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State Space Transformations and State Space Realization in the Max Algebra*

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Abstract

The topics of this paper are state space transformations and the (partial) state space problem in the max algebra, which is one of the modeling frameworks that can be used to model discrete event systems. We use the fact that a system of multivariate max-algebraic polynomial equations can be transformed into an Extended Linear Complementarity Problem to perform state space transformations and to find all equivalent fixed order state space realizations of a multiple input multiple output max-linear discrete event system starting from its impulse response matrices. We also give a geometrical description of the set of all equivalent state space realizations.

1. Introduction

Typical examples of discrete event systems (DES) are flexible manufacturing systems, telecommunication networks, parallel processing systems and railroad traffic networks. There exists a wide range of frameworks to model and to analyze DES: Petri nets, formal languages, computer simulation, perturbation analysis and so on. We concentrate on a subclass of DES that can be described with the max algebra [1, 2]. Although the description of these systems is nonlinear in linear algebra, the model becomes "linear" when we formulate it in the max algebra. In this paper we only consider systems that can be described by max-linear time-invariant state space models. One of the main advantages of an analytic max-algebraic model of a DES is that it allows us to derive some properties of the system (in particular the steady state behavior) fairly easily, whereas in some cases brute force simulation might require a large amount of computation time.

Although there are many analogies between max algebra and linear algebra (there exist max-algebraic equivalents of Cramer's rule, the Cayley-Hamilton theorem, eigenvectors and eigenvalues, . . .), there are also some major differences that prevent a straightforward translation of properties and algorithms from linear algebra to max algebra. As a result many problems that can be solved rather easily in linear system theory are not that easy to solve in max-algebraic system theory. In this paper we address two such problems: state space transformations and state space realizations of impulse responses.

This paper is organized as follows. First we give a short introduction to the max algebra and we briefly discuss the problem of solving a system of multivariate max-algebraic polynomial equations. Next we use the results to perform state space transformations for systems described by a max-linear state space model and to solve the (partial) state space realization problem for multiple input multiple output max-linear DES. We also give a geometric characterization of the set of all equivalent state space realizations. Finally we illustrate the procedure with an example, in which we also show how the different state space realizations are linked by state space transformations.

2. The max algebra

In this section we give a short introduction to the max algebra. A more complete overview of the max algebra can be found in [1, 3]. The basic max-algebraic operations are defined as follows:

$$a \oplus b = \max(a, b)$$
$$a \otimes b = a + b$$

where $a, b \in \mathbb{R} \cup \{-\infty\}$. The resulting structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the max algebra. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$ and $\varepsilon = -\infty$. Note that ε is the zero element for $\oplus$ in $\mathbb{R}_\varepsilon$.
Let $r \in \mathbb{R}$. The rth max-algebraic power of $a \in \mathbb{R}$ is denoted by $a^{\otimes^r}$ and corresponds to ra in conventional algebra. So $a^{\otimes^0} = 0$ and if $a \neq \varepsilon$ then $a^{\otimes^{-1}} = -a$ is the inverse element of a w.r.t. $\otimes$. There is no inverse element for ε since ε is absorbing for $\otimes$. If $r > 0$ then $\varepsilon^{\otimes^r} = \varepsilon$ and if $r \leqslant 0$ then $\varepsilon^{\otimes^r}$ is not defined.

The basic max-algebraic operations are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$; if $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then

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$(A\otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$. The m by n zero matrix in the max algebra is denoted by $\mathcal{E}_{m\times n}$: $(\mathcal{E}_{m\times n})_{ij} = \varepsilon$ for all i, j. A square matrix D is a max-algebraic diagonal matrix if $d_{ij} = \varepsilon$ for all $i \neq j$. A max-algebraic permutation matrix is square matrix P with exactly one 0 entry in each row and each column and where the other entries are equal to ε.

