus} \ \nabla \ (U^{\oplus} \ominus U^{\ominus}) \otimes \Sigma \otimes (V^{\oplus} \ominus V^{\ominus})^T$$

$$(U^{\oplus} \ominus U^{\ominus})^T \otimes (U^{\oplus} \ominus U^{\ominus}) \ \nabla \ E_m$$

$$(V^{\oplus} \ominus V^{\ominus})^T \otimes (V^{\oplus} \ominus V^{\ominus}) \ \nabla \ E_n$$

or

$$A^{\oplus} \ \oplus \ U^{\oplus} \otimes \Sigma \otimes (V^{\ominus})^T \ \oplus \ U^{\ominus} \otimes \Sigma \otimes (V^{\oplus})^T \ \nabla \ A^{\ominus} \ \oplus \ U^{\oplus} \otimes \Sigma \otimes (V^{\oplus})^T \ \oplus \ U^{\ominus} \otimes \Sigma \otimes (V^{\ominus})^T \tag{33}$$

$$(U^{\oplus})^T \otimes U^{\oplus} \ \oplus \ (U^{\ominus})^T \otimes U^{\ominus} \ \nabla \ E_m \ \oplus \ (U^{\oplus})^T \otimes U^{\ominus} \ \oplus \ (U^{\ominus})^T \otimes U^{\oplus} \tag{34}$$

$$(V^{\oplus})^T \otimes V^{\oplus} \ \oplus \ (V^{\ominus})^T \otimes V^{\ominus} \ \nabla \ E_n \ \oplus \ (V^{\oplus})^T \otimes V^{\ominus} \ \oplus \ (V^{\ominus})^T \otimes V^{\oplus} \tag{35}$$

by Proposition 1.7. Both sides of all the balances are now signed. So by Proposition 1.8 we can replace the balances by equalities. Define three matrices $T \in \mathbb{R}_{\varepsilon}^{m \times n}$, $P \in \mathbb{R}_{\varepsilon}^{m \times m}$ and $Q \in \mathbb{R}_{\varepsilon}^{n \times n}$ such that

$$\begin{aligned} A^{\oplus} \ \oplus \ U^{\oplus} \otimes \Sigma \otimes (V^{\ominus})^T \ \oplus \ U^{\ominus} \otimes \Sigma \otimes (V^{\oplus})^T &= T \\ (U^{\oplus})^T \otimes U^{\oplus} \ \oplus \ (U^{\ominus})^T \otimes U^{\ominus} &= P \\ (V^{\oplus})^T \otimes V^{\oplus} \ \oplus \ (V^{\ominus})^T \otimes V^{\ominus} &= Q \ . \end{aligned}$$

Note that P and Q are symmetric. Since the max-algebraic singular values are finite, their max-algebraic inverses are defined. The entries of the matrices T, P and Q are also finite. So their max-algebraic inverses are also defined.
If we work out the matrix multiplications in (33) and if we transfer the entries of T to the left hand side, we get

$$a_{ij}^{\oplus} \otimes t_{ij}^{\otimes^{-1}} \ \oplus \ \bigoplus_{k=1}^{r} u_{ik}^{\oplus} \otimes \sigma_k \otimes v_{jk}^{\ominus} \otimes t_{ij}^{\otimes^{-1}} \ \oplus \ \bigoplus_{k=1}^{r} u_{ik}^{\ominus} \otimes \sigma_k \otimes v_{jk}^{\oplus} \otimes t_{ij}^{\otimes^{-1}} \ = \ 0 \tag{36}$$

$$a_{ij}^{\ominus} \otimes t_{ij}^{\otimes^{-1}} \ \oplus \ \bigoplus_{k=1}^{r} u_{ik}^{\oplus} \otimes \sigma_k \otimes v_{jk}^{\oplus} \otimes t_{ij}^{\otimes^{-1}} \ \oplus \ \bigoplus_{k=1}^{r} u_{ik}^{\ominus} \otimes \sigma_k \otimes v_{jk}^{\ominus} \otimes t_{ij}^{\otimes^{-1}} \ = \ 0 \tag{37}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.
Since P is symmetric and since $e_{ij} = 0$ if $i = j$ and $e_{ij} = \varepsilon$ if $i \neq j$, (34) leads to

$$\bigoplus_{k=1}^{m} u_{ki}^{\oplus} \otimes u_{kj}^{\oplus} \otimes p_{ij}^{\otimes^{-1}} \ \oplus \ \bigoplus_{k=1}^{m} u_{ki}^{\ominus} \otimes u_{kj}^{\ominus} \otimes p_{ij}^{\otimes^{-1}} \ = \ 0 \tag{38}$$

$$\bigoplus_{k=1}^{m} u_{ki}^{\oplus} \otimes u_{kj}^{\ominus} \otimes p_{ij}^{\otimes^{-1}} \ \oplus \ \bigoplus_{k=1}^{m} u_{ki}^{\ominus} \otimes u_{kj}^{\oplus} \otimes p_{ij}^{\otimes^{-1}} \ = \ 0 \tag{39}$$

for $i = 1, 2, \ldots, m$ and $j = i+1, i+2, \ldots, m$, and

$$\bigoplus_{k=1}^{m} u_{ki}^{\oplus} \otimes u_{ki}^{\oplus} \oplus \bigoplus_{k=1}^{m} u_{ki}^{\ominus} \otimes u_{ki}^{\ominus} \quad = \quad 0 \oplus \bigoplus_{k=1}^{m} u_{ki}^{\oplus} \otimes u_{ki}^{\ominus} \oplus \bigoplus_{k=1}^{m} u_{ki}^{\ominus} \otimes u_{ki}^{\oplus} \quad = \quad p_{ii}$$

