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THE QR DECOMPOSITION AND THE SINGULAR VALUE DECOMPOSITION IN THE SYMMETRIZED MAX-PLUS ALGEBRA

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Abstract. In this paper we discuss matrix decompositions in the symmetrized max-plus algebra. The max-plus algebra has maximization and addition as basic operations. In contrast to linear algebra many fundamental problems in the max-plus algebra still have to be solved. In this paper we discuss max-algebraic analogues of some basic matrix decompositions from linear algebra. We show that we can use algorithms from linear algebra to prove the existence of max-algebraic analogues of the QR decomposition, the singular value decomposition, the Hessenberg decomposition, the LU decomposition and so on.

Key words. max-plus algebra, matrix decompositions, QR decomposition, singular value decomposition

AMS subject classifications. 15A23, 16Y99

1. Introduction. In recent years both industry and the academic world have become more and more interested in techniques to model, analyze and control complex systems such as flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems, logistic systems and so on. These systems are typical examples of *discrete event systems* (DESs), the subject of an emerging discipline in system and control theory. The class of the DESs essentially contains man-made systems that consist of a finite number of resources (e.g., machines, communications channels or processors) that are shared by several users (e.g., product types, information packets or jobs) all of which contribute to the achievement of some common goal (e.g., the assembly of products, the end-to-end transmission of a set of information packets or a parallel computation). Although in general DESs lead to a non-linear description in conventional algebra, there exists a subclass of DESs for which this model becomes “linear” when we formulate it in the max-plus algebra [1, 5]. DESs that belong to this subclass are called *max-linear DESs*.

The basic operations of the max-plus algebra are maximization and addition. There exists a remarkable analogy between the basic operations of the max-plus algebra on the one hand, and the basic operations of conventional algebra (addition and multiplication) on the other hand. As a consequence many concepts and properties of conventional algebra (such as the Cayley–Hamilton theorem, eigenvectors, eigenvalues and Cramer’s rule) also have a max-algebraic analogue. This analogy also allows us to translate many concepts, properties and techniques from conventional linear system theory to system theory for max-linear DESs. However, there are also some major differences that prevent a straightforward translation of properties, concepts and algorithms from conventional linear algebra and linear system theory to max-plus algebra and max-algebraic system theory for DESs.

Compared to linear algebra and linear system theory, the max-plus algebra and the max-algebraic system theory for DESs is at present far from fully developed, and much research on this topic is still needed in order to get a complete system theory. The main goal of this paper is to fill one of the gaps in the theory of the max-plus

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algebra by showing that there exist max-algebraic analogues of many fundamental matrix decompositions from linear algebra such as the QR decomposition and the singular value decomposition. These matrix decompositions are important tools in many linear algebra algorithms (see [14] and the references cited therein) and in many contemporary algorithms for the identification of linear systems (see [21, 22, 33, 34, 35] and the references cited therein).

In [30], Olsder and Roos have used asymptotic equivalences to show that every matrix has at least one max-algebraic eigenvalue and to prove a max-algebraic version of Cramer's rule and of the Cayley–Hamilton theorem. We shall use an extended and formalized version of their technique to prove the existence of the QR decomposition and the singular value decomposition in the symmetrized max-plus algebra. In our existence proof we shall use algorithms from linear algebra. This proof technique can easily be adapted to prove the existence of max-algebraic analogues of many other matrix decompositions from linear algebra such as the Hessenberg decomposition, the LU decomposition, the eigenvalue decomposition, the Schur decomposition, and so on.

This paper is organized as follows. After introducing some concepts and definitions in §2, we give a short introduction to the max-plus algebra and the symmetrized max-plus algebra in §3. Next we establish a link between a ring of real functions (with conventional addition and multiplication as basic operations) and the symmetrized max-plus algebra. In §5 we use this link to define the QR decomposition and the singular value decomposition of a matrix in the symmetrized max-plus algebra and to prove the existence of these decompositions. We conclude with an example.

2. Notations and definitions. In this section we give some definitions that will be needed in the next sections.

The set of all reals except for 0 is represented by $\mathbb{R}_0$ ($\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$). The set of the nonnegative real numbers is denoted by $\mathbb{R}^+$, and the set of the nonpositive real numbers is denoted by $\mathbb{R}^-$. We have $\mathbb{R}_0^+ = \mathbb{R}^+ \setminus \{0\}$.

We shall use "vector" as a synonym for "n-tuple". Furthermore, all vectors are assumed to be column vectors. If a is a vector, then a_i is the ith component of a. If A is a matrix, then a_{ij} or $(A)_{ij}$ is the entry on the ith row and the jth column. The n by n identity matrix is denoted by I_n and the m by n zero matrix is denoted by $O_{m\times n}$.

The matrix $A \in \mathbb{R}^{n\times n}$ is called orthogonal if $A^T A = I_n$.

The Frobenius norm of the matrix $A \in \mathbb{R}^{m\times n}$ is represented by

$$\|A\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} \quad .$$

The 2-norm of the vector a is defined by $\|a\|_2 = \sqrt{a^T a}$ and the 2-norm of the matrix A is defined by

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 \quad .$$

If $A \in \mathbb{R}^{m\times n}$, then there exist an orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ and an upper triangular matrix $R \in \mathbb{R}^{m\times n}$ such that $A = QR$. We say that QR is a *QR decomposition* of A.

Let $A \in \mathbb{R}^{m\times n}$ and let $r = \min(m, n)$. Then there exist a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ and two orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that

$$A = U\,\Sigma\,V^T \tag{1}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$ where $\sigma_i = (\Sigma)_{ii}$ for $i = 1, 2, \ldots, r$. Factorization (1) is called a *singular value decomposition* (SVD) of A. The diagonal entries of Σ are the singular values of A. We have $\sigma_1 = \|A\|_2$. The columns of U are the left singular vectors of A and the columns of V are the right singular vectors of A. For more information on the QR decomposition and the SVD the interested reader is referred to [14, 18].

We use f, $f(\cdot)$ or $x \mapsto f(x)$ to represent a function. The domain of definition of the function f is denoted by $\mathrm{dom}\, f$ and the value of f at $x \in \mathrm{dom}\, f$ is denoted by $f(x)$.

Definition 2.1 (Analytic function). *Let f be a real function and let $\alpha \in \mathbb{R}$ be an interior point of* $\mathrm{dom}\, f$*. Then f is analytic in α if the Taylor series of f with center α exists and if there is a neighborhood of α where this Taylor series converges to f.*
A real function f is analytic in an interval $[\alpha, \beta] \subseteq \mathrm{dom}\, f$ if it is analytic in every point of that interval.
A real matrix-valued function $\tilde{F}$ is analytic in $[\alpha, \beta] \subseteq \mathrm{dom}\, \tilde{F}$ if all its entries are analytic in $[\alpha, \beta]$.

Definition 2.2 (Asymptotic equivalence in the neighborhood of ∞). *Let f and g be real functions such that ∞ is an accumulation point of* $\mathrm{dom}\, f$ *and* $\mathrm{dom}\, g$*.*
If there is no real number K such that g is identically zero in $[K, \infty)$ then we say that f is asymptotically equivalent to g in the neighborhood of ∞, denoted by $f(x) \sim g(x)$, $x \to \infty$, if $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$.
If there exists a real number L such that both f and g are identically zero in $[L, \infty)$ then we also say that $f(x) \sim g(x)$, $x \to \infty$.
Let $\tilde{F}$ and $\tilde{G}$ be real m by n matrix-valued functions such that ∞ is an accumulation point of $\mathrm{dom}\, \tilde{F}$ *and* $\mathrm{dom}\, \tilde{G}$*. Then $\tilde{F}(x) \sim \tilde{G}(x)$, $x \to \infty$ if $\tilde{f}_{ij}(x) \sim \tilde{g}_{ij}(x)$, $x \to \infty$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.*
The main difference with the conventional definition of asymptotic equivalence is that Definition 2.2 also allows us to say that a function is asymptotically equivalent to 0 in the neighborhood of ∞: $f(x) \sim 0$, $x \to \infty$ if there exists a real number L such that $f(x) = 0$ for all $x \geq L$.

3. The max-plus algebra and the symmetrized max-plus algebra. In this section we give a short introduction to the max-plus algebra and the symmetrized max-plus algebra. A complete overview of the max-plus algebra can be found in [1, 5, 12].

3.1. The max-plus algebra. The basic max-algebraic operations are defined as follows:

$$x \oplus y = \max(x, y) \tag{2}$$
$$x \otimes y = x + y \tag{3}$$

for $x, y \in \mathbb{R} \cup \{-\infty\}$ with, by definition, $\max(x, -\infty) = x$ and $x + (-\infty) = -\infty$ for all $x \in \mathbb{R} \cup \{-\infty\}$. The reason for using the symbols $\oplus$ and $\otimes$ to represent maximization and addition is that there is a remarkable analogy between $\oplus$ and addition, and between $\otimes$ and multiplication: many concepts and properties from conventional

linear algebra (such as the Cayley–Hamilton theorem, eigenvectors, eigenvalues and Cramer's rule) can be translated to the (symmetrized) max-plus algebra by replacing $+$ by $\oplus$ and $\times$ by $\otimes$ (see also §4). Therefore, we also call $\oplus$ the max-algebraic addition. Likewise, we call $\otimes$ the max-algebraic multiplication. The resulting algebraic structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the *max-plus algebra*. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$. The zero element for $\oplus$ in $\mathbb{R}_\varepsilon$ is represented by $\varepsilon \stackrel{\text{def}}{=} -\infty$. So $x \oplus \varepsilon = x = \varepsilon \oplus x$ for all $x \in \mathbb{R}_\varepsilon$.
Let $r \in \mathbb{R}$. The rth max-algebraic power of $x \in \mathbb{R}$ is denoted by $x^{\otimes^r}$ and corresponds to rx in conventional algebra. If $x \in \mathbb{R}$ then $x^{\otimes^0} = 0$ and the inverse element of x w.r.t. $\otimes$ is $x^{\otimes^{-1}} = -x$. There is no inverse element for ε since ε is absorbing for $\otimes$. If $r > 0$ then $\varepsilon^{\otimes^r} = \varepsilon$. If $r \leq 0$ then $\varepsilon^{\otimes^r}$ is not defined.
The rules for the order of evaluation of the max-algebraic operators are similar to those of conventional algebra. So max-algebraic power has the highest priority, and max-algebraic multiplication has a higher priority than max-algebraic addition.
Consider the finite sequence $a_1, a_2, \ldots, a_n$ with $a_i \in \mathbb{R}_\varepsilon$ for all i. We define

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus a_2 \oplus \ldots \oplus a_n \ .$$

The matrix E_n is the n by n max-algebraic identity matrix:

$$\begin{aligned} (E_n)_{ii} &= 0 && \text{for } i = 1, 2, \ldots, n\,, \\ (E_n)_{ij} &= \varepsilon && \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n \text{ with } i \neq j\,. \end{aligned}$$

The m by n max-algebraic zero matrix is represented by $\mathcal{E}_{m \times n}$: we have $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j.
The off-diagonal entries of a max-algebraic diagonal matrix $D \in \mathbb{R}_\varepsilon^{m \times n}$ are equal to ε: $d_{ij} = \varepsilon$ for all i, j with $i \neq j$. A matrix $R \in \mathbb{R}_\varepsilon^{m \times n}$ is a max-algebraic upper triangular matrix if $r_{ij} = \varepsilon$ for all i, j with $i > j$. If we permute the rows or the columns of the max-algebraic identity matrix, we obtain a max-algebraic permutation matrix.
The operations $\oplus$ and $\otimes$ are extended to matrices as follows. If $\alpha \in \mathbb{R}_\varepsilon$ and if $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then

$$(\alpha \otimes A)_{ij} = \alpha \otimes a_{ij} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

and

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\,.$$

If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\,.$$

3.2. The symmetrized max-plus algebra. One of the major differences between conventional algebra and the max-plus algebra is that there exist no inverse elements w.r.t. $\oplus$ in $\mathbb{R}_\varepsilon$: if $x \in \mathbb{R}_\varepsilon$ then there does not exist an element $y_x \in \mathbb{R}_\varepsilon$ such that $x \oplus y_x = \varepsilon = y_x \oplus x$, except when x is equal to ε. So $(\mathbb{R}_\varepsilon, \oplus)$ is not a group. Therefore, we now introduce $\mathbb{S}_{\max}$ [1, 12, 25], which is a kind of symmetrization of the max-plus algebra. This can be compared with the extension of $(\mathbb{N}, +, \times)$ to $(\mathbb{Z}, +, \times)$.

In §4 we shall show that $\mathbb{R}_{\max}$ corresponds to a set of nonnegative real functions with addition and multiplication as basic operations and that $\mathbb{S}_{\max}$ corresponds to a set of real functions with addition and multiplication as basic operations. Since the $\oplus$ operation is idempotent, we cannot use the conventional symmetrization technique since every idempotent group reduces to a trivial group [1, 25]. Nevertheless, it is possible to adapt the method of the construction of $\mathbb{Z}$ from $\mathbb{N}$ to obtain "balancing" elements rather than inverse elements.