One of the major differences between linear algebra and max algebra is that in general there do not exist inverse elements w.r.t. $\oplus$. This also means that in general matrices are not invertible either: the only matrices that are invertible in the max algebra are matrices of the form $D\otimes P$ where D is a max-algebraic diagonal matrix with non-ε diagonal entries and P is a max-algebraic permutation matrix [3].

3. Systems of multivariate max-algebraic polynomial equations

In this section we address systems of multivariate max-algebraic polynomial equations, which can be seen as a generalized framework for many important max-algebraic problems such as matrix decompositions, state space transformations, state space realization of impulse responses, construction of matrices with a given characteristic polynomial, ... [5, 7]. Consider the following problem:

> Given integers $m_1, \ldots, m_p$ and real numbers a_{ki}, b_k and c_{kij} for $k = 1, \ldots, p$, $i = 1, \ldots, m_k$ and $j = 1, \ldots, n$, find $x \in \mathbb{R}^n$ such that
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} = b_k \quad \text{for } k = 1, \ldots, p\,. \quad (1)$$

We call (1) a system of multivariate max-algebraic polynomial equations. Note that the exponents c_{kij} can be negative or real.

In [5] we have shown that this problem can be reformulated as an Extended Linear Complementarity Problem (ELCP) [4]. This leads to an algorithm that yields the entire solution set of (1). In general the solution set consists of the union of faces of a polyhedron $\mathcal{P}$ and is defined by three sets of vectors $\mathcal{X}^{\mathrm{c}}$, $\mathcal{X}^{\mathrm{i}}$, $\mathcal{X}^{\mathrm{f}}$ and a set Λ. These sets can be characterized as follows:

- $\mathcal{X}^{\mathrm{c}}$ is the set of "central rays" of $\mathcal{P}$. It is a basis for the linear subspace associated with the largest affine subspace of $\mathcal{P}$. Let $\mathcal{P}_{\mathrm{red}}$ be the polyhedron obtained by subtracting this linear subspace from $\mathcal{P}$.
- $\mathcal{X}^{\mathrm{i}}$ is the set of the extreme rays or "vertices at infinity" of the polyhedron $\mathcal{P}_{\mathrm{red}}$.
- $\mathcal{X}^{\mathrm{f}}$ is the set of finite vertices of $\mathcal{P}_{\mathrm{red}}$.
- Λ is a set of ordered pairs $\left(\mathcal{X}_s^{\mathrm{i}}, \mathcal{X}_s^{\mathrm{f}}\right)$ with $\mathcal{X}_s^{\mathrm{i}} \subset \mathcal{X}^{\mathrm{i}}$, $\mathcal{X}_s^{\mathrm{f}} \neq \emptyset$ and $\mathcal{X}_s^{\mathrm{f}} \subset \mathcal{X}^{\mathrm{f}}$. Each pair determines a face $\mathcal{F}_s$ of $\mathcal{P}$ that belongs to the solution set: $\mathcal{X}_s^{\mathrm{i}}$ contains the extreme rays of $\mathcal{F}_s$, if any, and $\mathcal{X}_s^{\mathrm{f}}$ contains the finite vertices of $\mathcal{F}_s$.

The solution set of (1) is characterized by the following theorems:

Theorem 3.1 *When $\mathcal{X}^{\mathrm{c}}$, $\mathcal{X}^{\mathrm{i}}$, $\mathcal{X}^{\mathrm{f}}$ and Λ are given, then x is a (finite) solution of the system of multivariate max-algebraic polynomial equations if and only if there exists a pair $\left(\mathcal{X}_s^{\mathrm{i}}, \mathcal{X}_s^{\mathrm{f}}\right) \in \Lambda$ such that*

$$x = \sum_{x_k \in \mathcal{X}^{\mathrm{c}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\mathrm{i}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\mathrm{f}}} \mu_k x_k \qquad (2)$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k, \mu_k \geqslant 0$ and $\sum_k \mu_k = 1$.