for $i = 1, 2, \ldots, m$, or

$$\bigoplus_{k=1}^{m} (u_{ki}^{\oplus})^{\otimes^2} \oplus \bigoplus_{k=1}^{m} (u_{ki}^{\ominus})^{\otimes^2} = 0 = p_{ii} \qquad \text{for } i = 1, 2, \ldots, m$$

since the entries of U are signed. If $x \in \mathbb{R}_\varepsilon$ then $x^{\otimes^2}$ is equal to $2 \cdot x$ in linear algebra. Hence,

$$\bigoplus_{k=1}^{m} u_{ki}^{\oplus} \oplus \bigoplus_{k=1}^{m} u_{ki}^{\ominus} \quad = \quad 0 \qquad \text{for } i = 1, 2, \ldots, m \ . \tag{40}$$

Note that $p_{ii} = 0$ for $i = 1, 2, \ldots, m$.
Analogously we obtain

$$\bigoplus_{k=1}^{n} v_{ki}^{\oplus} \otimes v_{kj}^{\oplus} \otimes q_{ij}{}^{\otimes^{-1}} \oplus \bigoplus_{k=1}^{n} v_{ki}^{\ominus} \otimes v_{kj}^{\ominus} \otimes q_{ij}{}^{\otimes^{-1}} \quad = \quad 0 \tag{41}$$

$$\bigoplus_{k=1}^{n} v_{ki}^{\oplus} \otimes v_{kj}^{\ominus} \otimes q_{ij}{}^{\otimes^{-1}} \oplus \bigoplus_{k=1}^{n} v_{ki}^{\ominus} \otimes v_{kj}^{\oplus} \otimes q_{ij}{}^{\otimes^{-1}} \quad = \quad 0 \tag{42}$$

for $i = 1, 2, \ldots, n$ and $j = i+1, i+2, \ldots, n$,

$$\bigoplus_{k=1}^{n} v_{ki}^{\oplus} \oplus \bigoplus_{k=1}^{n} v_{ki}^{\ominus} \quad = \quad 0 \qquad \text{for } i = 1, 2, \ldots, n \tag{43}$$

and $q_{ii} = 0$ for $i = 1, 2, \ldots, n$.
The condition $\sigma_1 \leqslant \|A\|_\oplus$ can be rewritten as

$$\|A\|_\oplus \otimes \sigma_1{}^{\otimes^{-1}} \geqslant 0 \ . \tag{44}$$

Finally we order the max-algebraic singular values by requiring that

$$\sigma_i \geqslant \sigma_{i+1} \qquad \text{for } i = 1, 2, \ldots, r-1$$

or

$$\sigma_i \otimes (\sigma_{i+1})^{\otimes^{-1}} \geqslant 0 \qquad \text{for } i = 1, 2, \ldots, r-1 \ . \tag{45}$$

Expressions (31) – (45) constitute a system of multivariate max-algebraic polynomial equalities and inequalities. Using the technique explained in Section 2 and taking Remark 2.3 into account they can be transformed into an ELCP. So we can use the ELCP algorithm of [4] to find all the solutions of (31) – (45).
An arbitrary solution of an ELCP is given by the sum of a linear combination of the central rays, a nonnegative combination of cross-complementary extreme rays and a convex combination of cross-complementary finite vertices that are also cross-complementary with these extreme rays. Since the max-algebraic singular values are bounded from above by $\|A\|_\oplus$ and since the max-absolute values of the components of the max-algebraic singular vectors are bounded from above by 0, there cannot be any central rays in the solution set of the ELCP

that corresponds to (31) – (45). Furthermore, the fact that all the components of the solutions are bounded from above also implies that the components of the extreme rays are less than or equal to 0. Therefore, the finite vertices that result from the ELCP algorithm will always correspond to a maximal max-algebraic SVD.
Since every matrix with finite entries has at least one max-algebraic SVD with finite singular values and finite singular vectors by Theorem 3.5, the solution set of the ELCP that corresponds to (31) – (45) cannot be empty. Now we can characterize the set of all the max-algebraic SVDs of a given matrix with finite entries:

Theorem 4.1 *Let $A \in \mathbb{S}^{m \times n}$ such that all the entries of A are finite. In general the set of all the max-algebraic SVDs of A with finite singular values and finite singular vectors consists of the union of faces of a polyhedron in the x-space, where x is the vector obtained by putting the diagonal entries of Σ and the entries of matrices U and V in one large vector.*

Max-algebraic SVDs for which some singular values are infinite or for which some singular vectors have infinite components correspond to points at infinity if this polyhedron.

How to reduce the number of variables and equations?