We shall restrict ourselves to a short introduction to the most important features of $\mathbb{S}_{\max}$. This introduction is based on [1, 12, 25]. First we introduce the "algebra of pairs". We consider the set of ordered pairs $\mathcal{P}_\varepsilon \stackrel{\text{def}}{=} \mathbb{R}_\varepsilon \times \mathbb{R}_\varepsilon$ with operations $\oplus$ and $\otimes$ that are defined as follows:

$$(x,y) \oplus (w,z) = (x \oplus w,\ y \oplus z) \tag{4}$$

$$(x,y) \otimes (w,z) = (x \otimes w \oplus y \otimes z,\ x \otimes z \oplus y \otimes w) \tag{5}$$

for (x,y), $(w,z) \in \mathcal{P}_\varepsilon$ and where the operations $\oplus$ and $\otimes$ on the right-hand side correspond to maximization and addition as defined in (2) and (3). The reason for also using $\oplus$ and $\otimes$ on the left-hand side is that these operations correspond to $\oplus$ and $\otimes$ as defined in $\mathbb{R}_{\max}$ as we shall see later on. It is easy to verify that in $\mathcal{P}_\varepsilon$ the $\oplus$ operation is associative, commutative and idempotent, and its zero element is $(\varepsilon,\varepsilon)$; that the $\otimes$ operation is associative, commutative and distributive w.r.t. $\oplus$; that the identity element of $\otimes$ is $(0,\varepsilon)$; and that the zero element $(\varepsilon,\varepsilon)$ is absorbing for $\otimes$. We call the structure $(\mathcal{P}_\varepsilon,\oplus,\otimes)$ the *algebra of pairs*.
If $u=(x,y) \in \mathcal{P}_\varepsilon$ then we define the max-absolute value of u as $|u|_\oplus = x \oplus y$ and we introduce two unary operators $\ominus$ (the max-algebraic minus operator) and $(\,\cdot\,)^\bullet$ (the balance operator) such that $\ominus u = (y,x)$ and $u^\bullet = u \oplus (\ominus u) = (|u|_\oplus\,, |u|_\oplus)$. We have

$$u^\bullet = (\ominus u)^\bullet = (u^\bullet)^\bullet \tag{6}$$

$$u \otimes v^\bullet = (u \otimes v)^\bullet \tag{7}$$

$$\ominus(\ominus u) = u \tag{8}$$

$$\ominus(u \oplus v) = (\ominus u) \oplus (\ominus v) \tag{9}$$

$$\ominus(u \otimes v) = (\ominus u) \otimes v \tag{10}$$

for all $u,v \in \mathcal{P}_\varepsilon$. The last three properties allow us to write $u \ominus v$ instead of $u \oplus (\ominus v)$. Since the properties (8) – (10) resemble properties of the $-$ operator in conventional algebra, we could say that the $\ominus$ operator of the algebra of pairs can be considered as the analogue of the $-$ operator of conventional algebra (see also §4). As for the order of evaluation of the max-algebraic operators, the max-algebraic minus operator has the same, i.e., the lowest, priority as the max-algebraic addition operator.
In conventional algebra we have $x - x = 0$ for all $x \in \mathbb{R}$, but in the algebra of pairs we have $u \ominus u = u^\bullet \neq (\varepsilon,\varepsilon)$ for all $u \in \mathcal{P}_\varepsilon$ unless u is equal to $(\varepsilon,\varepsilon)$, the zero element for $\oplus$ in $\mathcal{P}_\varepsilon$. Therefore, we introduce a new relation:

Definition 3.1 (Balance relation). *Consider $u=(x,y)$, $v=(w,z) \in \mathcal{P}_\varepsilon$. We say that u balances v, denoted by $u \,\nabla\, v$, if $x \oplus z = y \oplus w$.*
We have $u \ominus u = u^\bullet = (|u|_\oplus\,, |u|_\oplus) \;\nabla\; (\varepsilon,\varepsilon)$ for all $u \in \mathcal{P}_\varepsilon$. The balance relation is reflexive and symmetric but it is not transitive since, e.g., $(2,1) \,\nabla\, (2,2)$ and $(2,2) \,\nabla\, (1,2)$ but $(2,1) \,\not\nabla\, (1,2)$. Hence, the balance relation is not an equivalence relation and we cannot use it to define the quotient set of $\mathcal{P}_\varepsilon$ by ∇ (as opposed to conventional algebra where $(\mathbb{N} \times \mathbb{N})/=$ yields $\mathbb{Z}$). Therefore, we introduce another relation that is closely

related to the balance relation and that is defined as follows: if (x,y), $(w,z) \in \mathcal{P}_\varepsilon$ then

$$(x,y)\mathcal{B}(w,z) \quad \text{if} \quad \begin{cases} (x,y) \,\nabla\, (w,z) & \text{if } x \neq y \text{ and } w \neq z\,, \\ (x,y) = (w,z) & \text{otherwise.} \end{cases}$$

Note that if $u \in \mathcal{P}_\varepsilon$ then we have $u \ominus u \not\mathcal{B}\, (\varepsilon,\varepsilon)$ unless u is equal to $(\varepsilon,\varepsilon)$. It is easy to verify that $\mathcal{B}$ is an equivalence relation that is compatible with $\oplus$ and $\otimes$, with the balance relation ∇ and with the $\ominus$, $|\cdot|_\oplus$ and $(\cdot)^\bullet$ operators. We can distinguish between three kinds of equivalence classes generated by $\mathcal{B}$:

1. $\overline{(w,-\infty)} = \{\,(w,x) \in \mathcal{P}_\varepsilon \,|\, x < w\,\}$, called max-positive;
2. $\overline{(-\infty,w)} = \{\,(x,w) \in \mathcal{P}_\varepsilon \,|\, x < w\,\}$, called max-negative;
3. $\overline{(w,w)} = \{\,(w,w) \in \mathcal{P}_\varepsilon\,\}$, called balanced.

The class $\overline{(\varepsilon,\varepsilon)}$ is called the max-zero class.
Now we define the quotient set $\mathbb{S} = \mathcal{P}_\varepsilon/\mathcal{B}$. The algebraic structure $\mathbb{S}_{\max} = (\mathbb{S},\oplus,\otimes)$ is called the *symmetrized max-plus algebra*. By associating $\overline{(w,-\infty)}$ with $w \in \mathbb{R}_\varepsilon$, we can identify $\mathbb{R}_\varepsilon$ with the set of max-positive or max-zero classes denoted by $\mathbb{S}^\oplus$. The set of max-negative or max-zero classes will be denoted by $\mathbb{S}^\ominus$, and the set of balanced classes will be represented by $\mathbb{S}^\bullet$. This results in the following decomposition: $\mathbb{S} = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$. Note that the max-zero class $\overline{(\varepsilon,\varepsilon)}$ corresponds to ε. The max-positive elements, the max-negative elements and ε are called signed. Define $\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$. Note that $\mathbb{S}^\oplus \cap \mathbb{S}^\ominus \cap \mathbb{S}^\bullet = \left\{\overline{(\varepsilon,\varepsilon)}\right\}$ and $\varepsilon = \ominus\varepsilon = \varepsilon^\bullet$.
These notations allow us to write, e.g., $2 \oplus (\ominus 4)$ instead of $\overline{(2,-\infty)} \oplus \overline{(-\infty,4)}$. Since $\overline{(2,-\infty)} \oplus \overline{(-\infty,4)} = \overline{(2,4)} = \overline{(-\infty,4)}$, we have $2 \oplus (\ominus 4) = \ominus 4$.
Let $x,y \in \mathbb{R}_\varepsilon$. Since we have

$$\begin{aligned} (x,-\infty) \oplus (y,-\infty) &= (x \oplus y,\, -\infty) \\ (x,-\infty) \otimes (y,-\infty) &= (x \otimes y,\, -\infty)\ , \end{aligned}$$

the operations $\oplus$ and $\otimes$ of the algebra of pairs as defined by (4) – (5) correspond to the operations $\oplus$ and $\otimes$ of the max-plus algebra as defined by (2) – (3).
In general, if $x,y \in \mathbb{R}_\varepsilon$ then we have

$$x \oplus (\ominus y) = x \quad \text{if } x > y\,, \tag{11}$$

$$x \oplus (\ominus y) = \ominus y \quad \text{if } x < y\,, \tag{12}$$

$$x \oplus (\ominus x) = x^\bullet\,. \tag{13}$$

Now we give some extra properties of balances that will be used in the next sections. An element with a $\ominus$ sign can be transferred to the other side of a balance as follows:

PROPOSITION 3.2. $\forall a,b,c \in \mathbb{S} : a \ominus c \,\nabla\, b$ *if and only if* $a \,\nabla\, b \oplus c$.

If both sides of a balance are signed, we may replace the balance by an equality:

PROPOSITION 3.3. $\forall a,b \in \mathbb{S}^\vee : a \,\nabla\, b \;\Rightarrow\; a = b$.

Let $a \in \mathbb{S}$. The max-positive part $a^\oplus$ and the max-negative part $a^\ominus$ of a are defined as follows:

- if $a \in \mathbb{S}^\oplus$ then $a^\oplus = a$ and $a^\ominus = \varepsilon$,
- if $a \in \mathbb{S}^\ominus$ then $a^\oplus = \varepsilon$ and $a^\ominus = \ominus a$,
- if $a \in \mathbb{S}^\bullet$ then there exists a number $x \in \mathbb{R}_\varepsilon$ such that $a = x^\bullet$ and then $a^\oplus = a^\ominus = x$.

So $a = a^{\oplus} \ominus a^{\ominus}$ and $a^{\oplus}, a^{\ominus} \in \mathbb{R}_\varepsilon$. Note that a decomposition of the form $a = x \ominus y$ with $x, y \in \mathbb{R}_\varepsilon$ is unique if it is required that either $x \neq \varepsilon$ and $y = \varepsilon$; $x = \varepsilon$ and $y \neq \varepsilon$; or $x = y$. Hence, the decomposition $a = a^{\oplus} \ominus a^{\ominus}$ is unique.

Note that $|a|_{\oplus} = a^{\oplus} \oplus a^{\ominus}$ for all $a \in \mathbb{S}$. We say that $a \in \mathbb{S}$ is *finite* if $|a|_{\oplus} \in \mathbb{R}$. If $|a|_{\oplus} = \varepsilon$ then we say that a is *infinite*.

Definition 3.1 can now be reformulated as follows:

PROPOSITION 3.4. $\forall a, b \in \mathbb{S} :$ $a \,\nabla\, b$ *if and only if* $a^{\oplus} \oplus b^{\ominus} = a^{\ominus} \oplus b^{\oplus}$.

The balance relation is extended to matrices in the usual way: if $A, B \in \mathbb{S}^{m\times n}$ then $A \,\nabla\, B$ if $a_{ij} \nabla b_{ij}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. Propositions 3.2 and 3.3 can be extended to the matrix case as follows:

PROPOSITION 3.5. $\forall A, B, C \in \mathbb{S}^{m\times n} : A \ominus C \,\nabla\, B$ *if and only if* $A \,\nabla\, B \oplus C$.

PROPOSITION 3.6. $\forall A, B \in (\mathbb{S}^{\vee})^{m\times n} :$ $A \,\nabla\, B \;\Rightarrow\; A = B$.

We conclude this section with a few extra examples that illustrate the concepts defined above.

EXAMPLE 3.7. We have $2^{\oplus} = 2$, $2^{\ominus} = \varepsilon$ and $(5^{\bullet})^{\oplus} = (5^{\bullet})^{\ominus} = 5$. Hence, $2 \,\nabla\, 5^{\bullet}$ since $2^{\oplus} \oplus (5^{\bullet})^{\ominus} = 2 \oplus 5 = 5 = \varepsilon \oplus 5 = 2^{\ominus} \oplus (5^{\bullet})^{\oplus}$.

We have $2 \,\not\nabla \ominus 5$ since $2^{\oplus} \oplus (\ominus 5)^{\ominus} = 2 \oplus 5 = 5 \neq \varepsilon = \varepsilon \oplus \varepsilon = 2^{\ominus} \oplus (\ominus 5)^{\oplus}$. ◇

EXAMPLE 3.8. Consider the balance $x \oplus 2 \,\nabla\, 5$. From Proposition 3.2 it follows that this balance can be rewritten as $x \,\nabla\, 5 \ominus 2$ or $x \,\nabla\, 5$ since $5 \ominus 2 = 5$ by (11).

If we want a signed solution, the balance $x \,\nabla\, 5$ becomes an equality by Proposition 3.3. This yields $x = 5$.

The balanced solutions of $x \,\nabla\, 5$ are of the form $x = t^{\bullet}$ with $t \in \mathbb{R}_\varepsilon$. We have $t^{\bullet} \,\nabla\, 5$ or equivalently $t = 5 \oplus t$ if and only if $t \geq 5$.

So the solution set of $x \oplus 2 \,\nabla\, 5$ is given by $\{\, 5 \,\} \cup \{\, t^{\bullet} \,|\, t \in \mathbb{R}_\varepsilon, t \geq 5 \,\}$. ◇

DEFINITION 3.9 (Max-algebraic norm). *Let $a \in \mathbb{S}^n$. The max-algebraic norm of a is defined by*

$$\|a\|_{\oplus} = \bigoplus_{i=1}^{n} |a_i|_{\oplus} \quad .$$

The max-algebraic norm of the matrix $A \in \mathbb{S}^{m\times n}$ is defined by

$$\|A\|_{\oplus} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_{\oplus} \quad .$$

The max-algebraic vector norm corresponds to the p-norms from linear algebra since

$$\|a\|_{\oplus} = \left(\bigoplus_{i=1}^{n} |a_i|_{\oplus}^{\otimes^p} \right)^{\otimes^{\frac{1}{p}}} \qquad \text{for every } a \in \mathbb{S}^n \text{ and every } p \in \mathbb{N}_0 \,.$$

The max-algebraic matrix norm corresponds to both the Frobenius norm and the p-norms from linear algebra since we have

$$\|A\|_{\oplus} = \left(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_{\oplus}^{\otimes^2} \right)^{\otimes^{\frac{1}{2}}} \qquad \text{for every } A \in \mathbb{S}^{m\times n}$$

and also $\|A\|_{\oplus} = \max\limits_{\|x\|_{\oplus}=0} \|A \otimes x\|_{\oplus}$ (the maximum is reached for $x = O_{n\times 1}$).

4. A link between conventional algebra and the symmetrized max-plus algebra. In [30] Olsder and Roos have used a kind of link between conventional algebra and the max-plus algebra based on asymptotic equivalences to show that every matrix has at least one max-algebraic eigenvalue and to prove a max-algebraic version of Cramer's rule and of the Cayley–Hamilton theorem. In [10] we have extended and formalized this link. Now we recapitulate the reasoning of [10] but in a slightly different form that is mathematically more rigorous.