Theorem 3.2 *In general the solution set of a system of multivariate max-algebraic polynomial equations consists of the union of faces of a polyhedron.*

In order to apply the ELCP technique we have to assume that x contains only finite components, but solutions with ε components can be retrieved by applying a limit argument in which we allow some of the λ_k's or κ_k's in (2) to become infinite, but in a controlled way, since we only allow infinite components that are equal to ε and since negative powers of ε are not defined. Solutions obtained in this way will correspond to points at infinity of the polyhedron $\mathcal{P}$. This technique will be demonstrated in Example 6.1. We refer the interested reader to [5] for more information on this topic.

4. State space transformations

Consider a DES that can be described by an nth order max-linear time invariant state space model:

$$x(k+1) = A \otimes x(k) \;\oplus\; B \otimes u(k) \qquad (3)$$

$$y(k) = C \otimes x(k) \qquad (4)$$

with $A \in \mathbb{R}_\varepsilon^{n\times n}$, $B \in \mathbb{R}_\varepsilon^{n\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times n}$. The vector x represents the state; u is the input vector and y is the output vector of the system.

If we apply a unit impulse: $e(k) = 0$ if $k = 0$, and $e(k) = \varepsilon$ if $k \neq 0$, to the ith input of the system and if $x(0) = \mathcal{E}_{n\times 1}$, we get $y(k) = C \otimes A^{\otimes^{k-1}} \otimes B_{.i}$ for $k = 1, 2, \ldots$ as the output of the system, where $B_{.i}$ is the ith column of B. We do this for all inputs $i = 1, \ldots, m$ and store the outputs in l by m matrices $G_k = C \otimes A^{\otimes^k} \otimes B$ for $k = 0, 1, \ldots$. The G_k's are called the *impulse response matrices* or *Markov parameters*.

Now we give some theorems on equivalent state space realizations. We shall again encounter these theorems when we look at the set of all equivalent state space realizations in Example 6.1. A part of this approach – the L-transformations – was hinted at, but not proven, in [10]. We have extended it such that the dimension of the state space vector can also change and we have added another type of transformations, the M-transformations.

Since we only consider impulse responses in this paper, we assume that always $x(0) = \mathcal{E}_{n\times 1}$. However, the theorems of this section can easily be extended to the case with arbitrary initial conditions. We shall characterize a model of the form (3) – (4) by the triplet (A, B, C) of system matrices.

Definition 4.1 (Equivalent state space realizations) *Two triplets $(A,\ B,\ C)$ and $(\tilde{A},\ \tilde{B},\ \tilde{C})$ are called equivalent if the corresponding state space models have the same impulse response, i.e. $\forall k \in \mathbb{N}$: $C \otimes A^{\otimes^k} \otimes B = \tilde{C} \otimes \tilde{A}^{\otimes^k} \otimes \tilde{B}$.*

Proposition 4.2 *Let $T \in \mathbb{R}_\varepsilon^{n\times n}$ be an invertible matrix. If the triplet $(A,\ B,\ C)$ is a realization of the impulse response of a max-linear time-invariant system then the triplet $(T\otimes A\otimes T^{\otimes^{-1}},\ T\otimes B,\ C\otimes T^{\otimes^{-1}})$ is an equivalent realization.*

The transformation of Proposition 4.2 is the max-algebraic equivalent of a similarity transformation. Since the class of invertible matrices in $\mathbb{R}_{\max}$ is rather small, max-algebraic similarity transformations have a limited scope. Furthermore, in contrast to linear systems, minimal state space realizations of a max-linear time-invariant system are not always related by a similarity transformation as will be shown in Example 6.1.