The time and memory space needed to solve an ELCP with the algorithm described in [4] increases rapidly as the number of variables and equations increases. Therefore, it is advantageous to reduce the number of variables and equations as much as possible.
If there is a signed entry in A that is equal to $\|A\|_\oplus$ in max-absolute value then we already know that $\sigma_1 = \|A\|_\oplus$ by Proposition 3.2.
Since

$$A = \bigoplus_{i=1}^{r} \sigma_i \otimes u_i \otimes v_i^T \ ,$$

a left singular vector stays a left singular vector if we max-multiply it (and the corresponding right singular vector) by $\ominus 0$. This means that we can further reduce the number of variables and equations by requiring that the diagonal entries of U (or V, depending on which one has the largest dimension) are max-positive:

$$u_{ii}^{\ominus} = \varepsilon \qquad \text{for } i = 1, 2, \ldots, m \ .$$

It is obvious that the max-algebraic QRD of a matrix $A \in \mathbb{S}^{m \times n}$ can also be calculated using this ELCP technique. In this case we can also reduce the number of variables by requiring that the diagonal entries of Q are max-positive or zero: If Q satisfies $Q \otimes Q^T \ \nabla \ E_m$ and if we replace q_i, the ith column of Q, by $\ominus q_i$, we still have $Q \otimes Q^T \ \nabla \ E_m$. Furthermore, we already know that $A \ \nabla \ Q \otimes R$ can be rewritten as:

$$A \ \nabla \ \bigoplus_{i=1}^{r} q_i \otimes R_{i.} \tag{46}$$

where q_i is the ith column of Q, $R_{i.}$ is the ith row of R and $r = \min(m, n)$. So if we replace q_i by $\ominus q_i$ and $R_{i.}$ by $\ominus R_{i.}$ in (46), we still have a max-algebraic QRD of A. This means that we can always assume that the diagonal entries of Q are max-positive or zero.

5 Extensions of the max-algebraic singular value decomposition and the max-algebraic QR decomposition

If U is a (real) m by m matrix then $U^T U = I_m$ if and only if $U U^T = I_m$. Furthermore, if $U^T U = I_m$ then the columns of U are linearly independent. However, in the extended max algebra $U^T \otimes U \nabla E_m$ does not always imply that $U \otimes U^T \nabla E_m$ or that the columns of U are max-linearly independent. Therefore, we have proposed some extensions of the definitions of the max-algebraic SVD and the max-algebraic QRD in [6]. Now we show that these extended decompositions can also be reformulated as an ELCP.

Theorem 5.1 (The extended SVD in $\mathbb{S}_{\mathbf{max}}$) *Let $A \in \mathbb{S}^{m \times n}$ and let $r = \min(m, n)$. Then there exist a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m \times n}$ and matrices $U \in (\mathbb{S}^\vee)^{m \times m}$ and $V \in (\mathbb{S}^\vee)^{n \times n}$ such that*

$$A \ \nabla \ U \otimes \Sigma \otimes V^T$$

with

$$\begin{aligned} U^T \otimes U &\ \nabla\ E_m \\ U \otimes U^T &\ \nabla\ E_m \\ V^T \otimes V &\ \nabla\ E_n \\ V \otimes V^T &\ \nabla\ E_n \ , \end{aligned}$$

where the rows and the columns of U and V are max-linearly independent or equivalently

$$\begin{aligned} \det\nolimits_\oplus U &\ \not\nabla\ \varepsilon \\ \det\nolimits_\oplus V &\ \not\nabla\ \varepsilon \end{aligned}$$

and with $\|A\|_\oplus \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r \geqslant \varepsilon$ where $\sigma_i = (\Sigma)_{ii}$.

Theorem 5.2 (The extended QRD in $\mathbb{S}_{\mathbf{max}}$) *If $A \in \mathbb{S}^{m \times n}$ then there exist a matrix $Q \in (\mathbb{S}^\vee)^{m \times m}$ and a max-algebraic upper triangular matrix $R \in (\mathbb{S}^\vee)^{m \times n}$ such that*

$$A \ \nabla \ Q \otimes R$$

with

$$\begin{aligned} Q^T \otimes Q &\ \nabla\ E_m \\ Q \otimes Q^T &\ \nabla\ E_m \\ \det\nolimits_\oplus Q &\ \not\nabla\ \varepsilon \end{aligned}$$

and $\|R\|_\oplus \leqslant \|A\|_\oplus$.

If we use a reasoning similar to the one made for $U^T \otimes U \nabla E_m$, then the conditions

$$U \otimes U^T \ \nabla\ E_m \qquad (47)$$

$$V \otimes V^T \ \nabla\ E_n \qquad (48)$$

also yield multivariate max-algebraic polynomial equalities that could be added to (31) – (45). If the matrix A has finite entries, we can use a reasoning analogous to the one of the proof

of Theorem 3.5 to show that there exists at least one extended max-algebraic SVD of A with finite singular values and finite singular vectors that also satisfies (47) and (48).
The conditions

$$
\begin{aligned}
&\det\nolimits_{\oplus} U \not\nabla \varepsilon \\
&\det\nolimits_{\oplus} V \not\nabla \varepsilon
\end{aligned}
$$

can be rewritten as

$$
\begin{aligned}
(\det\nolimits_{\oplus} U)^{\oplus} \otimes (\det\nolimits_{\oplus} U)^{\ominus} &= \varepsilon && (49)\\
(\det\nolimits_{\oplus} V)^{\oplus} \otimes (\det\nolimits_{\oplus} V)^{\ominus} &= \varepsilon && (50)
\end{aligned}
$$

where

$$
\begin{aligned}
(\det\nolimits_{\oplus} U)^{\oplus} = & \bigoplus_{\substack{\varphi \cup \psi \in \mathcal{P}_{n,\text{even}} \\ D_\varphi \cap D_\psi = \emptyset,\ \#D_\psi \text{ is even}}} \bigotimes_i u^{\oplus}_{i\varphi(i)} \otimes \bigotimes_j u^{\ominus}_{j\psi(j)} \quad \oplus \\
& \bigoplus_{\substack{\varphi \cup \psi \in \mathcal{P}_{n,\text{odd}} \\ D_\varphi \cap D_\psi = \emptyset,\ \#D_\psi \text{ is odd}}} \bigotimes_i u^{\oplus}_{i\varphi(i)} \otimes \bigotimes_j u^{\ominus}_{j\psi(j)}
\end{aligned}
$$