In the next section we shall encounter functions that are asymptotically equivalent to an exponential of the form νe^{xs} for $s \to \infty$. Since we want to allow exponents that are equal to ε, we set $e^{\varepsilon s}$ equal to 0 for all positive real values of s by definition. We also define the following classes of functions:

$$\begin{aligned} \mathcal{R}_{\mathrm{e}}^{+} &= \Big\{ f : \mathbb{R}_0^+ \to \mathbb{R}^+ \,\Big|\, f(s) = \sum_{i=0}^{n} \mu_i e^{x_i s} \text{ with } n \in \mathbb{N}, \\ &\qquad\qquad \mu_i \in \mathbb{R}_0^+ \text{ and } x_i \in \mathbb{R}_\varepsilon \text{ for all } i \Big\} \\ \mathcal{R}_{\mathrm{e}} &= \Big\{ f : \mathbb{R}_0^+ \to \mathbb{R} \,\Big|\, f(s) = \sum_{i=0}^{n} \nu_i e^{x_i s} \text{ with } n \in \mathbb{N}, \\ &\qquad\qquad \nu_i \in \mathbb{R}_0 \text{ and } x_i \in \mathbb{R}_\varepsilon \text{ for all } i \Big\} \ . \end{aligned}$$

It is easy to verify that $(\mathcal{R}_{\mathrm{e}}, +, \times)$ is a ring.
For all $x, y, z \in \mathbb{R}_\varepsilon$ we have

$$x \oplus y = z \quad \Leftrightarrow \quad e^{xs} + e^{ys} \sim c e^{zs} \ , \ s \to \infty \tag{14}$$

$$x \otimes y = z \quad \Leftrightarrow \quad e^{xs} \cdot e^{ys} = e^{zs} \qquad \text{for all } s \in \mathbb{R}_0^+ \tag{15}$$

where $c = 1$ if $x \neq y$ and $c = 2$ if $x = y$. The relations (14) and (15) show that there exists a connection between the operations $\oplus$ and $\otimes$ performed on the real numbers and $-\infty$, and the operations $+$ and $\times$ performed on exponentials. We shall extend this link between $(\mathcal{R}_{\mathrm{e}}^{+}, +, \times)$ and $\mathbb{R}_{\max}$ that has already been used in [26, 27, 28, 29, 30] — and under a slightly different form in [6] — to $\mathbb{S}_{\max}$.
We define a mapping $\mathcal{F}$ with domain of definition $\mathbb{S} \times \mathbb{R}_0 \times \mathbb{R}_0^+$ and with

$$\begin{aligned} \mathcal{F}(a, \mu, s) &= |\mu| e^{as} && \text{if } a \in \mathbb{S}^{\oplus} \\ \mathcal{F}(a, \mu, s) &= -|\mu| e^{|a|_{\oplus} s} && \text{if } a \in \mathbb{S}^{\ominus} \\ \mathcal{F}(a, \mu, s) &= \mu e^{|a|_{\oplus} s} && \text{if } a \in \mathbb{S}^{\bullet} \end{aligned}$$

where $a \in \mathbb{S}$, $\mu \in \mathbb{R}_0$ and $s \in \mathbb{R}_0^+$.
In the remainder of this paper the first two arguments of $\mathcal{F}$ will most of the time be fixed and we shall only consider $\mathcal{F}$ in function of the third argument, i.e., for a given $a \in \mathbb{S}$ and $\mu \in \mathbb{R}_0$ we consider the function $\mathcal{F}(a, \mu, \cdot)$. Note that if $x \in \mathbb{R}_\varepsilon$ and $\mu \in \mathbb{R}_0$ then we have

$$\begin{aligned} \mathcal{F}(x, \mu, s) &= |\mu| e^{xs} \\ \mathcal{F}(\ominus x, \mu, s) &= -|\mu| e^{xs} \\ \mathcal{F}(x^{\bullet}, \mu, s) &= \mu e^{xs} \end{aligned}$$

for all $s \in \mathbb{R}_0^+$. Furthermore, $\mathcal{F}(\varepsilon, \mu, \cdot) = 0$ for all $\mu \in \mathbb{R}_0$ since we have $e^{\varepsilon s} = 0$ for all $s \in \mathbb{R}_0^+$ by definition.

For a given $\mu \in \mathbb{R}_0$ the number $a \in \mathbb{S}$ will be mapped by $\mathcal{F}$ to an exponential function $s \mapsto \nu e^{|a|_\oplus s}$ where $\nu = |\mu|$, $\nu = -|\mu|$ or $\nu = \mu$ depending on the max-algebraic sign of a. In order to reverse this process, we define the mapping $\mathcal{R}$, which we shall call the *reverse mapping* and which will map a function that is asymptotically equivalent to an exponential function $s \mapsto \nu e^{|a|_\oplus s}$ in the neighborhood of ∞ to the number $|a|_\oplus$ or $\ominus |a|_\oplus$ depending on the sign of ν. More specifically, if f is a real function, if $x \in \mathbb{R}_\varepsilon$ and if $\mu \in \mathbb{R}_0$ then we have

$$\begin{aligned} f(s) \sim |\mu| e^{xs}, \ s \to \infty \quad &\Rightarrow \quad \mathcal{R}(f) = x \\ f(s) \sim -|\mu| e^{xs}, \ s \to \infty \quad &\Rightarrow \quad \mathcal{R}(f) = \ominus x \ . \end{aligned}$$

Note that $\mathcal{R}$ will always map a function that is asymptotically equivalent to an exponential function in the neighborhood of ∞ to a signed number and never to a balanced number that is different from ε. Furthermore, for a fixed $\mu \in \mathbb{R}_0$ the mappings $a \mapsto \mathcal{F}(a, \mu, \cdot)$ and $\mathcal{R}$ are not each other's inverse since these mappings are not bijections as is shown by the following example.

EXAMPLE 4.1. Let $\mu = 1$. We have $\mathcal{F}(2, \mu, s) = e^{2s}$ and $\mathcal{F}(2^\bullet, \mu, s) = e^{2s}$ for all $s \in \mathbb{R}_0^+$. So $\mathcal{R}(\mathcal{F}(2^\bullet, \mu, \cdot)) = 2 \neq 2^\bullet$.

Consider the real functions f and g defined by $f(s) = e^{2s}$ and $g(s) = e^{2s} + 1$. We have $f(s) \sim g(s) \sim e^{2s}$, $s \to \infty$. Hence, $\mathcal{R}(f) = \mathcal{R}(g) = 2$. So $\mathcal{F}(\mathcal{R}(g), \mu, \cdot) = f \neq g$. $\diamond$

Let $\mu \in \mathbb{R}_0$. It is easy to verify that in general we have $\mathcal{R}(\mathcal{F}(a, \mu, \cdot)) = a$ if $a \in \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$, $\mathcal{R}(\mathcal{F}(a, \mu, \cdot)) = |a|_\oplus$ if $a \in \mathbb{S}^\bullet$ and $\mu > 0$, and $\mathcal{R}(\mathcal{F}(a, \mu, \cdot)) = \ominus |a|_\oplus$ if $a \in \mathbb{S}^\bullet$ and $\mu < 0$. Furthermore, if f is a real function that is asymptotically equivalent to an exponential function in the neighborhood of ∞, then we have $\mathcal{F}(\mathcal{R}(f), \mu, s) \sim f(s)$, $s \to \infty$.

For all $a, b, c \in \mathbb{S}$ we have

$$a \oplus b = c \ \Rightarrow \ \begin{cases} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \mathcal{F}(a, \mu_a, s) + \mathcal{F}(b, \mu_b, s) \sim \mathcal{F}(c, \mu_c, s), \ s \to \infty \end{cases} \tag{16}$$

$$\left.\begin{array}{l} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \mathcal{F}(a, \mu_a, s) + \mathcal{F}(b, \mu_b, s) \sim \mathcal{F}(c, \mu_c, s), \ s \to \infty \end{array}\right\} \Rightarrow \ a \oplus b \ \nabla \ c \tag{17}$$

$$a \otimes b = c \ \Rightarrow \ \begin{cases} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \mathcal{F}(a, \mu_a, s) \cdot \mathcal{F}(b, \mu_b, s) = \mathcal{F}(c, \mu_c, s) \quad \text{for all } s \in \mathbb{R}_0^+ \end{cases} \tag{18}$$

$$\left.\begin{array}{l} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \mathcal{F}(a, \mu_a, s) \cdot \mathcal{F}(b, \mu_b, s) = \mathcal{F}(c, \mu_c, s) \quad \text{for all } s \in \mathbb{R}_0^+ \end{array}\right\} \Rightarrow \ a \otimes b \ \nabla \ c \ . \tag{19}$$

As a consequence, we could say that the mapping $\mathcal{F}$ provides a link between the structure $(\mathcal{R}_e^+, +, \times)$ and $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$, and a link between the structure $(\mathcal{R}_e, +, \times)$ and $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$.

REMARK 4.2. The balance in (17) results from the fact that we can have cancellation of equal terms with opposite sign in $(\mathcal{R}_e^+, +, \times)$ whereas this is in general not possible in the symmetrized max-plus algebra since $\forall a \in \mathbb{S} \setminus \{\varepsilon\} : a \ominus a \neq \varepsilon$.

The following example shows that the balance on the right-hand side of (19) is also necessary: we have $\mathcal{F}(0, 1, s) \cdot \mathcal{F}(0, 1, s) = 1 \cdot 1 = 1 = \mathcal{F}(0^\bullet, 1, s)$ for all $s \in \mathbb{R}_0^+$, but $0 \otimes 0 = 0 \neq 0^\bullet$.

We have $1 \oplus 2 = 2 \,\nabla\, 3^\bullet$. However, there do not exist real numbers μ_1, μ_2, $\mu_3 \in \mathbb{R}_0$

such that

$$\mathcal{F}(1,\mu_1,s)+\mathcal{F}(2,\mu_2,s) \ \sim \ \mathcal{F}(3^\bullet,\mu_3,s)\ ,\ s\to\infty$$

or equivalently

$$|\mu_1|\,e^{s} \ + \ |\mu_2|\,e^{2s} \ \sim \ \mu_3\,e^{3s}\ ,\ s\to\infty\ .$$

This implies that in general (16) does not hold any more if we replace the equality on the left-hand side by a balance.
In a similar way we can show that in general $a\otimes b\,\nabla\, c$ with $a,b,c,\in\mathbb{S}$ does not imply that there exist real numbers $\mu_a,\ \mu_b,\ \mu_c\in\mathbb{R}_0$ such that $\mathcal{F}(a,\mu_a,s)\cdot\mathcal{F}(b,\mu_b,s)=\mathcal{F}(c,\mu_c,s)$ for all $s\in\mathbb{R}_0^+$. $\diamond$

We extend the mapping $\mathcal{F}$ to matrices as follows. If $A\in\mathbb{S}^{m\times n}$ and if $M\in\mathbb{R}_0^{m\times n}$ then $\tilde{A}=\mathcal{F}(A,M,\cdot)$ is a real m by n matrix-valued function with domain of definition $\mathbb{R}_0^+$ and with $\tilde{a}_{ij}(s)=\mathcal{F}(a_{ij},m_{ij},s)$ for all i,j. Note that the mapping is performed entrywise. The reverse mapping $\mathcal{R}$ is extended to matrices in a similar way: if $\tilde{A}$ is a real matrix-valued function with entries that are asymptotically equivalent to an exponential in the neighborhood of ∞, then $(\mathcal{R}(\tilde{A}))_{ij}=\mathcal{R}(\tilde{a}_{ij})$ for all i,j. If A, B and C are matrices with entries in $\mathbb{S}$, we have

$$A\oplus B=C \ \Rightarrow\ \left\{\begin{array}{l}\exists M_A,M_B,M_C \text{ such that}\\ \mathcal{F}(A,M_A,s)+\mathcal{F}(B,M_B,s)\ \sim\ \mathcal{F}(C,M_C,s)\ ,\ s\to\infty\end{array}\right. \tag{20}$$

$$\left.\begin{array}{l}\exists M_A,M_B,M_C \text{ such that}\\ \mathcal{F}(A,M_A,s)+\mathcal{F}(B,M_B,s)\ \sim\ \mathcal{F}(C,M_C,s)\ ,\ s\to\infty\end{array}\right\}\ \Rightarrow\ A\oplus B\ \nabla\ C \tag{21}$$

$$A\otimes B=C \ \Rightarrow\ \left\{\begin{array}{l}\exists M_A,M_B,M_C \text{ such that}\\ \mathcal{F}(A,M_A,s)\cdot\mathcal{F}(B,M_B,s)\ \sim\ \mathcal{F}(C,M_C,s)\ ,\ s\to\infty\end{array}\right. \tag{22}$$

$$\left.\begin{array}{l}\exists M_A,M_B,M_C \text{ such that}\\ \mathcal{F}(A,M_A,s)\cdot\mathcal{F}(B,M_B,s)\ \sim\ \mathcal{F}(C,M_C,s)\ ,\ s\to\infty\end{array}\right\}\ \Rightarrow\ A\otimes B\ \nabla\ C\ . \tag{23}$$