Another way to construct equivalent state space realizations is the following:

Theorem 4.3 (*L*-transformation) *Let the triplet $(A,\ B,\ C)$ be an nth order state space realization of the impulse response of a max-linear time-invariant system. Let $L \in \mathbb{R}_\varepsilon^{p\times n}$ be a common factor of A and C such that $A = \hat{A} \otimes L$ and $C = \hat{C} \otimes L$. Then the triplet $(\tilde{A},\ \tilde{B},\ \tilde{C})$ with $\tilde{A} = L \otimes \hat{A}$, $\tilde{B} = L \otimes B$ and $\tilde{C} = \hat{C}$ is an equivalent realization.*

Proof: For each integer $k \geqslant 0$ we have

$$\begin{aligned} C \otimes A^{\otimes^k} \otimes B &= \hat{C} \otimes L \otimes \left(\hat{A} \otimes L\right)^{\otimes^k} \otimes B \\ &= \hat{C} \otimes \left(L \otimes \hat{A}\right)^{\otimes^k} \otimes L \otimes B \\ &= \tilde{C} \otimes \tilde{A}^{\otimes^k} \otimes \tilde{B} \ . \qquad \Box \end{aligned}$$

We can also use the dual of this theorem:

Theorem 4.4 (*M*-transformation) *Let the triplet $(A,\ B,\ C)$ be an nth order state space realization of the impulse response of a max-linear time-invariant system. Let $M \in \mathbb{R}_\varepsilon^{n\times p}$ be a common factor of A and B such that $A = M \otimes \hat{A}$ and $B = M \otimes \hat{B}$. Then the triplet $(\tilde{A},\ \tilde{B},\ \tilde{C})$ with $\tilde{A} = \hat{A} \otimes M$, $\tilde{B} = \hat{B}$ and $\tilde{C} = C \otimes M$ is an equivalent realization.*

So to get another equivalent state space model of a system characterized by the triplet $(A,\ B,\ C)$ we have to find a decomposition

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \otimes L \quad \text{or} \quad \begin{bmatrix} A & B \end{bmatrix} = M \otimes \begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix}$$

with $L \in \mathbb{R}_\varepsilon^{p\times n}$ or $M \in \mathbb{R}_\varepsilon^{n\times p}$. These matrix decompositions can be considered as systems of multivariate max-algebraic equations with the entries of $\hat{A}$, $\hat{C}$ and L, or $\hat{A}$, $\hat{B}$ and M as unknowns and can thus be solved using the ELCP approach.

Even if $p = n$ the matrices L and M are in general not invertible, so in general L- and M-transformations are not similarity transformations. If $p = n$ then L or M will be square and then $(\tilde{A},\ \tilde{B},\ \tilde{C})$ will also be an nth order realization. If we take a rectangular L or M matrix, we can change the dimension of the state space vector and get a pth order state space model. It is obvious that p always has to be larger than or equal to the minimal system order otherwise we cannot find a common factor of A and C or of A and B.

Note that L- and M-transformations can be considered as inverse transformations: if we can construct $(A_1,\ B_1,\ C_1)$ from $(A_2,\ B_2,\ C_2)$ with an L-transformation, we can go back from $(A_2,\ B_2,\ C_2)$ to $(A_1,\ B_1,\ C_1)$ with an M-transformation with $M = L$ and with the same $\hat{A}$ as for the L-transformation. However, as will be shown in Example 6.1 L- and M-transformations in general do not yield the entire set of all equivalent state space realizations in one step.

5. State space realization of impulse responses

Suppose that A, B and C are unknown, and that we only know the Markov parameters. How can we construct A, B and C from the G_k's? This problem is called state space realization.

We assume that the max-linear system can be described by an rth order state space system (see e.g. [8, 9] for methods to determine lower and upper bounds for the minimal system order). For sake of simplicity we also assume that the entries of all G_k's are finite and that the system exhibits a periodic steady state behavior of the following kind:

$$\exists n_0, d \in \mathbb{N}, \exists c \in \mathbb{R} \text{ such that}$$
$$\forall n > n_0 \ : \ G_{n+d} = c^{\otimes^d} \otimes G_n \ . \qquad (5)$$

It can be shown [1, 8] that a sufficient condition for (5) to hold is that the system matrix A is irreducible, i.e. $(A \oplus A^{\otimes^2} \oplus \ldots \oplus A^{\otimes^n})_{ij} \neq \varepsilon$ for all i, j. This will e.g. be the case for DES without separate independent subsystems and with a cyclic behavior or with feedback from the output to the input like e.g. flexible production systems in which the parts are carried around on a limited number of pallets that circulate in the system. As will be shown in Example 6.1 the kind of steady state behavior mentioned above can also occur if the system matrix A is not irreducible.