and

$$
\begin{aligned}
(\det\nolimits_{\oplus} U)^{\ominus} = & \bigoplus_{\substack{\varphi \cup \psi \in \mathcal{P}_{n,\text{even}} \\ D_\varphi \cap D_\psi = \emptyset,\ \#D_\psi \text{ is odd}}} \bigotimes_i u^{\oplus}_{i\varphi(i)} \otimes \bigotimes_j u^{\ominus}_{j\psi(j)} \quad \oplus \\
& \bigoplus_{\substack{\varphi \cup \psi \in \mathcal{P}_{n,\text{odd}} \\ D_\varphi \cap D_\psi = \emptyset,\ \#\psi \text{ is even}}} \bigotimes_i u^{\oplus}_{i\varphi(i)} \otimes \bigotimes_j u^{\ominus}_{j\psi(j)}
\end{aligned}
$$

where $\mathcal{P}_{n,\text{even}}$ is the set of even permutations of $\{1, 2, \ldots, n\}$ and $\mathcal{P}_{n,\text{odd}}$ is the set of odd permutations of $\{1, 2, \ldots, n\}$. Analogous expressions exist for $(\det_{\oplus} V)^{\oplus}$ and $(\det_{\oplus} V)^{\ominus}$. So (49) and (50) can be considered as multivariate max-algebraic polynomial equalities. If we also add these constraints to the system (31) − (45), we still have a system of multivariate max-algebraic polynomial equalities and inequalities.
As a direct consequence of (21) the max-algebraic determinant of the matrix U of the proof of Theorem 3.5 satisfies $|\det_{\oplus} U|_{\oplus} = 0$. Since $M < 0$ and since the entries of U are less than or equal to 0 in max-absolute value, the value of $\det_{\oplus} U$ will not change if we replace the infinite entries of U by M or $\ominus M$. This also holds for V. So we can still use the procedure of the proof of Theorem 3.5 to obtain an extended max-algebraic SVD with finite singular values and finite singular vectors for a matrix with finite entries.
This means that in theory we can still use the ELCP algorithm to solve the extended system of multivariate max-algebraic polynomial equalities and inequalities. However, we have to point out that the conditions (49) and (50) would yield such a large number of extra inequalities that in practice it will be impossible to solve the resulting ELCP in a reasonable amount of CPU time with the algorithm of [4], especially if m and n are large.

Using a similar reasoning as for the extended max-algebraic SVD it can be shown that we can still use the ELCP algorithm to solve the system of multivariate max-algebraic polynomial equalities and inequalities that corresponds to the extended max-algebraic QRD of a matrix.

6 Worked examples

In this section we calculate all the max-algebraic SVDs and all the max-algebraic QRDs of the example of [8].

Example 6.1 Consider

$$A = \begin{bmatrix} 2 & \ominus 5 \\ \ominus 0 & 3 \end{bmatrix} .$$

In [8] we have used the link between $\mathbb{S}_{\max}$ and linear algebra to calculate a max-algebraic SVD of A. Now we use the transformation to a system of multivariate max-algebraic polynomial equalities and inequalities to calculate all the max-algebraic SVDs of A. We always have $u_{ij}^{\oplus} \otimes u_{ij}^{\ominus} = \varepsilon$ and thus $u_{ij}^{\oplus} = \varepsilon$ or $u_{ij}^{\ominus} = \varepsilon$ for all i, j. This also holds for the entries of V. However, since the ELCP algorithm normally only yields finite solutions, we can no longer work with the max-positive and the max-negative parts of the entries of U and V. Therefore, we apply the technique of Remark 2.3 and we introduce new (finite) variables $u_{ij}^{\boxplus}$ and $u_{ij}^{\boxminus}$ such that $u_{ij}^{\boxplus} \otimes u_{ij}^{\boxminus} \leqslant -\xi$ where ξ is a large positive real number. We still have $u_{ij} = u_{ij}^{\boxplus} \ominus u_{ij}^{\boxminus}$ provided that ξ is large enough. In a similar way we also define $v_{ij}^{\boxplus}$ and $v_{ij}^{\boxminus}$. Now we put all the variables in one large column vector x:

$$x = [\ \sigma_1 \ \ \sigma_2 \ \ u_{11}^{\boxplus} \ \ u_{12}^{\boxplus} \ \ u_{21}^{\boxplus} \ \ u_{22}^{\boxplus} \ \ u_{11}^{\boxminus} \ \ u_{12}^{\boxminus} \ \ u_{21}^{\boxminus} \ \ u_{22}^{\boxminus} \ \ v_{11}^{\boxplus} \ \ v_{12}^{\boxplus} \ \ v_{21}^{\boxplus} \ \ v_{22}^{\boxplus}$$
$$v_{11}^{\boxminus} \ \ v_{12}^{\boxminus} \ \ v_{21}^{\boxminus} \ \ v_{22}^{\boxminus} \ \ t_{11} \ \ t_{12} \ \ t_{21} \ \ t_{22} \ \ p_{12} \ \ q_{12}\ ]^T \ .$$