EXAMPLE 4.3. Let $A=\begin{bmatrix}0 & \varepsilon\\ \ominus 1 & \ominus 2\end{bmatrix}$ and $B=\begin{bmatrix}-3 & 1\\ 2^\bullet & \ominus 0\end{bmatrix}$. Hence, $A\otimes B=\begin{bmatrix}-3 & 1\\ 4^\bullet & 2^\bullet\end{bmatrix}$. Let M, N and $P\in\mathbb{R}_0^{2\times 2}$. In general we have

$$\begin{aligned}
\mathcal{F}(A,M,s) &= \begin{bmatrix}|m_{11}| & 0\\ -|m_{21}|\,e^{s} & -|m_{22}|\,e^{2s}\end{bmatrix}\\
\mathcal{F}(B,N,s) &= \begin{bmatrix}|n_{11}|\,e^{-3s} & |n_{12}|\,e^{s}\\ n_{21}\,e^{2s} & -|n_{22}|\end{bmatrix}\\
\mathcal{F}(A\otimes B,P,s) &= \begin{bmatrix}|p_{11}|\,e^{-3s} & |p_{12}|\,e^{s}\\ p_{21}\,e^{4s} & p_{22}\,e^{2s}\end{bmatrix}
\end{aligned}$$

for all $s\in\mathbb{R}_0^+$. Furthermore,

$$\mathcal{F}(A,M,s)\cdot\mathcal{F}(B,N,s)=$$
$$\begin{bmatrix}|m_{11}|\,|n_{11}|\,e^{-3s} & |m_{11}|\,|n_{12}|\,e^{s}\\ -|m_{21}|\,|n_{11}|\,e^{-2s}\ -\ |m_{22}|\,n_{21}\,e^{4s} & (-|m_{21}|\,|n_{12}|\ +\ |m_{22}|\,|n_{22}|)\,e^{2s}\end{bmatrix}$$

for all $s \in \mathbb{R}_0^+$.
If $-|m_{21}||n_{12}| + |m_{22}||n_{22}| \neq 0$ and if we take

$$p_{11} = |m_{11}|\,|n_{11}|\,, \quad p_{12} = |m_{11}|\,|n_{12}|\,,$$
$$p_{21} = -|m_{22}|\,n_{21}\,, \quad p_{22} = -|m_{21}|\,|n_{12}| + |m_{22}|\,|n_{22}|\,,$$

then we have $\mathcal{F}(A, M, s) \cdot \mathcal{F}(B, N, s) \sim \mathcal{F}(A \otimes B, P, s)$, $s \to \infty$.
If we take $m_{ij} = n_{ij} = 1$ for all i, j, we get

$$\mathcal{F}(A, s) \cdot \mathcal{F}(B, s) \sim \begin{bmatrix} e^{-3s} & e^{s} \\ -e^{4s} & 0 \end{bmatrix} \stackrel{\text{def}}{=} \tilde{C}(s)\ ,\ s \to \infty\ .$$

The reverse mapping results in $C = \mathcal{R}(\tilde{C}) = \begin{bmatrix} -3 & 1 \\ \ominus 4 & \varepsilon \end{bmatrix}$. Note that $A \otimes B \,\nabla\, C$.
Taking $m_{ij} = n_{ij} = (-1)^{(i+j)}(i + j)$ for all i, j leads to

$$\mathcal{F}(A, s) \cdot \mathcal{F}(B, s) \sim \begin{bmatrix} 4e^{-3s} & 6e^{s} \\ 12e^{4s} & 7e^{2s} \end{bmatrix} \stackrel{\text{def}}{=} \tilde{D}(s)\ ,\ s \to \infty\ .$$

The reverse mapping results in $D = \mathcal{R}(\tilde{D}) = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$ and again we have $A \otimes B \,\nabla\, D$. $\diamond$

5. The QR decomposition and the singular value decomposition in the symmetrized max-plus algebra. In [10] we have used the mapping from $\mathbb{S}_{\max}$ to $(\mathcal{R}_{\mathrm{e}}, +, \times)$ and the reverse mapping $\mathcal{R}$ to prove the existence of a kind of singular value decomposition (SVD) in $\mathbb{S}_{\max}$. The proof of [10] is based on the *analytic SVD*. In this section we present an alternative proof for the existence theorem of the max-algebraic SVD. The major advantage of the new proof technique that will be developed in this section over the one of [10] is that it can easily be extended to prove the existence of many other matrix decompositions in the symmetrized max-plus algebra such as the max-algebraic QR decomposition, the max-algebraic LU decomposition, the max-algebraic eigenvalue decomposition (for symmetric matrices) and so on. This proof technique consists in transforming a matrix with entries in $\mathbb{S}$ to a matrix-valued function with exponential entries (using the mapping $\mathcal{F}$), applying an algorithm from linear algebra and transforming the result back to the symmetrized max-plus algebra (using the mapping $\mathcal{R}$).

5.1. Sums and series of exponentials. The entries of the matrices that are used in the existence proofs for the max-algebraic QR decomposition and the max-algebraic SVD that will be presented in this section are sums or series of exponentials. Therefore, we first study some properties of this kind of functions.

Definition 5.1 (Sum or series of exponentials). *Let $\mathcal{S}_{\mathrm{e}}$ be the set of real functions that are analytic and that can be written as a (possibly infinite, but absolutely convergent) sum of exponentials in a neighborhood of ∞:*

$$\mathcal{S}_{\mathrm{e}} = \Big\{ f : \quad A \to \mathbb{R} \;\Big|\; A \subseteq \mathbb{R},\ \exists K \in \mathbb{R}_0^+ \text{ such that } [K, \infty) \subseteq A \text{ and}$$
$$f \text{ is analytic in } [K, \infty) \text{ and either}$$

$$\forall x \geq K : f(x) = \sum_{i=0}^{n} \alpha_i e^{a_i x} \tag{24}$$

with $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}_0$, $a_i \in \mathbb{R}_\varepsilon$ for all i and $a_0 > a_1 > \ldots > a_n$; or

$$\forall x \geq K : f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} \tag{25}$$

with $\alpha_i \in \mathbb{R}_0$, $a_i \in \mathbb{R}$, $a_i > a_{i+1}$ for all i, $\lim_{i\to\infty} a_i = \varepsilon$ and

where the series converges absolutely for every $x \geq K \Big\}$.

If $f \in \mathcal{S}_\mathrm{e}$ then the largest exponent in the sum or the series of exponentials that corresponds to f is called the *dominant exponent* of f.
Recall that by definition we have $e^{\varepsilon s} = 0$ for all $s \in \mathbb{R}_0^+$. Since we allow exponents that are equal to $\varepsilon = -\infty$ in the definition of $\mathcal{S}_\mathrm{e}$, the zero function also belongs to $\mathcal{S}_\mathrm{e}$. Since we require that the sequence of the exponents that appear in (24) or (25) is decreasing and since the coefficients cannot be equal to 0, any sum of exponentials of the form (24) or (25) that corresponds to the zero function consists of exactly one term: e.g., $1 \cdot e^{\varepsilon x}$.
If $f \in \mathcal{S}_\mathrm{e}$ is a series of the form (25) then the set $\{a_i \,|\, i = 0, 1, \ldots, \infty\}$ has no finite accumulation point since the sequence $\{a_i\}_{i=0}^{\infty}$ is decreasing and unbounded from below. Note that series of the form (25) are related to (generalized) Dirichlet series [23].
The behavior of the functions in $\mathcal{S}_\mathrm{e}$ in the neighborhood of ∞ is given by the following property:

LEMMA 5.2. *Every function $f \in \mathcal{S}_\mathrm{e}$ is asymptotically equivalent to an exponential in the neighborhood of ∞:*

$$f \in \mathcal{S}_\mathrm{e} \;\; \Rightarrow \;\; f(x) \; \sim \; \alpha_0 e^{a_0 x} \; , \; x \to \infty$$

for some $\alpha_0 \in \mathbb{R}_0$ and some $a_0 \in \mathbb{R}_\varepsilon$.

Proof. See Appendix A. □

The set $\mathcal{S}_\mathrm{e}$ is closed under elementary operations such as additions, multiplications, subtractions, divisions, square roots and absolute values:

PROPOSITION 5.3. *If f and g belong to $\mathcal{S}_\mathrm{e}$ then ρf, $f + g$, $f - g$, fg, f^l and $|f|$ also belong to $\mathcal{S}_\mathrm{e}$ for any $\rho \in \mathbb{R}$ and any $l \in \mathbb{N}$.*
Furthermore, if there exists a real number P such that $f(x) \neq 0$ for all $x \geq P$ then the functions $\dfrac{1}{f}$ and $\dfrac{g}{f}$ restricted to $[P, \infty)$ also belong to $\mathcal{S}_\mathrm{e}$.
If there exists a real number Q such that $f(x) > 0$ for all $x \geq Q$ then the function $\sqrt{f}$ restricted to $[Q, \infty)$ also belongs to $\mathcal{S}_\mathrm{e}$.

Proof. See Appendix B. □

5.2. The max-algebraic QR decomposition. Let $\tilde{A}$ and $\tilde{R}$ be real m by n matrix-valued functions and let $\tilde{Q}$ be a real m by m matrix-valued function. Suppose that these matrix-valued functions are defined in $J \subseteq \mathbb{R}$. If $\tilde{Q}(s)\,\tilde{R}(s) = \tilde{A}(s)$, $\tilde{Q}^T(s)\,\tilde{Q}(s) = I_m$ and $\tilde{R}(s)$ is an upper triangular matrix for all $s \in J$ then we say that $\tilde{Q}\tilde{R}$ is a *path of QR decompositions* of $\tilde{A}$ on J. A path of singular value decompositions is defined in a similar way.
Note that if $\tilde{Q}\tilde{R}$ is a path of QR decompositions of $\tilde{A}$ on J then we have $\|\tilde{R}(s)\|_\mathrm{F} = \|\tilde{A}(s)\|_\mathrm{F}$ for all $s \in J$. Now we prove that for a matrix with entries in $\mathcal{S}_\mathrm{e}$ there exists a path of QR decompositions with entries that also belong to $\mathcal{S}_\mathrm{e}$. Next we use this result to prove the existence of a max-algebraic analogue of the QR decomposition.

PROPOSITION 5.4. *If $\tilde{A} \in \mathcal{S}_{\mathrm{e}}^{m\times n}$ then there exists a path of QR decompositions $\tilde{Q}\tilde{R}$ of $\tilde{A}$ for which the entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_{\mathrm{e}}$.*

Proof. To compute the QR decomposition of a matrix with real entries we can use the Givens QR algorithm (see [14]). The operations used in this algorithm are additions, multiplications, subtractions, divisions and square roots. Furthermore, the number of operations used in this algorithm is finite.
So if we apply this algorithm to a matrix-valued function $\tilde{A}$ with entries in $\mathcal{S}_{\mathrm{e}}$ then the entries of the resulting matrix-valued functions $\tilde{Q}$ and $\tilde{R}$ will also belong to $\mathcal{S}_{\mathrm{e}}$ by Proposition 5.3. ☐

THEOREM 5.5 (Max-algebraic QR decomposition). *If $A \in \mathbb{S}^{m\times n}$ then there exist a matrix $Q \in (\mathbb{S}^{\vee})^{m\times m}$ and a max-algebraic upper triangular matrix $R \in (\mathbb{S}^{\vee})^{m\times n}$ such that*

$$A \,\nabla\, Q \otimes R \tag{26}$$

with $Q^T \otimes Q \,\nabla\, E_m$ and $\|R\|_{\oplus} = \|A\|_{\oplus}$.
Every decomposition of the form (26) that satisfies the above conditions is called a max-algebraic QR decomposition of A.

Proof. If $A \in \mathbb{S}^{m\times n}$ has entries that are not signed, we can always define a matrix $\hat{A} \in (\mathbb{S}^{\vee})^{m\times n}$ such that

$$\hat{a}_{ij} = \begin{cases} a_{ij} & \text{if } a_{ij} \text{ is signed,} \\ |a_{ij}|_{\oplus} & \text{if } a_{ij} \text{ is not signed,} \end{cases}$$

for all i, j. Since $|\hat{a}_{ij}|_{\oplus} = |a_{ij}|_{\oplus}$ for all i, j, we have $\|\hat{A}\|_{\oplus} = \|A\|_{\oplus}$. Moreover, we have

$$\forall a, b \in \mathbb{S} : \; a \,\nabla\, b \;\Rightarrow\; a^{\bullet} \,\nabla\, b \;\; ,$$

which means that if $\hat{A} \,\nabla\, Q \otimes R$ then also $A \,\nabla\, Q \otimes R$. Therefore, it is sufficient to prove this theorem for signed matrices A.
So from now on we assume that A is signed. We construct $\tilde{A} = \mathcal{F}(A, M, \cdot)$ where $M \in \mathbb{R}^{m\times n}$ with $m_{ij} = 1$ for all i, j. Hence, $\tilde{a}_{ij}(s) = \gamma_{ij} e^{c_{ij} s}$ for all $s \in \mathbb{R}_0^+$ and for all i, j with $\gamma_{ij} \in \{-1, 1\}$ and $c_{ij} = |a_{ij}|_{\oplus} \in \mathbb{R}_{\varepsilon}$ for all i, j. Note that the entries of $\tilde{A}$ belong to $\mathcal{S}_{\mathrm{e}}$. By Proposition 5.4 there exists a path of QR decompositions of $\tilde{A}$. So there exists a positive real number L and matrix-valued functions $\tilde{Q}$ and $\tilde{R}$ with entries in $\mathcal{S}_{\mathrm{e}}$ such that

$$\tilde{A}(s) = \tilde{Q}(s)\, \tilde{R}(s) \qquad \text{for all } s \geq L \tag{27}$$

$$\tilde{Q}^T(s)\, \tilde{Q}(s) = I_m \qquad \text{for all } s \geq L \tag{28}$$

$$\|\tilde{R}(s)\|_{\mathrm{F}} = \|\tilde{A}(s)\|_{\mathrm{F}} \qquad \text{for all } s \geq L \;. \tag{29}$$

The entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_{\mathrm{e}}$ and are thus asymptotically equivalent to an exponential in the neighborhood of ∞ by Lemma 5.2.
If we define $Q = \mathcal{R}(\tilde{Q})$ and $R = \mathcal{R}(\tilde{R})$, then Q and R have signed entries. If we apply the reverse mapping $\mathcal{R}$ to (27) – (29), we get

$$A \,\nabla\, Q \otimes R, \quad Q^T \otimes Q \,\nabla\, E_m \;\text{ and }\; \|R\|_{\oplus} = \|A\|_{\oplus} \;\; . \qquad ☐$$

If f, g and h belong to $\mathcal{S}_{\mathrm{e}}$ then they are asymptotically equivalent to an exponential in the neighborhood of ∞ by Lemma 5.2. So if L is large enough, then $f(L) \geq 0$

and $g(L) \geq h(L)$ imply that $f(s) \geq 0$ and $g(s) \geq h(s)$ for all $s \in [L, \infty)$. This fact and the fact that $\mathcal{S}_\mathrm{e}$ is closed under some elementary algebraic operations explain why many algorithms from linear algebra — such as the Givens QR algorithm and Kogbetliantz's SVD algorithm (see §5.3) — also work for matrices with entries that belong to $\mathcal{S}_\mathrm{e}$ instead of $\mathbb{R}$. If we apply an algorithm from linear algebra to a matrix-valued function $\tilde{A}$ with entries in $\mathcal{S}_\mathrm{e}$ that is defined on some interval $[L, \infty)$, we are in fact applying this algorithm on the (constant) matrix $\tilde{A}(s)$ for every value of $s \in [L, \infty)$ in parallel.