If the assumptions stated above hold, then it can be proved [6] that the partial realization problem – in which we look for a realization that only fits the first, say, N Markov parameters – if solvable, always admits a solution with finite components (which is necessary in order to apply the ELCP approach). Once we have the set of all finite partial realizations, we can obtain the set of all realizations of the given impulse response by putting all components that are smaller than some threshold equal to ε or by applying a limit argument if necessary. This will be demonstrated in Example 6.1. It is obvious that we have to take N large enough. In practice it appears that we should at least include the transient behavior and the first cycles of the steady state behavior.

If the system does not exhibit the steady state behavior of (5) then in most cases the ELCP technique can still be applied if an analogous but more complicated threshold or limit procedure is used [6].

Now we try to find an rth order state space realization of the first N Markov parameters: we look for matrices $A \in \mathbb{R}_\varepsilon^{r\times r}$, $B \in \mathbb{R}_\varepsilon^{r\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times r}$ such that $C \otimes A^{\otimes^k} \otimes B = G_k$ for $k = 0, \ldots, N-1$.
If we work out these equations, we get for $k = 0$:

$$\bigoplus_{p=1}^{r} c_{ip} \otimes b_{pj} = (G_0)_{ij} \quad \text{for all } i, j \ .$$

For $k > 0$ we obtain

$$\bigoplus_{p=1}^{r} \bigoplus_{q=1}^{r} c_{ip} \otimes (A^{\otimes^k})_{pq} \otimes b_{qj} = (G_k)_{ij} \qquad (6)$$

for all i, j. Since

$$(A^{\otimes^k})_{pq} = \bigoplus_{i_1=1}^{r} \ldots \bigoplus_{i_{k-1}=1}^{r} a_{pi_1} \otimes \ldots \otimes a_{i_{k-1}q} \ ,$$

equation (6) can be rewritten as

$$\bigoplus_{p,q=1}^{r} \bigoplus_{s=1}^{r^{k-1}} c_{ip} \otimes \bigotimes_{u,v=1}^{r} a_{uv}{}^{\otimes^{\gamma_{kpqsuv}}} \otimes b_{qj} = G_k \qquad (7)$$

where γ_{kpqsuv} is the number of times that a_{uv} appears in the sth term of $(A^{\otimes^k})_{pq}$. If a_{uv} does not appear in that term, we take $\gamma_{kpqsuv} = 0$, since $a^{\otimes^0} = 0 \cdot a = 0$, the identity element for $\otimes$. If we use the fact that $\forall x, y \in \mathbb{R}_\varepsilon : x \oplus x = x$ and $x \otimes y \leqslant x \otimes x \oplus y \otimes y$, we can remove many redundant terms. There are then w_{kij} terms in (7) with $w_{kij} \leqslant r^{k+1}$.
If we put all unknowns in one large vector x of length $r(r+m+l)$, we get a system of multivariate max-algebraic polynomial equations of the following form:

$$\bigoplus_{p=1}^{r} \bigotimes_{q=1}^{r(r+m+l)} x_q{}^{\otimes^{\delta_{0ijpq}}} = (G_0)_{ij} \qquad (8)$$

$$\bigoplus_{p=1}^{w_{kij}} \bigotimes_{q=1}^{r(r+m+l)} x_q{}^{\otimes^{\delta_{kijpq}}} = (G_k)_{ij} \qquad (9)$$