Note that p_{11}, p_{22}, q_{11}, q_{22}, p_{21} and q_{21} are not considered as unknowns since we already know that $p_{11} = p_{22} = q_{11} = q_{22} = 0$ and $p_{21} = p_{12}$ and $q_{21} = q_{12}$.
If we set ξ equal to 1000, the ELCP algorithm of [4] yields the rays and vertices of Tables 1 and 2 and the pairs of subsets of Table 3. Note that there are no central rays. Any arbitrary solution of the system of multivariate polynomial equalities and inequalities can now be expressed as

$$x = x_s^{\mathrm{f}} + \sum_{x_k^{\mathrm{i}} \in \mathcal{X}_s^{\mathrm{inf}}} \kappa_k x_k^{\mathrm{i}}$$

with $s \in \{1, 2, \ldots, 8\}$ and $\kappa_k \geqslant 0$.
Consider ray x_1^{f}. Since $u_{11}^{\boxplus}$, $u_{12}^{\boxplus}$, $u_{22}^{\boxplus}$, $u_{21}^{\boxminus}$, $v_{11}^{\boxplus}$, $v_{12}^{\boxminus}$, $v_{21}^{\boxminus}$ and $v_{22}^{\boxminus}$ are negative numbers of the same order of magnitude as ξ and since there are no positive components of the same order of magnitude as ξ – as was to be expected since the max-algebraic singular values are bounded from above by $\|A\|_{\oplus} = 5$ and since the max-absolute values of the components of the max-algebraic singular vectors are bounded from above by 0 – these entries can be replaced by ε as explained in Remark 2.3. Then we get the following decomposition:

$$A \ \nabla \ \begin{bmatrix} \ominus 0 & \ominus(-2) \\ -2 & \ominus 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & \varepsilon \\ \varepsilon & 0 \end{bmatrix} \otimes \begin{bmatrix} \ominus(-3) & 0 \\ 0 & -3 \end{bmatrix}^T = \begin{bmatrix} 2 & \ominus 5 \\ \ominus 0 & 3 \end{bmatrix} . \tag{51}$$

This solution can also be obtained as the following combination of the extreme rays and the vertex of the pair $\left\{\mathcal{X}_1^{\mathrm{inf}}, \mathcal{X}_1^{\mathrm{fin}}\right\}$:

$$x = x_1^{\mathrm{f}} + \eta x_1^{\mathrm{i}} + \eta x_3^{\mathrm{i}} + \eta x_4^{\mathrm{i}} + \eta x_5^{\mathrm{i}} + \eta x_6^{\mathrm{i}} + \eta x_8^{\mathrm{i}} + \eta x_{15}^{\mathrm{i}} + \eta x_{16}^{\mathrm{i}}$$

for $\eta \to \infty$, or by using the fact that

$$\begin{aligned} u_{ij} \;=\; u_{ij}^{\boxplus} \ominus u_{ij}^{\boxminus} \;&=\; u_{ij}^{\boxplus} \qquad \text{if } u_{ij}^{\boxplus} > u_{ij}^{\boxminus} \,, \\ &=\; u_{ij}^{\boxminus} \qquad \text{if } u_{ij}^{\boxminus} > u_{ij}^{\boxplus} \,, \end{aligned}$$

and an analogous expression for v_{ij}.
Since the extreme ray x_{11}^{i} belongs to the set $\mathcal{X}_3^{\mathrm{inf}}$, we can replace σ_2 in (51) by any negative real number or by ε. So

$$A \;\nabla\; \begin{bmatrix} \ominus 0 & \ominus(-2) \\ -2 & \ominus 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & \varepsilon \\ \varepsilon & \sigma_2 \end{bmatrix} \otimes \begin{bmatrix} \ominus(-3) & 0 \\ 0 & -3 \end{bmatrix}^T = \begin{bmatrix} 2 & \ominus 5 \\ \ominus 0 & 3 \end{bmatrix} \tag{52}$$

is a max-algebraic SVD of A for every $\sigma_2 \in \mathbb{R}_\varepsilon$ with $\sigma_2 \leqslant 0$.
The decompositions that correspond to the other finite vertices of Table 2 can be obtained from decomposition (51) by replacing u_2 by $\ominus u_2$, or by replacing v_2 by $\ominus v_2$ or by replacing u_1 and v_1 by $\ominus u_1$ and $\ominus v_1$ respectively, or by a combination of these replacements. Furthermore, since the extreme ray x_{11}^{i} belongs to every set of cross-complementary extreme rays, we can replace σ_2 in all these compositions by any negative real number or by ε. Since $A \not\nabla \mathcal{E}_{2\times 2}$, this means that the max-algebraic SVD rank of A is equal to 1.
The solution of [8]:

$$A \;\nabla\; \begin{bmatrix} 0 & -2 \\ \ominus(-2) & 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} -3 & 0 \\ \ominus 0 & -3 \end{bmatrix}^T = \begin{bmatrix} 2 & \ominus 5 \\ \ominus 0 & 3 \end{bmatrix}$$

corresponds to the following combination of the extreme rays and the vertex of the pair $\left\{\mathcal{X}_4^{\mathrm{inf}}, \mathcal{X}_4^{\mathrm{fin}}\right\}$:

$$x_4^{\mathrm{f}} + \eta x_2^{\mathrm{i}} + \eta x_3^{\mathrm{i}} + \eta x_5^{\mathrm{i}} + \eta x_7^{\mathrm{i}} + \eta x_9^{\mathrm{i}} + \eta x_{11}^{\mathrm{i}} + \eta x_{12}^{\mathrm{i}} + \eta x_{13}^{\mathrm{i}} + \eta x_{17}^{\mathrm{i}}$$