If QR is a QR decomposition of a matrix $A \in \mathbb{R}^{m \times n}$ then we always have $\|R\|_\mathrm{F} = \|A\|_\mathrm{F}$ since Q is an orthogonal matrix. However, the following example shows that $A \nabla Q \otimes R$ and $Q^T \otimes Q \nabla E_m$ do not always imply that $\|R\|_\oplus = \|A\|_\oplus$.

EXAMPLE 5.6. Consider

$$A = \begin{bmatrix} \ominus 0 & 0 & 0 \\ 0 & \ominus 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Without the condition $\|R\|_\oplus = \|A\|_\oplus$ every max-algebraic product of the form

$$Q \otimes R(\rho) = \begin{bmatrix} \ominus 0 & 0 & 0 \\ 0 & \ominus 0 & 0 \\ 0 & 0 & \ominus 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & \varepsilon & \rho \\ \varepsilon & 0 & \rho \\ \varepsilon & \varepsilon & \rho \end{bmatrix} = \begin{bmatrix} \ominus 0 & 0 & \rho^\bullet \\ 0 & \ominus 0 & \rho^\bullet \\ 0 & 0 & \rho^\bullet \end{bmatrix}$$

with $\rho \geq 0$ would have been a max-algebraic QR decomposition of A. However, since $\|R(\rho)\|_\oplus = \rho$ if $\rho \geq 0$ and since $\|A\|_\oplus = 0$, we do not have $\|R\|_\oplus = \|A\|_\oplus$ if $\rho > 0$. $\diamond$

This example explains why we have included the condition $\|R\|_\oplus = \|A\|_\oplus$ in the definition of the max-algebraic QR decomposition.

Now we explain why we really need the symmetrized max-plus algebra $\mathbb{S}_{\max}$ to define the max-algebraic QR decomposition: we shall show that the class of matrices with entries in $\mathbb{R}_\varepsilon$ that have max-algebraic QR decompositions for which the entries of Q and R belong to $\mathbb{R}_\varepsilon$ is rather limited. Let $A \in \mathbb{R}_\varepsilon^{m \times n}$ and let $Q \otimes R$ be a max-algebraic QR decomposition of A for which the entries of Q and R belong to $\mathbb{R}_\varepsilon$. Since the entries of A, Q and R are signed, it follows from Proposition 3.6 that the balances $A \nabla Q \otimes R$ and $Q^T \otimes Q \nabla E_m$ result in $A = Q \otimes R$ and $Q^T \otimes Q = E_m$. It is easy to verify that we can only have $Q^T \otimes Q = E_m$ if every column and every row of Q contains exactly one entry that is equal to 0 and if all the other entries of Q are equal to ε. Hence, Q is max-algebraic permutation matrix. As a consequence, A has to be a row-permuted max-algebraic upper triangular matrix.

So only row-permuted max-algebraic upper triangular matrices with entries in $\mathbb{R}_\varepsilon$ have a max-algebraic QR decomposition with entries in $\mathbb{R}_\varepsilon$. This could be compared with the class of real matrices in linear algebra that have a QR decomposition with only nonnegative entries: using an analogous reasoning one can prove that this class coincides with the set of the real row-permuted upper triangular matrices. Furthermore, it is obvious that every QR decomposition in $\mathbb{R}_{\max}$ is also a QR decomposition in $\mathbb{S}_{\max}$.

5.3. The max-algebraic singular value decomposition. Now we give an alternative proof for the existence theorem of the max-algebraic SVD. In this proof we shall use Kogbetliantz's SVD algorithm [20], which can be considered as an extension of Jacobi's method for the computation of the eigenvalue decomposition of

a real symmetric matrix. We now state the main properties of this algorithm. The explanation below is mainly based on [4] and [17].
A Givens matrix is a square matrix of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cos(\theta) & \cdots & \sin(\theta) & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -\sin(\theta) & \cdots & \cos(\theta) & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \end{bmatrix} .$$

The off-norm of the matrix $A \in \mathbb{R}^{m \times n}$ is defined by

$$\|A\|_{\text{off}} = \sqrt{\sum_{i=1}^{n} \sum_{j=1,\, j \neq i}^{n} a_{ij}^2}$$

where the empty sum is equal to 0 by definition (so if A is a 1 by 1 matrix then we have $\|A\|_{\text{off}} = 0$). Let $A \in \mathbb{R}^{m \times n}$. Since USV^T is an SVD of A if and only if VS^TU^T is an SVD of A^T, we may assume without loss of generality that $m \geq n$. Before applying Kogbetliantz's SVD algorithm we compute a QR decomposition of A:

$$A = Q \begin{bmatrix} R \\ O_{(m-n)\times n} \end{bmatrix}$$

where R is an n by n upper triangular matrix.
Now we apply Kogbetliantz's SVD algorithm to R. In this algorithm a sequence of matrices is generated as follows:

$$\begin{aligned} &U_0 = I_n\,, \quad V_0 = I_n\,, \quad S_0 = R\,, \\ &U_k = U_{k-1}G_k\,, \quad V_k = V_{k-1}H_k\,, \quad S_k = G_k^T S_{k-1} H_k \qquad \text{for } k = 1, 2, 3, \ldots \end{aligned}$$

such that $\|S_k\|_{\text{off}}$ decreases monotonously as k increases. So S_k tends more and more to a diagonal matrix as the iteration process progresses. The absolute values of the diagonal entries of S_k will converge to the singular values of R as k goes to ∞.
The matrices G_k and H_k are Givens matrices that are chosen such that $(S_k)_{i_k j_k} = (S_k)_{j_k i_k} = 0$ for some ordered pair of indices (i_k, j_k). As a result we have

$$\|S_k\|_{\text{off}}^2 = \|S_{k-1}\|_{\text{off}}^2 - (S_{k-1})_{i_k j_k}^2 - (S_{k-1})_{j_k i_k}^2 \ .$$

Since the matrices G_k and H_k are orthogonal for all $k \in \mathbb{N}_0$, we have

$$\|S_k\|_{\text{F}} = \|R\|_{\text{F}}\,, \quad R = U_k S_k V_k^T, \quad U_k^T U_k = I_n \ \text{ and } \ V_k^T V_k = I_n \tag{30}$$

for all $k \in \mathbb{N}$.
We shall use the row-cyclic version of Kogbetliantz's SVD algorithm: in each cycle the indices i_k and j_k are chosen such that the entries in the strictly upper triangular

part of the S_k's are selected row by row. This yields the following sequence for the ordered pairs of indices (i_k, j_k):

$$(1,2) \to (1,3) \to \ldots \to (1,n) \to (2,3) \to (2,4) \to \ldots \to (n-1,n) \ .$$

A full cycle $(1,2) \to \ldots \to (n-1,n)$ is called a *sweep*. Note that a sweep corresponds to $N = \dfrac{(n-1)n}{2}$ iterations. Sweeps are repeated until S_k becomes diagonal. If we have an upper triangular matrix at the beginning of a sweep then we shall have a lower triangular matrix after the sweep and vice versa.
For triangular matrices the row-cyclic Kogbetliantz algorithm is globally convergent [11, 17]. Furthermore, for triangular matrices the convergence of this algorithm is quadratic if k is large enough [2, 3, 15, 16, 31]:

$$\exists K \in \mathbb{N} \text{ such that } \forall k \geq K : \|S_{k+N}\|_{\text{off}} \leq \gamma \, \|S_k\|_{\text{off}}^2 \tag{31}$$

for some constant γ that does not depend on k, under the assumption that diagonal entries that correspond to the same singular value or that are affiliated with the same cluster of singular values occupy successive positions on the diagonal. This assumption is not restrictive since we can always reorder the diagonal entries of S_k by inserting an extra step in which we select a permutation matrix $\hat{P} \in \mathbb{R}^{n \times n}$ such that the diagonal entries of $S_{k+1} = \hat{P}^T S_k \hat{P}$ exhibit the required ordering. Note that $\|S_{k+1}\|_{\text{F}} = \|S_k\|_{\text{F}}$. If we define $U_{k+1} = U_k \hat{P}$ and $V_{k+1} = V_k \hat{P}$ then U_{k+1} and V_{k+1} are orthogonal since $\hat{P}^T \hat{P} = I_n$. We also have

$$U_{k+1} S_{k+1} V_{k+1}^T \ = \ \left(U_k \hat{P}\right) \left(\hat{P}^T S_k \hat{P}\right) \left(\hat{P}^T V_k^T\right) \ = \ U_k S_k V_k^T \ = \ R \ .$$

Furthermore, once the diagonal entries have the required ordering, they hold it provided that k is sufficiently large [15].
If we define $S = \lim\limits_{k\to\infty} S_k$, $U = \lim\limits_{k\to\infty} U_k$ and $V = \lim\limits_{k\to\infty} V_k$ then S is a diagonal matrix, U and V are orthogonal matrices and $USV^T = R$. We make all the diagonal entries of S nonnegative by multiplying S with an appropriate diagonal matrix D. Next we construct a permutation matrix P such that the diagonal entries of $P^T SDP$ are arranged in descending order. If we define $U_R = UP$, $S_R = P^T SDP$ and $V_R = VD^{-1}P$, then U_R and V_R are orthogonal, the diagonal entries of S_R are nonnegative and ordered and

$$U_R S_R V_R^T = (UP)\left(P^T SDP\right)\left(P^T D^{-1} V^T\right) = USV^T = R \ .$$

Hence, $U_R S_R V_R^T$ is an SVD of R. If we define

$$U_A = Q \begin{bmatrix} U_R & O_{n\times(m-n)} \\ O_{(m-n)\times n} & I_{m-n} \end{bmatrix}, \ S_A = \begin{bmatrix} S_R \\ O_{(m-n)\times n} \end{bmatrix} \ \text{and} \ V_A = V_R \ ,$$

then $U_A S_A V_A^T$ is an SVD of A.

Theorem 5.7 (Max-algebraic singular value decomposition). *Let $A \in \mathbb{S}^{m\times n}$ and let $r = \min(m,n)$. Then there exist a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m\times n}$ and matrices $U \in (\mathbb{S}^\vee)^{m\times m}$ and $V \in (\mathbb{S}^\vee)^{n\times n}$ such that*

$$A \ \nabla \ U \otimes \Sigma \otimes V^T \tag{32}$$

with $U^T \otimes U \ \nabla \ E_m$, $V^T \otimes V \ \nabla \ E_n$ and $\|A\|_\oplus = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ where $\sigma_i = (\Sigma)_{ii}$ for $i = 1, 2, \ldots, r$.

Every decomposition of the form (32) that satisfies the above conditions is called a max-algebraic singular value decomposition of A.

Proof. Using a reasoning that is similar to the one that has been used at the beginning of the proof of Theorem 5.5, we can show that it is sufficient to prove this theorem for signed matrices A. So from now on we assume that A is signed.
Define $c = \|A\|_{\oplus}$. If $c = \varepsilon$ then $A = \mathcal{E}_{m\times n}$. If we take $U = E_m$, $\Sigma = \mathcal{E}_{m\times n}$ and $V = E_n$, we have $A = U \otimes \Sigma \otimes V^T$, $U^T \otimes U = E_m$, $V^T \otimes V = E_n$ and $\sigma_1 = \sigma_2 = \ldots = \sigma_r = \varepsilon = \|A\|_{\oplus}$. So $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A.
From now on we assume that $c \neq \varepsilon$. We may assume without loss of generality that $m \geq n$: if $m < n$, we can apply the subsequent reasoning to A^T since $A \,\nabla\, U \otimes \Sigma \otimes V^T$ if and only if $A^T \,\nabla\, V \otimes \Sigma^T \otimes U^T$. So $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A if and only if $V \otimes \Sigma^T \otimes U^T$ is a max-algebraic SVD of A^T.
Now we distinguish between two different situations depending on whether or not all the a_{ij}'s have a finite max-absolute value. In Remark 5.8 we shall explain why this distinction is necessary.