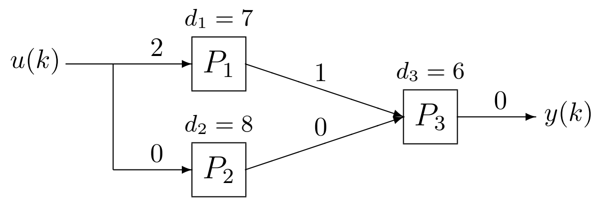
for $i = 1, \ldots, l$, $j = 1, \ldots, m$ and $k = 1, \ldots, N-1$.
If we find a solution x of (8) – (9), we extract the entries of the system matrices A, B and C from x. This results in a partial realization of the given impulse response matrices. If we do not get any solutions, this means that r is less than the minimal system order, i.e. it is not possible to describe the given impulse response with an rth order state space model.

Now we can characterize the set of all (partial) state space realizations of a given impulse response:

Theorem 5.1 *Let $r \in \mathbb{N}$. In general the set of all rth order (partial) state space realizations of the impulse response of a max-linear time-invariant DES consists of the union of faces of a polyhedron in the x-space, where x is the vector obtained by putting the components of the system matrices in one large vector.*

6. Example

Example 6.1 Consider the following production system:



This system consists of 3 processing units P_1, P_2 and P_3. Raw material is fed to P_1 and P_2, processed and sent to P_3 where assembly takes place. The processing times for P_1, P_2 and P_3 are $d_1 = 7$, $d_2 = 8$ and $d_3 = 6$ time units respectively. We assume that the raw material needs 2 time units to get from the input source to P_1 and that it takes 1 time unit for the finished product of processing unit P_1 to reach P_3. The other transportation times are assumed to be negligible. At the input of the system and between the processing units there are buffers with a capacity that is large enough to ensure that no buffer overflow will occur. Now we define:

- $u(k)$: time instant at which raw material is fed to the system for the $k+1$st time,
- $x_i(k)$: time instant at which the ith processing unit starts working for the kth time,
- $y(k)$: time instant at which the kth finished product leaves the system.

A processing unit can only start working on a new product if it has finished processing the previous one. If we assume that each processing unit starts working as soon as all parts are available, we get the following evolution equations for the system:

$$\begin{aligned}
x_1(k+1) &= \max(\, x_1(k)+7,\, u(k)+2\,) \\
x_2(k+1) &= \max(\, x_2(k)+8,\, u(k)\,) \\
x_3(k+1) &= \max(\, x_1(k+1)+7+1,\, x_2(k+1)+8, \\
&\qquad x_3(k)+6\,) \\
&= \max(\, x_1(k)+15,\, x_2(k)+16, \\
&\qquad x_3(k)+6,\, u(k)+10\,) \\
y(k) &= x_3(k)+6 \ ,
\end{aligned}$$

or in max-algebraic matrix notation:

$$\begin{aligned}
x(k+1) &= \begin{bmatrix} 7 & \varepsilon & \varepsilon \\ \varepsilon & 8 & \varepsilon \\ 15 & 16 & 6 \end{bmatrix} \otimes x(k) \,\oplus\, \begin{bmatrix} 2 \\ 0 \\ 10 \end{bmatrix} \otimes u(k) \\
y(k) &= \begin{bmatrix} \varepsilon & \varepsilon & 6 \end{bmatrix} \otimes x(k)
\end{aligned}$$

where $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix}^T$. Now we construct state space realizations of this system starting from its impulse response, which is given by

$$\{G_k\}_{k=0}^{\infty} = 16, 23, 30, 38, 46, 54, 62, 70, 78, 86, \ldots \ .$$

Note that this system does exhibit a periodic steady state behavior of the form (5) although the system matrix A of this system is not irreducible. We try to find a 2nd order state space realization of this impulse response – this corresponds to the lower bound for the minimal system order given by [8, p. 173] or [9, Theorem 2.2.4]. Let us take $N = 5$. Using the ELCP algorithm of [4] we find the rays and vertices of Table 1 and the pairs of subsets of Table 2. If we take $N > 5$, we get the same result, but if we take $N < 5$, some combinations of the rays lead to a partial realization of the given impulse response: they only fit the first N Markov parameters.