for $\eta \to \infty$. □

Example 6.2 Now we calculate all the max-algebraic QR decompositions of

$$A \;=\; \begin{bmatrix} 2 & \ominus 5 \\ \ominus 0 & 3 \end{bmatrix}$$

using the ELCP technique. To reduce the number of variables and equations we assume that the diagonal entries of Q are max-positive or zero. We introduce a matrix $T \in \mathbb{R}_\varepsilon^{m\times n}$ such that $T = A^\oplus \;\oplus\; Q^\oplus \otimes R^\ominus \;\oplus\; Q^\ominus \otimes R^\oplus$ and a symmetric matrix $P \in \mathbb{R}_\varepsilon^{m\times m}$ such that $P = (Q^\oplus)^T \otimes Q^\oplus \;\oplus\; (Q^\ominus)^T \otimes Q^\ominus$. Note that $p_{11} = p_{22} = 0$ since we also have $P = E_2 \;\oplus\; (Q^\oplus)^T \otimes Q^\ominus \;\oplus\; (Q^\ominus)^T \otimes Q^\oplus$. So the variables of the system of multivariate max-algebraic polynomial equalities and inequalities that correspond to the max-algebraic QRD of A are the off-diagonal entries of $Q^\ominus$, the entries of $Q^\oplus$ and T, the entries of the upper triangular parts of $R^\oplus$ and $R^\ominus$, and p_{21}. Just as in Example 6.1 we introduce new (finite) variables $q_{ij}^{\boxplus}$, $q_{ij}^{\boxminus}$, $r_{ij}^{\boxplus}$ and $r_{ij}^{\boxminus}$ such that

$$\begin{aligned} q_{ij} &= q_{ij}^{\boxplus} \ominus q_{ij}^{\boxminus} \quad \text{and} \quad q_{ij}^{\boxplus} \otimes q_{ij}^{\boxminus} \;\leqslant\; -\xi \\ r_{ij} &= r_{ij}^{\boxplus} \ominus r_{ij}^{\boxminus} \quad \text{and} \quad r_{ij}^{\boxplus} \otimes r_{ij}^{\boxminus} \;\leqslant\; -\xi \end{aligned}$$

where ξ is a large positive real number. All the variables are put in one large column vector x:

$$\begin{aligned} x \;=\; \big[\; & q_{11}^{\boxplus} \;\; q_{12}^{\boxplus} \;\; q_{21}^{\boxplus} \;\; q_{22}^{\boxplus} \;\; q_{12}^{\boxminus} \;\; q_{21}^{\boxminus} \;\; r_{11}^{\boxplus} \;\; r_{12}^{\boxplus} \;\; r_{22}^{\boxplus} \\ & \qquad\qquad\qquad r_{11}^{\boxminus} \;\; r_{12}^{\boxminus} \;\; r_{22}^{\boxminus} \;\; t_{11} \;\; t_{12} \;\; t_{21} \;\; t_{22} \;\; p_{12} \;\big]^T \;. \end{aligned}$$

If we set ξ equal to 1000, the ELCP algorithm of [4] yields the rays and vertices of Tables 4. All the extreme rays and all the finite vertices are cross-complementary:

$$\Lambda \;=\; \Big\{ \{x_1^{\mathrm{i}}, x_2^{\mathrm{i}}, x_3^{\mathrm{i}}, x_4^{\mathrm{i}}, x_5^{\mathrm{i}}, x_6^{\mathrm{i}}\} \;,\; \{x_1^{\mathrm{f}}, x_2^{\mathrm{f}}\} \Big\} \;.$$

Note that there are no central rays. Any arbitrary solution of the system of multivariate polynomial equalities and inequalities can now be expressed as

$$x \;=\; \mu_1 x_1^{\mathrm{f}} + \mu_2 x_2^{\mathrm{f}} + \sum_{k=1}^{6} \kappa_k x_k^{\mathrm{i}}$$

with $\mu_1, \mu_2, \kappa_k \geqslant 0$ and $\mu_1 + \mu_2 = 1$.
Consider ray x_1^{f}. Since $q_{21}^{\boxplus}$, $q_{12}^{\boxminus}$, $r_{12}^{\boxplus}$, $r_{22}^{\boxplus}$ and $r_{11}^{\boxminus}$ are negative numbers of the same order of magnitude as ξ and since there are no positive components of the same order of magnitude as ξ these entries can be replaced by ε. This yields

$$Q \;=\; \begin{bmatrix} 0 & -2 \\ \ominus(-2) & 0 \end{bmatrix} \quad \text{and} \quad R \;=\; \begin{bmatrix} 2 & \ominus 5 \\ \varepsilon & \ominus 3 \end{bmatrix} \;. \tag{53}$$

We have

$$Q \otimes R \;=\; \begin{bmatrix} 2 & \ominus 5 \\ \ominus 0 & 3^{\bullet} \end{bmatrix} \;\nabla\; A$$

$$Q^T \otimes Q \;=\; \begin{bmatrix} 0 & (-2)^{\bullet} \\ (-2)^{\bullet} & 0 \end{bmatrix} \;\nabla\; E_2$$

and $\|R\|_{\oplus} = 5 = \|A\|_{\oplus}$.
Ray x_2^{f} corresponds to

$$Q \;=\; \begin{bmatrix} 0 & -2 \\ \ominus(-2) & 0 \end{bmatrix} \quad \text{and} \quad R \;=\; \begin{bmatrix} 2 & \ominus 5 \\ \varepsilon & 3 \end{bmatrix} \;. \tag{54}$$