Case 1: All the a_{ij}'s have a finite max-absolute value.
We construct $\tilde{A} = \mathcal{F}(A, M, \cdot)$ where $M \in \mathbb{R}^{m\times n}$ with $m_{ij} = 1$ for all i, j. The entries of $\tilde{A}$ belong to $\mathcal{S}_{\mathrm{e}}$.
In order to determine a path of SVDs of $\tilde{A}$, we first compute a path of QR decompositions of $\tilde{A}$ on $\mathbb{R}_0^+$:

$$\tilde{A} = \tilde{Q} \begin{bmatrix} \tilde{R} \\ O_{(m-n)\times n} \end{bmatrix}$$

where $\tilde{R}$ is an n by n upper triangular matrix-valued function. By Proposition 5.4 the entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_{\mathrm{e}}$.
Now we use the row-cyclic Kogbetliantz algorithm to compute a path of SVDs of $\tilde{R}$. The operations used in this algorithm are additions, multiplications, subtractions, divisions, square roots and absolute values. So if we apply this algorithm to a matrix with entries in $\mathcal{S}_{\mathrm{e}}$, the entries of all the matrices generated during the iteration process also belong to $\mathcal{S}_{\mathrm{e}}$ by Proposition 5.3.
In theory we should run the row-cyclic Kogbetliantz algorithm forever in order to produce a path of exact SVDs of $\tilde{A}$. However, since we are only interested in the asymptotic behavior of the singular values and the entries of the singular vectors of $\tilde{A}$, we may stop the iteration process after a finite number of sweeps:
Let $\tilde{S}_k$, $\tilde{U}_k$ and $\tilde{V}_k$ be the matrix-valued functions that are computed in the kth step of the algorithm. Let $\tilde{\Delta}_p$ be the diagonal matrix-valued function obtained by removing the off-diagonal entries of $\tilde{S}_{pN}$ (where $N = \dfrac{n(n-1)}{2}$ is the number of iterations per sweep), making all diagonal entries nonnegative and arranging them in descending order, and adding $m - n$ zero rows (cf. the transformations used to go from S to S_A in the explanation of Kogbetliantz's algorithm given above). Let $\tilde{X}_p$ and $\tilde{Y}_p$ be the matrix-valued functions obtained by applying the corresponding transformations to $\tilde{U}_{pN}$ and $\tilde{V}_{pN}$ respectively. If we define a matrix-valued function $\tilde{C}_p = \tilde{X}_p \tilde{\Delta}_p \tilde{Y}_p^T$, we have a path of *exact* SVDs of $\tilde{C}_p$ on some interval $[L, \infty)$. This means that we may stop the iteration process as soon as

$$\mathcal{F}(A, N, s) \sim \tilde{C}_p(s),\ s \to \infty \tag{33}$$

for some $N \in \mathbb{R}_0^{m\times n}$. Note that eventually this condition will always be satisfied due to the fact that Kogbetliantz's SVD algorithm is globally convergent and — for triangular matrices — also quadratically convergent if p is large enough, and due to the fact that the entries of $\tilde{A}$ — to which the entries of $\tilde{C}_p$ should converge — are not identically zero since they have a finite dominant exponent.

Let $\tilde{U}\tilde{S}\tilde{V}^T$ be a path of approximate SVDs of $\tilde{A}$ on some interval $[L,\infty)$ that was obtained by the procedure given above. Since we have performed a *finite* number of elementary operations on the entries of $\tilde{A}$, the entries of $\tilde{U}$, $\tilde{S}$ and $\tilde{V}$ belong to $\mathcal{S}_{\mathrm{e}}$. We have

$$\mathcal{F}(A,N,s) \ \sim \ \tilde{U}(s)\,\tilde{\Sigma}(s)\,\tilde{V}^T(s)\ ,\ s\to\infty \tag{34}$$

for some $N \in \mathbb{R}_0^{m\times n}$. Furthermore,

$$\tilde{U}^T(s)\,\tilde{U}(s) = I_m \qquad \text{for all } s \geq L \tag{35}$$

$$\tilde{V}^T(s)\,\tilde{V}(s) = I_n \qquad \text{for all } s \geq L\,. \tag{36}$$

The diagonal entries of $\tilde{\Sigma}$ and the entries of $\tilde{U}$ and $\tilde{V}$ belong to $\mathcal{S}_{\mathrm{e}}$ and are thus asymptotically equivalent to an exponential in the neighborhood of ∞ by Lemma 5.2. Define $\tilde{\sigma}_i = \tilde{\Sigma}_{ii}$ for $i = 1,2,\ldots,r$.
Now we apply the reverse mapping $\mathcal{R}$ in order to obtain a max-algebraic SVD of A. If we define

$$\Sigma = \mathcal{R}(\tilde{\Sigma})\,,\ U = \mathcal{R}(\tilde{U})\,,\ V = \mathcal{R}(\tilde{V}) \ \text{ and } \ \sigma_i = (\Sigma)_{ii} = \mathcal{R}(\tilde{\sigma}_i) \text{ for all } i\,,$$

then Σ is a max-algebraic diagonal matrix and U and V have signed entries. If we apply the reverse mapping $\mathcal{R}$ to (34) – (36), we get

$$A \,\nabla\, U \otimes \Sigma \otimes V^T,\quad U^T \otimes U \,\nabla\, E_m \ \text{ and } \ V^T \otimes V \,\nabla\, E_n \ \ .$$

The $\tilde{\sigma}_i$'s are nonnegative in $[L,\infty)$ and therefore we have $\sigma_i \in \mathbb{R}_\varepsilon$ for all i. Since the $\tilde{\sigma}_i$'s are ordered in $[L,\infty)$, their dominant exponents are also ordered. Hence, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$.
We have $\|\tilde{A}(s)\|_{\mathrm{F}} \ \sim \ \gamma e^{cs}$, $s\to\infty$ for some $\gamma > 0$ since $c = \|A\|_\oplus$ is the largest exponent that appears in the entries of $\tilde{A}$. Hence, $\mathcal{R}\Big(\|\tilde{A}\|_{\mathrm{F}}\Big) = c = \|A\|_\oplus$.
If $P \in \mathbb{R}^{m\times n}$ then $\dfrac{1}{\sqrt{n}}\ \|P\|_{\mathrm{F}} \ \leq \ \|P\|_2 \ \leq \ \|P\|_{\mathrm{F}}\,$. As a consequence, we have

$$\frac{1}{\sqrt{n}}\ \|\tilde{A}\|_{\mathrm{F}} \leq \|\tilde{A}\|_2 \leq \|\tilde{A}\|_{\mathrm{F}} \qquad \text{for all } s \geq L\,.$$

Since $\tilde{\sigma}_1(s) \ \sim \ \|\tilde{A}(s)\|_2$, $s\to\infty$ and since the mapping $\mathcal{R}$ preserves the order, this leads to $\|A\|_\oplus \leq \sigma_1 \leq \|A\|_\oplus$ and consequently, $\sigma_1 = \|A\|_\oplus$.

Case 2: Not all the a_{ij}'s have a finite max-absolute value.
First we construct a sequence $\{A_l\}_{l=1}^{\infty}$ of m by n matrices such that

$$(A_l)_{ij} = \begin{cases} a_{ij} & \text{if } |a_{ij}|_\oplus \neq \varepsilon\,, \\ \|A\|_\oplus - l & \text{if } |a_{ij}|_\oplus = \varepsilon\,, \end{cases}$$

for all i,j. So the entries of the matrices A_l are finite and $\|A\|_\oplus = \|A_l\|_\oplus$ for all $l \in \mathbb{N}_0$. Furthermore, $\lim\limits_{l\to\infty} A_l = A$.

Now we construct the sequence $\{\tilde{A}_l\}_{l=1}^{\infty}$ with $\tilde{A}_l = \mathcal{F}(A_l, M, \cdot)$ for $l = 1, 2, 3, \ldots$ with $M \in \mathbb{R}^{m \times n}$ and $m_{ij} = 1$ for all i, j. We compute a path of approximate SVDs $\tilde{U}_l \, \tilde{\Sigma}_l \, \tilde{V}_l^T$ of each $\tilde{A}_l$ using the method of Case 1 of this proof.

In general, it is possible that for some of the entries of the $\tilde{U}_l$'s and the $\tilde{V}_l$'s the sequence of the dominant exponents and the sequence of the corresponding coefficients have more than one accumulation point (since if two or more singular values coincide the corresponding left and right singular vectors are not uniquely defined). However, since we use a fixed computation scheme (the row-cyclic Kogbetliantz algorithm), all the sequences will have exactly one accumulation point. So some of the dominant exponents will reach a finite limit as l goes to ∞, while the other dominant exponents will tend to $-\infty$. If we take the reverse mapping $\mathcal{R}$, we get a sequence of max-algebraic SVDs $\{U_l \otimes \Sigma_l \otimes V_l^T\}_{l=1}^{\infty}$ where some of the entries, viz. those that correspond to dominant exponents that tend to $-\infty$, tend to ε as l goes to ∞.

If we define

$$U = \lim_{l \to \infty} U_l, \;\; \Sigma = \lim_{l \to \infty} \Sigma_l \;\text{ and }\; V = \lim_{l \to \infty} V_l$$

then we have

$$A \,\nabla\, U \otimes \Sigma \otimes V^T, \;\; U^T \otimes U \,\nabla\, E_m \;\text{ and }\; V^T \otimes V \,\nabla\, E_n \;\; .$$

Since the diagonal entries of all the Σ_l's belong to $\mathbb{R}_\varepsilon$ and are ordered, the diagonal entries of Σ also belong to $\mathbb{R}_\varepsilon$ and are also ordered. Furthermore, $(\Sigma)_{11} = \|A\|_\oplus$ since $(\Sigma_l)_{11} = \|A\|_\oplus$ for all l. Hence, $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A. □

REMARK 5.8. Now we explain why we have distinguished between two different cases in the proof of Theorem 5.7.

If there exist indices i and j such that $a_{ij} = \varepsilon$ then $\tilde{a}_{ij}(s) = 0$ for all $s \in \mathbb{R}_0^+$, which means that we cannot guarantee that condition (33) will be satisfied after a finite number of sweeps. This is why we make a distinction between the case where all the entries of A are finite and the case where at least one entry of A is equal to ε.

Let us now show that we do not have to take special precautions if $\tilde{A}$ has singular values that are identically zero in the neighborhood of ∞. If $\tilde{\Psi}$ is a real matrix-valued function that is analytic in some interval $J \subseteq \mathbb{R}$ then the rank of $\tilde{\Psi}$ is constant in J except in some isolated points where the rank drops [13]. If the rank of $\tilde{\Psi}(s)$ is equal to ρ for all $s \in J$ except for some isolated points then we say that the *generic rank* of $\tilde{\Psi}$ in J is equal to ρ. The entries of all the matrix-valued functions created in the row-cyclic Kogbetliantz algorithm when applied to $\tilde{A}$ are real and analytic in some interval $[L^*, \infty)$. Furthermore, for a fixed value of s the matrices $\tilde{A}(s)$, $\tilde{R}(s)$, $\tilde{S}_1(s)$, $\tilde{S}_2(s)$, $\ldots$ all have the same rank since they are related by orthogonal transformations. So if ρ is the generic rank of $\tilde{A}$ in $[L^*, \infty)$ then the generic rank of $\tilde{R}$, $\tilde{S}_1$, $\tilde{S}_2$, $\ldots$ in $[L^*, \infty)$ is also equal to ρ. If $\rho < n$ then the $n - \rho$ smallest singular values of $\tilde{R}$ will be identically zero in $[L^*, \infty)$. However, since $\tilde{R}$, $\tilde{S}_N$, $\tilde{S}_{2N}$, $\ldots$ are triangular matrices, they have at least $n - \rho$ diagonal entries that are identically zero in $[L^*, \infty)$ since otherwise their generic rank would be greater than ρ. In fact this also holds for $\tilde{S}_1$, $\tilde{S}_2$, $\ldots$ since these matrix-valued functions are hierarchically triangular, i.e., block triangular such that the diagonal blocks are again block triangular, etc. [17]. Furthermore, if k is large enough, diagonal entries do not change their affiliation any more, i.e., if a diagonal entry corresponds to a specific singular value in the kth iteration then it will also correspond to that singular value in all the next iterations. Since the diagonal entries of $\tilde{S}_k$ are asymptotically equivalent to an exponential in the neighborhood of ∞, this means that at least $n-\rho$ diagonal entries (with a fixed position) of $\tilde{S}_k$, $\tilde{S}_{k+1}$, ... will be identically zero in some interval $[L,\infty) \subseteq [L^*,\infty)$ if k is large enough. Hence, we do not have to take special precautions if $\tilde{A}$ has singular values that are identically zero in the neighborhood of ∞ since convergence to these singular values in a finite number of iteration steps is guaranteed.

For inner products of two different columns of $\tilde{U}$ there are no problems either: these inner products are equal to 0 by construction since the matrix-valued function $\tilde{U}_k$ is orthogonal on $[L,\infty)$ for all $k \in \mathbb{N}$. This also holds for inner products of two different columns of $\tilde{V}$. $\diamond$

If $U\Sigma V^T$ is an SVD of a matrix $A \in \mathbb{R}^{m\times n}$ then we have $\sigma_1 = (\Sigma)_{11} = \|A\|_2$. However, in $\mathbb{S}_{\max}$ the balances $A \,\nabla\, U \otimes \Sigma \otimes V^T$, $U^T \otimes U \,\nabla\, E_m$ and $V^T \otimes V \,\nabla\, E_n$ where Σ is a diagonal matrix with entries in $\mathbb{R}_\varepsilon$ and where the entries of U and V are signed do not always imply that $(\Sigma)_{11} = \|A\|_\oplus$ [10]. Therefore, we have included the extra condition $\sigma_1 = \|A\|_\oplus$ in the definition of the max-algebraic SVD.

Using a reasoning that is similar to the one that has been used at the end of §5.2 we can show that only permuted max-algebraic diagonal matrices with entries in $\mathbb{R}_\varepsilon$ have a max-algebraic SVD with entries in $\mathbb{R}_\varepsilon$ [7, 10].

For properties of the max-algebraic SVD and for a possible application of this decomposition in a method to solve the identification problem for max-linear DESs the interested reader is referred to [7, 10]. In [7] we have also proposed some possible extensions of the definitions of the max-algebraic QR decomposition and the max-algebraic singular value decomposition.

The proof technique that has been used in this section essentially consists in applying an algorithm from linear algebra to a matrix with entries in $\mathcal{S}_\mathrm{e}$. This proof technique can also be used to prove the existence of many other max-algebraic matrix decompositions: it can easily be adapted to prove the existence of a max-algebraic eigenvalue decomposition for symmetric matrices (by using the Jacobi algorithm for the computation of the eigenvalue decomposition of a real symmetric matrix), a max-algebraic LU decomposition, a max-algebraic Schur decomposition, a max-algebraic Hessenberg decomposition and so on.

6. A worked example of the max-algebraic QR decomposition and the max-algebraic singular value decomposition. Now we give an example of the computation of a max-algebraic QR decomposition and a max-algebraic singular value decomposition of a matrix using the mapping $\mathcal{F}$.