Any arbitrary finite 2nd order state space realization can now be expressed as

$$\begin{bmatrix} a_{11} & a_{12} & a_{21} & a_{22} & b_1 & b_2 & c_1 & c_2 \end{bmatrix}^T = \lambda_1 x_1^{\mathrm{c}} + \lambda_2 x_2^{\mathrm{c}} + \kappa_1 x_{i_1}^{\mathrm{i}} + \kappa_2 x_{i_2}^{\mathrm{i}} + x_{j_1}^{\mathrm{f}} \qquad (10)$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\kappa_1, \kappa_2 \geqslant 0$ and $x_{i_1}^{\mathrm{i}}, x_{i_2}^{\mathrm{i}} \in \mathcal{X}_s^{\mathrm{i}}$, $x_{j_1}^{\mathrm{f}} \in \mathcal{X}_s^{\mathrm{f}}$ for some $s \in \{1, \ldots, 8\}$. Hence the set of all 2nd order state space realizations of the given impulse response is a union of 8 faces of a polyhedron.

Now we show how we can obtain a state space realization with ε components by allowing some coefficients in (10) to become infinite. The combination $-\eta x_2^{\mathrm{c}} + \eta x_1^{\mathrm{i}} + \eta x_4^{\mathrm{i}} + x_1^{\mathrm{f}}$ with $\eta \geqslant 0$ of the central rays and the vertices of the pair $(\mathcal{X}_2^{\mathrm{i}}, \mathcal{X}_2^{\mathrm{f}})$ corresponds to

$$A = \begin{bmatrix} 8 & 14-\eta \\ -\eta & 7 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ -6 \end{bmatrix},\ C = \begin{bmatrix} 14 & 22 \end{bmatrix} \ .$$

If we take the limit for η going to ∞, we get

$$A = \begin{bmatrix} 8 & \varepsilon \\ \varepsilon & 7 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ -6 \end{bmatrix},\ C = \begin{bmatrix} 14 & 22 \end{bmatrix} \ ,$$

which also is a realization of the given impulse response.

Finally, we give an interpretation of the solution set of the state space realization problem in terms of the theorems on state space transformations of Section 4. Ray x_1^{c} corresponds to a similarity transformation with $T_1 = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$. Ray x_2^{c} corresponds to a similarity transformation with $T_2 = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$. Ray x_2^{f} can be obtained from x_1^{f} by a similarity transformation with $T_3 = \begin{bmatrix} \varepsilon & 6 \\ -8 & \varepsilon \end{bmatrix}$. Since $\{T_1, T_2, T_3\}$ can be considered as a basis for the set of 2 by 2 invertible matrices in the max algebra, the set

$$\mathcal{S} = \{\, x \,|\, x = \lambda_1 x_1^{\mathrm{c}} + \lambda_2 x_2^{\mathrm{c}} + x_1^{\mathrm{f}} \text{ or } x = \lambda_1 x_1^{\mathrm{c}} + \lambda_2 x_2^{\mathrm{c}} + x_2^{\mathrm{f}} \text{ with } \lambda_1, \lambda_2 \in \mathbb{R} \}$$