The other max-algebraic QR decompositions of A can be obtained from (53) or (54) by replacing q_1 by $\ominus q_1$ and $R_{1.}$ by $\ominus R_{1.}$, by replacing q_2 by $\ominus q_2$ and $R_{2.}$ by $\ominus R_{2.}$, or by a combination of these replacements.
Since x_1^{f} and x_3^{i} are cross-complementary, we can replace r_{22} in (53) by $\ominus\rho$ with $\rho \leqslant 3$ or by ε. Likewise, we can replace r_{22} in (54) by ρ with $\rho \leqslant 3$ or by ε. Hence, the max-algebraic QR rank of A is 1. $\square$

7 Conclusions and future research

We have used the fact that a system of multivariate polynomial equations in the max algebra can be transformed into an Extended Linear Complementarity Problem (ELCP) to derive a method to calculate all max-algebraic singular value decompositions of a matrix. The ELCP technique of this paper can also be used to calculate many other max-algebraic matrix decompositions (such as the max-algebraic QR decomposition, the max-algebraic LU decomposition and the max-algebraic eigenvalue decomposition of a symmetric matrix).

One of the main characteristics of the ELCP algorithm of [4] that we have used to solve a system of multivariate max-algebraic polynomial equations is that it finds all (finite) solutions. This provides us a geometrical insight in the set of all max-algebraic singular value decompositions of a given matrix. On the other hand this also leads to large computation times and storage space requirements even if the size of the matrix is small. Therefore, it might be interesting to develop (heuristic) algorithms that only find one max-algebraic singular value decomposition. The knowledge about the geometric structure of the set of all the max-algebraic singular value decompositions, as revealed in this paper, might be helpful to develop such algorithms. This also holds for the other max-algebraic matrix decompositions mentioned above.

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Set	$\mathcal{X}^{\mathrm{inf}}$														
Ray	x^{i}_1	x^{i}_2	x^{i}_3	x^{i}_4	x^{i}_5	x^{i}_6	x^{i}_7	x^{i}_8	x^{i}_9	x^{i}_{10}	x^{i}_{11}	x^{i}_{12}	x^{i}_{13}	x^{i}_{14}	x^{i}_{15}
σ_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
σ_2	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0
$u^{\boxplus}_{11}$	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0
$u^{\boxplus}_{12}$	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0
$u^{\boxplus}_{21}$	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
$u^{\boxplus}_{22}$	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$u^{\boxminus}_{11}$	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0
$u^{\boxminus}_{12}$	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
$u^{\boxminus}_{21}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1
$u^{\boxminus}_{22}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v^{\boxplus}_{11}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v^{\boxplus}_{12}$	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0
$v^{\boxplus}_{21}$	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0
$v^{\boxplus}_{22}$	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0
$v^{\boxminus}_{11}$	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0
$v^{\boxminus}_{12}$	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0
$v^{\boxminus}_{21}$	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0
$v^{\boxminus}_{22}$	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
t_{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_{21}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_{22}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 1: The rays and vertices for Example 6.1 .

Set	$\mathcal{X}^{\text{inf}}$		$\mathcal{X}^{\text{fin}}$							
Ray	x^{i}_{16}	x^{i}_{17}	x^{f}_{1}	x^{f}_{2}	x^{f}_{3}	x^{f}_{4}	x^{f}_{5}	x^{f}_{6}	x^{f}_{7}	x^{f}_{8}
σ_1	0	0	5	5	5	5	5	5	5	5
σ_2	0	0	0	0	0	0	0	0	0	0
$u^{\boxplus}_{11}$	0	0	-1000	-1000	0	0	-1000	-1000	0	0
$u^{\boxplus}_{12}$	0	0	-998	-2	-998	-2	-998	-2	-998	-2
$u^{\boxplus}_{21}$	0	0	-2	-2	-998	-998	-2	-2	-998	-998
$u^{\boxplus}_{22}$	0	0	-1000	0	-1000	0	-1000	0	-1000	0
$u^{\boxminus}_{11}$	0	0	0	0	-1000	-1000	0	0	-1000	-1000
$u^{\boxminus}_{12}$	0	0	-2	-998	-2	-998	-2	-998	-2	-998
$u^{\boxminus}_{21}$	0	0	-998	-998	-2	-2	-998	-998	-2	-2
$u^{\boxminus}_{22}$	0	-1	0	-1000	0	-1000	0	-1000	0	-1000
$v^{\boxplus}_{11}$	-1	0	-997	-997	-3	-3	-997	-997	-3	-3
$v^{\boxplus}_{12}$	0	0	0	0	0	0	-1000	-1000	-1000	-1000
$v^{\boxplus}_{21}$	0	0	0	0	-1000	-1000	0	0	-1000	-1000
$v^{\boxplus}_{22}$	0	0	-3	-3	-3	-3	-997	-997	-997	-997
$v^{\boxminus}_{11}$	0	0	-3	-3	-997	-997	-3	-3	-997	-997
$v^{\boxminus}_{12}$	0	0	-1000	-1000	-1000	-1000	0	0	0	0
$v^{\boxminus}_{21}$	0	0	-1000	-1000	0	0	-1000	-1000	0	0
$v^{\boxminus}_{22}$	0	0	-997	-997	-997	-997	-3	-3	-3	-3
t_{11}	0	0	2	2	2	2	2	2	2	2
t_{12}	0	0	5	5	5	5	5	5	5	5
t_{21}	0	0	0	0	0	0	0	0	0	0
t_{22}	0	0	3	3	3	3	3	3	3	3
p_{12}	0	0	-2	-2	-2	-2	-2	-2	-2	-2
q_{12}	0	0	-3	-3	-3	-3	-3	-3	-3	-3