EXAMPLE 6.1. Consider the matrix

$$A = \begin{bmatrix} \ominus 0 & 3^\bullet & \ominus(-1) \\ 1 & \ominus(-2) & \varepsilon \end{bmatrix} .$$

Let us first compute a max-algebraic QR decomposition of A using the mapping $\mathcal{F}$. Let $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and define $\tilde{A} = \mathcal{F}(A, M, \cdot)$. Hence,

$$\tilde{A}(s) = \begin{bmatrix} -1 & e^{3s} & -e^{-s} \\ e^{s} & -e^{-2s} & 0 \end{bmatrix} \qquad \text{for all } s \in \mathbb{R}_0^+ .$$

If we use the Givens QR algorithm, we get a path of QR decompositions $\tilde{Q}\tilde{R}$ of $\tilde{A}$ with

$$\tilde{Q}(s) = \begin{bmatrix} \dfrac{-e^{-s}}{\sqrt{1+e^{-2s}}} & \dfrac{-1}{\sqrt{1+e^{-2s}}} \\ \dfrac{1}{\sqrt{1+e^{-2s}}} & \dfrac{-e^{-s}}{\sqrt{1+e^{-2s}}} \end{bmatrix}$$

$$\tilde{R}(s) = \begin{bmatrix} e^{s}\sqrt{1+e^{-2s}} & \dfrac{-e^{2s}-e^{-2s}}{\sqrt{1+e^{-2s}}} & \dfrac{e^{-2s}}{\sqrt{1+e^{-2s}}} \\ 0 & \dfrac{-e^{3s}+e^{-3s}}{\sqrt{1+e^{-2s}}} & \dfrac{e^{-s}}{\sqrt{1+e^{-2s}}} \end{bmatrix}$$

for all $s \in \mathbb{R}_0^+$. Hence,

$$\tilde{Q}(s) \sim \begin{bmatrix} -e^{-s} & -1 \\ 1 & -e^{-s} \end{bmatrix}, \ s \to \infty$$

$$\tilde{R}(s) \sim \begin{bmatrix} e^{s} & -e^{2s} & e^{-2s} \\ 0 & -e^{3s} & e^{-s} \end{bmatrix}, \ s \to \infty \ .$$

If we define $Q = \mathcal{R}(\tilde{Q})$ and $R = \mathcal{R}(\tilde{R})$, we obtain

$$Q = \begin{bmatrix} \ominus(-1) & \ominus 0 \\ 0 & \ominus(-1) \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & \ominus 2 & -2 \\ \varepsilon & \ominus 3 & -1 \end{bmatrix} \ .$$

We have

$$Q \otimes R = \begin{bmatrix} \ominus 0 & 3 & \ominus(-1) \\ 1 & 2^{\bullet} & (-2)^{\bullet} \end{bmatrix} \nabla A$$

$$Q^T \otimes Q = \begin{bmatrix} 0 & (-1)^{\bullet} \\ (-1)^{\bullet} & 0 \end{bmatrix} \nabla E_2$$

and $\|R\|_{\oplus} = 3 = \|A\|_{\oplus}$.

Let us now compute a max-algebraic SVD of A. Since $\tilde{A}$ is a 2 by 3 matrix-valued function, we can compute a path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$ analytically, e.g., via the eigenvalue decomposition of $\tilde{A}^T\tilde{A}$ (see [14, 32]). This yields[1]

$$\tilde{U}(s) \sim \begin{bmatrix} 1 & 2\,e^{-5s} \\ -2\,e^{-5s} & 1 \end{bmatrix}, \ s \to \infty$$

$$\tilde{\Sigma}(s) \sim \begin{bmatrix} e^{3s} & 0 & 0 \\ 0 & e^{s} & 0 \end{bmatrix}, \ s \to \infty$$

$$\tilde{V}(s) \sim \begin{bmatrix} -e^{-3s} & 1 & e^{-7s} \\ 1 & e^{-3s} & e^{-4s} \\ -e^{-4s} & -2\,e^{-7s} & 1 \end{bmatrix}, \ s \to \infty \ .$$

[1] We have used the symbolic computation tool MAPLE to compute a path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$. However, since the full expressions for the entries of $\tilde{U}$, $\tilde{S}$ and $\tilde{V}$ are too long and too intricate to display here, we only give the dominant exponentials.

If we apply the reverse mapping $\mathcal{R}$, we get a max-algebraic SVD $U \otimes \Sigma \otimes V^T$ of A with

$$U = \mathcal{R}(\tilde{U}) = \begin{bmatrix} 0 & -5 \\ \ominus(-5) & 0 \end{bmatrix}$$

$$\Sigma = \mathcal{R}(\tilde{\Sigma}) = \begin{bmatrix} 3 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \end{bmatrix}$$

$$V = \mathcal{R}(\tilde{V}) = \begin{bmatrix} \ominus(-3) & 0 & -7 \\ 0 & -3 & -4 \\ \ominus(-4) & \ominus(-7) & 0 \end{bmatrix} .$$

We have

$$U \otimes \Sigma \otimes V^T = \begin{bmatrix} \ominus 0 & 3 & \ominus(-1) \\ 1 & (-2)^\bullet & (-6)^\bullet \end{bmatrix} \nabla A$$

$$U^T \otimes U = \begin{bmatrix} 0 & (-5)^\bullet \\ (-5)^\bullet & 0 \end{bmatrix} \nabla E_2$$

$$V^T \otimes V = \begin{bmatrix} 0 & (-3)^\bullet & (-4)^\bullet \\ (-3)^\bullet & 0 & (-7)^\bullet \\ (-4)^\bullet & (-7)^\bullet & 0 \end{bmatrix} \nabla E_3$$

and $\sigma_1 = 3 = \|A\|_\oplus \geq 1 = \sigma_2$. $\diamond$

Another example of the computation of a max-algebraic SVD can be found in [7, 10].

REMARK 6.2. In [7] we have shown that the max-algebraic QR decomposition and the max-algebraic SVD of a matrix can also be computed by solving an Extended Linear Complementarity Problem — which is a kind of mathematical programming problem. Although it would lead us too far to explain this procedure in detail, we shall now give a brief outline of how the equations that appear in the definition of the max-algebraic QR decomposition and the max-algebraic SVD can be transformed into a system of multivariate max-algebraic polynomial equalities.

Consider the equation $A \nabla Q \otimes R$. If we extract the max-positive and the max-negative parts of each matrix, we obtain

$$A^\oplus \ominus A^\ominus \ \nabla \ (Q^\oplus \ominus Q^\ominus) \otimes (R^\oplus \ominus R^\ominus)$$

or

$$A^\oplus \ominus A^\ominus \nabla Q^\oplus \otimes R^\oplus \ \ominus \ Q^\oplus \otimes R^\ominus \ \ominus \ Q^\ominus \otimes R^\oplus \ \oplus \ Q^\ominus \otimes R^\ominus \ .$$

By Proposition 3.5 this can be rewritten as

$$A^\oplus \ \oplus \ Q^\oplus \otimes R^\ominus \ \oplus \ Q^\ominus \otimes R^\oplus \nabla A^\ominus \ \oplus \ Q^\oplus \otimes R^\oplus \ \oplus \ Q^\ominus \otimes R^\ominus \ .$$

Both sides of this balance are signed. So by Proposition 3.6 we may replace the balance by an equality. If we introduce a matrix T of auxiliary variables, we obtain:

$$A^\oplus \ \oplus \ Q^\oplus \otimes R^\ominus \ \oplus \ Q^\ominus \otimes R^\oplus \ = \ T \tag{37}$$

$$A^\ominus \ \oplus \ Q^\oplus \otimes R^\oplus \ \oplus \ Q^\ominus \otimes R^\ominus \ = \ T \ . \tag{38}$$

If we write out the max-algebraic matrix multiplications in (37) and if we transfer the entries of T to the opposite side, we get

$$a_{ij}^{\oplus} \otimes {t_{ij}}^{\otimes^{-1}} \oplus \bigoplus_{k=1}^{m} q_{ik}^{\oplus} \otimes r_{kj}^{\ominus} \otimes {t_{ij}}^{\otimes^{-1}} \oplus$$
$$\bigoplus_{k=1}^{m} q_{ik}^{\ominus} \otimes r_{kj}^{\oplus} \otimes {t_{ij}}^{\otimes^{-1}} = 0 \qquad \text{for all } i,j \ . \tag{39}$$

Equation (38) can be rewritten in a similar way. The condition $Q^T \otimes Q \nabla E_m$ also leads to similar equations.
The condition that the entries of Q and R should be signed can be written as

$$q_{ij}^{\oplus} \otimes q_{ij}^{\ominus} = \varepsilon \qquad \text{for all } i,j \tag{40}$$
$$r_{ij}^{\oplus} \otimes r_{ij}^{\ominus} = \varepsilon \qquad \text{for all } i,j \ . \tag{41}$$

The condition $\|R\|_{\oplus} = \|A\|_{\oplus}$ is equivalent to

$$\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \left(r_{ij}^{\oplus} \oplus r_{ij}^{\ominus} \right) = \|A\|_{\oplus} \qquad \text{for all } i,j \ . \tag{42}$$

So if we combine all equations of the form (39) – (42), we obtain a system of multivariate max-algebraic polynomial equalities of the following form:

Given l integers $m_1,\ m_2,\ \ldots,\ m_l \in \mathbb{N}_0$ and real numbers $a_{ki},\ b_k$ and c_{kij} for $k = 1,2,\ldots,l,\ i = 1,2,\ldots,m_l$ and $j = 1,2,\ldots,r$, find $x \in \mathbb{R}_{\varepsilon}^{r}$ such that

$$\bigoplus_{i=1}^{m_l} a_{ki} \otimes \bigotimes_{j=1}^{r} {x_j}^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1,2,\ldots,l \, ,$$

or show that no such x exists;

where the vector x contains the max-positive and the max-negative parts of the entries of Q and R and the auxiliary variables.
Using a similar reasoning we can also show that the equations that appear in the definition of the max-algebraic SVD also lead to a system of multivariate max-algebraic polynomial equalities.
In [7, 9] we have shown that a system of multivariate max-algebraic polynomial equalities can be rewritten as a mathematical programming problem of the following form:

Given two matrices $A \in \mathbb{R}^{p \times r}$, $B \in \mathbb{R}^{q \times r}$, two vectors $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and s subsets $\phi_1,\ \phi_2,\ \ldots,\ \phi_s$ of $\{1,2,\ldots,p\}$, find $x \in \mathbb{R}^r$ such that

$$\sum_{j=1}^{s} \prod_{i \in \phi_j} (Ax - c)_i = 0$$

subject to $Ax \geq c$ and $Bx = d$, or show that no such x exists.

This problem is called an *Extended Linear Complementarity Problem* (ELCP). In [7, 8] we have developed an algorithm to find all solutions of a general ELCP. However, the execution time of this algorithm increases exponentially as the number of equations and variables of the ELCP increases. Furthermore, in [7, 8] we have shown that the general ELCP is an NP-hard problem. As a consequence, the ELCP approach can only be used to compute max-algebraic QR decompositions and max-algebraic SVDs of small-sized matrices. So there certainly is a need for efficient algorithms to compute max-algebraic QR decompositions and max-algebraic SVDs: this will be one of the most important topics for further research. An important question is whether we can develop efficient algorithms for special classes of matrices, e.g., is it possible to come up with more efficient algorithms by making use of the nonzero structure (sparsity, bandedness, ...) of the matrix? $\diamond$

7. Conclusions and future research. In this paper we have tried to fill one of the gaps in the theory of the (symmetrized) max-plus algebra by showing that there exist max-algebraic analogues of many fundamental matrix decompositions from linear algebra.

We have established a link between a ring of real functions (with addition and multiplication as basic operations) and the symmetrized max-plus algebra. Next we have introduced a class of functions that are analytic and that can be written as a sum or a series of exponentials in a neighborhood of ∞. This class is closed under basic operations such as additions, subtractions, multiplications, divisions, powers, square roots and absolute values. This fact has then been used to prove the existence of a QR decomposition and a singular value decomposition of a matrix in the symmetrized max-plus algebra. These decompositions are max-algebraic analogues of basic matrix decompositions from linear algebra. The proof technique that has been used to prove the existence of these max-algebraic matrix decompositions can also be used to prove the existence of max-algebraic analogues of other real matrix decompositions from linear algebra such as the LU decomposition, the Hessenberg decomposition, the eigenvalue decomposition (for symmetric matrices), the Schur decomposition and so on.

In [7, 10] we have introduced a further extension of the symmetrized max-plus algebra: the max-complex structure $\mathbb{T}_{\max}$, which corresponds to a ring of complex functions (with addition and multiplication as basic operations). We could also define max-algebraic matrix decompositions in $\mathbb{T}_{\max}$. These decompositions would then be analogues of matrix decompositions from linear algebra for complex matrices (such as the eigenvalue decomposition or the Jordan decomposition).

Topics for future research are: further investigation of the properties of the max-algebraic matrix decompositions that have been introduced in this paper, development of efficient algorithms to compute these max-algebraic matrix decompositions, investigation of the computational complexity of computing max-algebraic matrix decompositions (in general and for special classes of matrices) and application of the max-algebraic singular value decomposition and other max-algebraic matrix decompositions in the system theory for max-linear discrete event systems.

Appendix A. Proof of Lemma 5.2.