corresponds to an entire class of 2nd order state space realizations that are linked by a similarity transformation. But in this way we cannot construct the complete set of all possible 2nd order realizations since e.g. $x = x_1^{\mathrm{f}} + x_1^{\mathrm{i}}$ does not belong to $\mathcal{S}$. However, $x_1^{\mathrm{f}} + x_1^{\mathrm{i}}$ can be obtained from x_1^{f} by an M-transformation with $M = \begin{bmatrix} 0 & 5 \\ -8 & -1 \end{bmatrix}$ and $\hat{A} = \begin{bmatrix} 8 & 14 \\ 0 & 8 \end{bmatrix}$. The realization $x_1^{\mathrm{f}} + x_2^{\mathrm{i}}$ can be obtained from $x_1^{\mathrm{f}} + x_3^{\mathrm{i}}$ with an M-transformation and $x_1^{\mathrm{f}} + x_3^{\mathrm{i}}$ can be obtained from $x_1^{\mathrm{f}} + x_1^{\mathrm{i}}$ with an L-transformation, but it is impossible to transform $x_1^{\mathrm{f}} + x_1^{\mathrm{i}}$ into $x_1^{\mathrm{f}} + x_2^{\mathrm{i}}$ with an L- or an M-transformation. So starting from an arbitrary realization, we cannot obtain the set of all equivalent 2nd order state space realizations in one step by applying L- or M-transformations.
It is also impossible to find an L- or M-transformation that transforms the original 3rd order state space description into a 2nd order model.

7. Conclusions and future research

We have used the fact that a system of multivariate polynomial equations in the max algebra can be transformed into an Extended Linear Complementarity Problem (ELCP) to perform state space transformations of max-linear state space models and to find all fixed order partial state space realizations of a max-linear multiple input multiple output discrete event system given its Markov parameters and we have illustrated these procedures with an example.

One of the main characteristics of the ELCP algorithm of [4] that was used to solve a system of multivariate max-algebraic polynomial equations is that it finds all finite solutions. For the state realization problem this provides us a geometrical insight in the set of all state space realizations of a given impulse response. On the other hand this also leads to large computation times and storage space requirements if the number of variables and equations is large. Therefore it might be interesting to develop (heuristic) algorithms that only find one solution. Furthermore, it is still an open question how to determine the minimal number N such that all partial realizations of the first N Markov parameters are also realizations of the entire impulse response.

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Ray	a_{11}	a_{12}	a_{21}	a_{22}	b_1	b_2	c_1	c_2
x_1^{c}	0	0	0	0	1	1	-1	-1
x_2^{c}	0	-1	1	0	0	1	0	-1
x_1^{i}	0	-1	0	0	0	1	0	-1
x_2^{i}	0	0	0	0	0	1	-1	-1
x_3^{i}	0	0	0	0	0	0	-1	0
x_4^{i}	0	-1	0	0	0	0	0	0
x_5^{i}	0	0	0	0	0	0	0	-1
x_6^{i}	0	0	0	0	0	-1	0	0
x_1^{f}	8	14	0	7	0	-6	14	22
x_2^{f}	7	14	0	8	0	-8	16	22

Table 1: The rays and vertices for Example 6.1 .

s	$\mathcal{X}_s^{\mathrm{i}}$	$\mathcal{X}_s^{\mathrm{f}}$	s	$\mathcal{X}_s^{\mathrm{i}}$	$\mathcal{X}_s^{\mathrm{f}}$
1	$\{x_1^{\mathrm{i}}, x_2^{\mathrm{i}}\}$	$\{x_1^{\mathrm{f}}\}$	5	$\{x_2^{\mathrm{i}}, x_3^{\mathrm{i}}\}$	$\{x_1^{\mathrm{f}}\}$
2	$\{x_1^{\mathrm{i}}, x_4^{\mathrm{i}}\}$	$\{x_1^{\mathrm{f}}\}$	6	$\{x_3^{\mathrm{i}}, x_4^{\mathrm{i}}\}$	$\{x_1^{\mathrm{f}}\}$
3	$\{x_1^{\mathrm{i}}, x_4^{\mathrm{i}}\}$	$\{x_2^{\mathrm{f}}\}$	7	$\{x_4^{\mathrm{i}}, x_6^{\mathrm{i}}\}$	$\{x_2^{\mathrm{f}}\}$
4	$\{x_1^{\mathrm{i}}, x_5^{\mathrm{i}}\}$	$\{x_2^{\mathrm{f}}\}$	8	$\{x_5^{\mathrm{i}}, x_6^{\mathrm{i}}\}$	$\{x_2^{\mathrm{f}}\}$

Table 2: The pairs of subsets for Example 6.1 .