Table 2: The rays and vertices for Example 6.1 (continued).

s	$\mathcal{X}_s^{\mathrm{inf}}$	$\mathcal{X}_s^{\mathrm{fin}}$
1	$\{x_1^{\mathrm{i}}, x_3^{\mathrm{i}}, x_4^{\mathrm{i}}, x_5^{\mathrm{i}}, x_6^{\mathrm{i}}, x_8^{\mathrm{i}}, x_{11}^{\mathrm{i}}, x_{15}^{\mathrm{i}}, x_{16}^{\mathrm{i}}\}$	$\{x_1^{\mathrm{f}}\}$
2	$\{x_3^{\mathrm{i}}, x_4^{\mathrm{i}}, x_5^{\mathrm{i}}, x_8^{\mathrm{i}}, x_9^{\mathrm{i}}, x_{11}^{\mathrm{i}}, x_{15}^{\mathrm{i}}, x_{16}^{\mathrm{i}}, x_{17}^{\mathrm{i}}\}$	$\{x_2^{\mathrm{f}}\}$
3	$\{x_1^{\mathrm{i}}, x_2^{\mathrm{i}}, x_3^{\mathrm{i}}, x_5^{\mathrm{i}}, x_6^{\mathrm{i}}, x_7^{\mathrm{i}}, x_{11}^{\mathrm{i}}, x_{12}^{\mathrm{i}}, x_{13}^{\mathrm{i}}\}$	$\{x_3^{\mathrm{f}}\}$
4	$\{x_2^{\mathrm{i}}, x_3^{\mathrm{i}}, x_5^{\mathrm{i}}, x_7^{\mathrm{i}}, x_9^{\mathrm{i}}, x_{11}^{\mathrm{i}}, x_{12}^{\mathrm{i}}, x_{13}^{\mathrm{i}}, x_{17}^{\mathrm{i}}\}$	$\{x_4^{\mathrm{f}}\}$
5	$\{x_1^{\mathrm{i}}, x_4^{\mathrm{i}}, x_6^{\mathrm{i}}, x_8^{\mathrm{i}}, x_{10}^{\mathrm{i}}, x_{11}^{\mathrm{i}}, x_{14}^{\mathrm{i}}, x_{15}^{\mathrm{i}}, x_{16}^{\mathrm{i}}\}$	$\{x_5^{\mathrm{f}}\}$
6	$\{x_4^{\mathrm{i}}, x_8^{\mathrm{i}}, x_9^{\mathrm{i}}, x_{10}^{\mathrm{i}}, x_{11}^{\mathrm{i}}, x_{14}^{\mathrm{i}}, x_{15}^{\mathrm{i}}, x_{16}^{\mathrm{i}}, x_{17}^{\mathrm{i}}\}$	$\{x_6^{\mathrm{f}}\}$
7	$\{x_1^{\mathrm{i}}, x_2^{\mathrm{i}}, x_6^{\mathrm{i}}, x_7^{\mathrm{i}}, x_{10}^{\mathrm{i}}, x_{11}^{\mathrm{i}}, x_{12}^{\mathrm{i}}, x_{13}^{\mathrm{i}}, x_{14}^{\mathrm{i}}\}$	$\{x_7^{\mathrm{f}}\}$
8	$\{x_2^{\mathrm{i}}, x_7^{\mathrm{i}}, x_9^{\mathrm{i}}, x_{10}^{\mathrm{i}}, x_{11}^{\mathrm{i}}, x_{12}^{\mathrm{i}}, x_{13}^{\mathrm{i}}, x_{14}^{\mathrm{i}}, x_{17}^{\mathrm{i}}\}$	$\{x_8^{\mathrm{f}}\}$

Table 3: The pairs of subsets for Example 6.1 .

Set	$\mathcal{X}^{\mathrm{inf}}$						$\mathcal{X}^{\mathrm{fin}}$	
Ray	x_1^{i}	x_2^{i}	x_3^{i}	x_4^{i}	x_5^{i}	x_6^{i}	x_1^{f}	x_2^{f}
$q_{11}^{\boxplus}$	0	0	0	0	0	0	0	0
$q_{12}^{\boxplus}$	0	0	0	0	0	0	-2	-2
$q_{21}^{\boxplus}$	0	0	0	-1	0	0	-998	-998
$q_{22}^{\boxplus}$	0	0	0	0	0	0	0	0
$q_{12}^{\boxminus}$	0	-1	0	0	0	0	-998	-998
$q_{21}^{\boxminus}$	0	0	0	0	0	0	-2	-2
$r_{11}^{\boxplus}$	0	0	0	0	0	0	2	2
$r_{12}^{\boxplus}$	0	0	0	0	0	-1	-1005	-1005
$r_{22}^{\boxplus}$	-1	0	0	0	0	0	-1003	3
$r_{11}^{\boxminus}$	0	0	0	0	-1	0	-1002	-1002
$r_{12}^{\boxminus}$	0	0	0	0	0	0	5	5
$r_{22}^{\boxminus}$	0	0	-1	0	0	0	3	-1003
t_{11}	0	0	0	0	0	0	2	2
t_{12}	0	0	0	0	0	0	5	5
t_{21}	0	0	0	0	0	0	0	0
t_{22}	0	0	0	0	0	0	3	3
p_{12}	0	0	0	0	0	0	-2	-2

Table 4: The rays and vertices for Example 6.2 .