In this section we show that functions that belong to the class $\mathcal{S}_{\mathrm{e}}$ are asymptotically equivalent to an exponential in the neighborhood of ∞. We shall use the following lemma:

LEMMA A.1. *If $f \in \mathcal{S}_{\mathrm{e}}$ is a series with a nonpositive dominant exponent, i.e., if there exists a positive real number K such that $f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ for all $x \geq K$ with $\alpha_i \in \mathbb{R}$, $a_i \in \mathbb{R}^-$, $a_i > a_{i+1}$ for all i, $\lim_{i \to \infty} a_i = \varepsilon$ and where the series converges* *absolutely for every $x \geq K$, then the series $\sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ converges uniformly in $[K, \infty)$.*

Proof. If $x \geq K$ then we have $e^{a_i x} \leq e^{a_i K}$ for all $i \in \mathbb{N}$ since $a_i \leq 0$ for all i. Hence, $|\alpha_i e^{a_i x}| \leq |\alpha_i e^{a_i K}|$ for all $x \geq K$ and for all $i \in \mathbb{N}$. We already know that $\sum_{i=0}^{\infty} |\alpha_i e^{a_i K}|$ converges. Now we can apply the Weierstrass M-test (see [19, 24]). As a consequence, the series $\sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ converges uniformly in $[K, \infty)$. □

Proof (Proof of Lemma 5.2)
If $f \in \mathcal{S}_{\rm e}$ then there exists a positive real number K such that $f(x) = \sum_{i=0}^{n} \alpha_i e^{a_i x}$ for all $x \geq K$ with $n \in \mathbb{N} \cup \{\infty\}$, $\alpha_i \in \mathbb{R}_0$ and $a_i \in \mathbb{R}_\varepsilon$ for all i. If $n = \infty$ then f is a series that converges absolutely in $[K, \infty)$.
If $a_0 = \varepsilon$ then there exists a real number K such that $f(x) = 0$ for all $x \geq K$ and then we have $f(x) \sim 0 = 1 \cdot e^{\varepsilon x}$, $x \to \infty$ by Definition 2.2.
If $n = 1$ then $f(x) = \alpha_0 e^{a_0 x}$ and thus $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$ with $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}_\varepsilon$.
From now on we assume that $n > 1$ and $a_0 \neq \varepsilon$. Then we can rewrite $f(x)$ as

$$f(x) = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{n} \frac{\alpha_i}{\alpha_0}\ e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} (\, 1 + p(x)\,)$$

with $p(x) = \sum_{i=1}^{n} \gamma_i e^{c_i x}$ where $\gamma_i = \dfrac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0 < 0$ for all i. Note that $p \in \mathcal{S}_{\rm e}$ and that p has a negative dominant exponent. Since $c_i < 0$ for all i, we have

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} \left(\sum_{i=1}^{n} \gamma_i e^{c_i x} \right) = \sum_{i=1}^{n} \left(\lim_{x \to \infty} \gamma_i e^{c_i x} \right) = 0 \ . \tag{43}$$

If $n = \infty$ then the series $\sum_{i=1}^{\infty} \gamma_i e^{c_i x}$ converges uniformly in $[K, \infty)$ by Lemma A.1. As a consequence, we may also interchange the summation and the limit in (43) if $n = \infty$ (cf. [19]).
Now we have

$$\lim_{x \to \infty} \frac{f(x)}{\alpha_0 e^{a_0 x}} = \lim_{x \to \infty} \frac{\alpha_0 e^{a_0 x}(1 + p(x))}{\alpha_0 e^{a_0 x}} = \lim_{x \to \infty} (1 + p(x)) = 1$$

and thus $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$ where $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}$. □

Appendix B. Proof of Proposition 5.3.

In this section we show that $\mathcal{S}_{\rm e}$ is closed under elementary operations such as additions, multiplications, subtractions, divisions, square roots and absolute values.

Proof (Proof of Proposition 5.3)
If f and g belong to $\mathcal{S}_{\rm e}$ then we may assume without loss of generality that the domains of definition of f and g coincide, since we can always restrict the functions

f and g to $\operatorname{dom} f \cap \operatorname{dom} g$ and since the restricted functions also belong to $\mathcal{S}_{\mathrm{e}}$.
Since f and g belong to $\mathcal{S}_{\mathrm{e}}$, there exists a positive real number K such that

$$f(x) = \sum_{i=0}^{n} \alpha_i e^{a_i x} \text{ and } g(x) = \sum_{j=0}^{m} \beta_j e^{b_j x} \qquad \text{for all } x \geq K$$

with $m, n \in \mathbb{N} \cup \{\infty\}$, $\alpha_i, \beta_j \in \mathbb{R}_0$ and $a_i, b_j \in \mathbb{R}_\varepsilon$ for all i, j. If m or n is equal to ∞ then the corresponding series converges absolutely in $[K, \infty)$.
We may assume without loss of generality that both m and n are equal to ∞. If m or n are finite then we can always add dummy terms of the form $0 \cdot e^{\varepsilon x}$ to $f(x)$ or $g(x)$. Afterwards we can remove terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$ to obtain an expression with nonzero coefficients and decreasing exponents. So from now on we assume that both f and g are absolute convergent series of exponentials.

If $a_0 = \varepsilon$ then we have $f(x) = 0$ for all $x \geq K$, which means that $|f(x)| = 0$ for all $x \geq K$. So if $a_0 = \varepsilon$ then $|f|$ belongs to $\mathcal{S}_{\mathrm{e}}$.
If $a_0 \neq \varepsilon$ then there exists a real number $L \geq K$ such that either $f(x) > 0$ or $f(x) < 0$ for all $x \geq L$ since $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$ with $\alpha_0 \neq 0$ by Lemma 5.2. Hence, either $|f(x)| = f(x)$ or $|f(x)| = -f(x)$ for all $x \geq L$. So in this case $|f|$ also belongs to $\mathcal{S}_{\mathrm{e}}$.

Since f and g are analytic in $[K, \infty)$, the functions ρf, $f + g$, $f - g$, $f \cdot g$ and f^l are also analytic in $[K, \infty)$ for any $\rho \in \mathbb{R}$ and any $l \in \mathbb{N}$.

Now we prove that these functions can be written as a sum of exponentials or as an absolutely convergent series of exponentials.
Consider an arbitrary $\rho \in \mathbb{R}$. If $\rho = 0$ then $\rho f(x) = 0$ for all $x \geq K$ and thus $\rho f \in \mathcal{S}_{\mathrm{e}}$.
If $\rho \neq 0$ then we have $\rho f(x) = \sum_{i=0}^{\infty} (\rho \alpha_i) e^{a_i x}$. The series $\sum_{i=0}^{\infty} (\rho \alpha_i) e^{a_i x}$ also converges absolutely in $[K, \infty)$ and has the same exponents as $f(x)$. Hence, $\rho f \in \mathcal{S}_{\mathrm{e}}$.
The sum function $f + g$ is a series of exponentials since

$$f(x) + g(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} + \sum_{j=0}^{\infty} \beta_j e^{b_j x} \quad .$$

Furthermore, this series converges absolutely for every $x \geq K$. Therefore, the sum of the series does not change if we rearrange the terms [19]. So $f(x) + g(x)$ can be written in the form of Definition 5.1 by reordering the terms, adding up terms with equal exponents and removing terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$, if there are any. If the result is a series then the sequence of exponents is decreasing and unbounded from below. So $f + g \in \mathcal{S}_{\mathrm{e}}$.
Since $f - g = f + (-1)g$, the function $f - g$ also belongs to $\mathcal{S}_{\mathrm{e}}$.
The series corresponding to f and g converge absolutely for every $x \geq K$. Therefore, their Cauchy product will also converge absolutely for every $x \geq K$ and it will be equal to fg [19]:

$$f(x)g(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \alpha_j \beta_{i-j} e^{(a_j + b_{i-j})x} \qquad \text{for all } x \geq K \, .$$

Using the same procedure as for $f + g$, we can also write this product in the form (24) or (25). So $fg \in \mathcal{S}_{\mathrm{e}}$.

Let $l \in \mathbb{N}$. If $l = 0$ then $f^l = 0 \in \mathcal{S}_{\rm e}$ and if $l = 1$ then $f^l = f \in \mathcal{S}_{\rm e}$. If $l > 1$, we can make repeated use of the fact that $fg \in \mathcal{S}_{\rm e}$ if $f, g \in \mathcal{S}_{\rm e}$ to prove that f^l also belongs to $\mathcal{S}_{\rm e}$.

If there exists a real number P such that $f(x) \neq 0$ for all $x \geq P$ then $\dfrac{1}{f}$ and $\dfrac{g}{f}$ are defined and analytic in $[P, \infty)$. Note that we may assume without loss of generality that $P \geq K$. Furthermore, since the function f restricted to the interval $[P, \infty)$ also belongs to $\mathcal{S}_{\rm e}$, we may assume without loss of generality that the domain of definition of f is $[P, \infty)$.

If $f(x) \neq 0$ for all $x \geq P$ then we have $a_0 \neq \varepsilon$. As a consequence, we can rewrite $f(x)$ as

$$f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{\infty} \frac{\alpha_i}{\alpha_0} \; e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} (\, 1 + p(x) \,)$$

with $p(x) = \displaystyle\sum_{i=1}^{\infty} \gamma_i e^{c_i x}$ where $\gamma_i = \dfrac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0 < 0$ for all i. Note that p is defined in $[P, \infty)$, that $p \in \mathcal{S}_{\rm e}$ and that p has a negative dominant exponent.

If $c_1 = \varepsilon$ then $p(x) = 0$ and $\dfrac{1}{f(x)} = \dfrac{1}{\alpha_0} \, e^{-a_0 x}$ for all $x \geq P$. Hence, $\dfrac{1}{f} \in \mathcal{S}_{\rm e}$.

Now assume that $c_1 \neq \varepsilon$. Since $\{c_i\}_{i=1}^{\infty}$ is a non-increasing sequence of negative numbers with $\displaystyle\lim_{i \to \infty} c_i = \varepsilon = -\infty$ and since p converges uniformly in $[P, \infty)$ by Lemma A.1, we have $\displaystyle\lim_{x \to \infty} p(x) = 0$ (cf. (43)). So $|\, p(x) \,|$ will be less than 1 if x is large enough, say if $x \geq M$. If we use the Taylor series expansion of $\dfrac{1}{1+x}$, we obtain

$$\frac{1}{1 + p(x)} = \sum_{k=0}^{\infty} (-1)^k p^k(x) \qquad \text{if } |p(x)| < 1 \,. \tag{44}$$

We already know that $p \in \mathcal{S}_{\rm e}$. Hence, $p^k \in \mathcal{S}_{\rm e}$ for all $k \in \mathbb{N}$. We have $|p(x)| < 1$ for all $x \geq M$. Moreover, for any $k \in \mathbb{N}$ the highest exponent in p^k is equal to kc_1, which implies that the dominant exponent of p^k tends to $-\infty$ as k tends to ∞. As a consequence, the coefficients and the exponents of more and more successive terms of the partial sum function s_n that is defined by $s_n(x) = \displaystyle\sum_{k=0}^{n} (-1)^k p^k(x)$ for $x \in [M, \infty)$ will not change any more as n becomes larger and larger. Therefore, the series on the right-hand side of (44) also is a sum of exponentials:

$$\frac{1}{1 + p(x)} = \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=1}^{\infty} \gamma_i e^{c_i x} \right)^k = \sum_{k=0}^{\infty} d_i e^{\delta_i x} \qquad \text{for all } x \geq M \,.$$

Note that the set of exponents of this series will have no finite accumulation point since the highest exponent in p^k is equal to kc_1. Let us now prove that this series also converges absolutely. Define $p^*(x) = \displaystyle\sum_{i=1}^{\infty} |\gamma_i| e^{c_i x}$ for all $x \geq P$. Since the terms of the series p^* are the absolute values of the terms of the series p and since p converges absolutely in $[P, \infty)$, p^* also converges absolutely in $[P, \infty)$. By Lemma A.1 the series p^* also converges uniformly in $[P, \infty)$. Furthermore, $\{c_i\}_{i=1}^{\infty}$ is a non-increasing and unbounded sequence of negative numbers. As a consequence, we have $\lim_{x\to\infty} p^*(x) = 0$ (cf. (43)). So $|\, p^*(x)\,|$ will be less than 1 if x is large enough, say if $x \geq N$. Therefore, we have

$$\frac{1}{1+p^*(x)} = \sum_{k=0}^{\infty} (-1)^k \left(p^*(x)\right)^k \qquad \text{for all } x \geq N\,.$$

This series converges absolutely in $[N, \infty)$. Since

$$\sum_{k=0}^{\infty} |d_i| e^{\delta_i x} \leq \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} |\gamma_i| e^{c_i x} \right)^k = \sum_{k=0}^{\infty} \left| \left(p^*(x)\right)^k \right| \ ,$$

the series $\sum_{k=0}^{\infty} d_i e^{\delta_i x}$ also converges absolutely for any $x \in [N, \infty)$. Since this series converges absolutely, we can reorder the terms. After reordering the terms, adding up terms with the same exponents and removing terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$ if necessary, the sequence of exponents will be decreasing and unbounded from below. This implies that $\frac{1}{1+p} \in \mathcal{S}_{\rm e}$ and thus also $\frac{1}{f} \in \mathcal{S}_{\rm e}$.

As a consequence, $\frac{g}{f} = g\frac{1}{f}$ also belongs to $\mathcal{S}_{\rm e}$.

If there exists a real number Q such that $f(x) > 0$ for all $x \geq Q$ then the function $\sqrt{f}$ is defined and analytic in $[Q, \infty)$. We may assume without loss of generality that $Q \geq K$ and that the domain of definition of f is $[Q, \infty)$.

If $a_0 = \varepsilon$ then we have $\sqrt{f(x)} = 0$ for all $x \geq Q$ and thus $\sqrt{f} \in \mathcal{S}_{\rm e}$.

If $a_0 \neq \varepsilon$ then $\alpha_0 > 0$ and then we can rewrite $\sqrt{f(x)}$ as

$$\sqrt{f(x)} = \sqrt{\alpha_0}\ e^{\frac{1}{2}a_0 x} \sqrt{1+p(x)}\ .$$

Now we can use the Taylor series expansion of $\sqrt{1+x}$. This leads to

$$\sqrt{1+p(x)} = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-k\right)\, k!}\ p^k(x) \qquad \text{if } |p(x)| < 1\,,$$

where Γ is the gamma function. If we apply the same reasoning as for $\frac{1}{1+p}$, we find that $\sqrt{1+p} \in \mathcal{S}_{\rm e}$ and thus also $\sqrt{f} \in \mathcal{S}_{\rm e}$. ☐

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