K.U.Leuven

Department of Electrical Engineering (ESAT) **SISTA**

Technical report 96-70

The QR decomposition and the singular value decomposition in the symmetrized max-plus algebra*

B. De Schutter and B. De Moor

If you want to cite this report, please use the following reference instead:
B. De Schutter and B. De Moor, "The QR decomposition and the singular value decomposition in the symmetrized max-plus algebra," *Proceedings of the European Control Conference (ECC'97)*, Brussels, Belgium, 6 pp., July 1997. Paper 295 / TH-E K6.

ESAT-SISTA
K.U.Leuven
Leuven, Belgium
phone: +32-16-32.17.09 (secretary)
fax: +32-16-32.19.70
URL: `http://www.esat.kuleuven.ac.be/sista-cosic-docarch`

*This report can also be downloaded via `http://pub.deschutter.info/abs/96_70.html`

THE QR DECOMPOSITION AND THE SINGULAR VALUE DECOMPOSITION IN THE SYMMETRIZED MAX-PLUS ALGEBRA

B. De Schutter, B. De Moor

ESAT-SISTA, Kardinaal Mercierlaan 94, B-3001 Heverlee (Leuven), Belgium
tel.: +32-16-32.17.99, fax: +32-16-32.19.70
bart.deschutter@esat.kuleuven.ac.be, bart.demoor@esat.kuleuven.ac.be

Keywords : discrete event systems, max-plus algebra, matrix decompositions.

Abstract

The max-plus algebra has maximization and addition as basic operations, and can be used to model a certain class of discrete event systems. In contrast to linear algebra and linear system theory many fundamental problems in the max-plus algebra and in max-plus-algebraic system theory still need to be solved. In this paper we discuss max-plus-algebraic analogues of some basic matrix decompositions from linear algebra that play an important role in linear system theory. We use algorithms from linear algebra to prove the existence of max-plus-algebraic analogues of the QR decomposition and the singular value decomposition.

1 Introduction

In the last decades both industry and the academic world have become more and more interested in techniques to model, analyze and control complex systems such as flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems, logistic systems and so on. This kind of systems are typical examples of *discrete event systems*. Although in general discrete event systems lead to a non-linear description in conventional algebra, there exists a subclass of discrete event systems for which this model becomes "linear" when we formulate it in the max-plus algebra [1, 2]. Discrete event systems that belong to this subclass are called max-linear discrete event systems.

The basic operations of the max-plus algebra are maximization and addition. There exists a remarkable analogy between these operations and the basic operations of conventional algebra (addition and multiplication). This analogy allows us to translate many concepts, properties and techniques from linear algebra and linear system theory to the max-plus algebra and system theory for max-linear discrete event systems. However, there are also some major differences that prevent a straightforward translation of properties, concepts and algorithms from linear algebra and linear system theory to the max-plus algebra and max-plus-algebraic system theory. The max-plus-algebraic system theory for discrete event systems is at present far from fully developed and much research on this topic is still needed in order to get a complete system theory. The main goal of this paper is to fill one of the gaps in the theory of the max-plus algebra by showing that there exist max-plus-algebraic analogues of many fundamental matrix decompositions from linear algebra such as, e.g., the QR decomposition and the singular value decomposition. These matrix decompositions are important tools in many linear algebra algorithms and in many contemporary algorithms for the identification of linear systems (see, e.g., [8, 10] and the references cited therein).

2 Notations and definitions

A matrix $A \in \mathbb{R}^{n \times n}$ is called orthogonal if $A^T A = I_n$, where I_n is the n by n identity matrix. The Frobenius norm of a matrix $A \in \mathbb{R}^{m \times n}$ is defined by

$$\|A\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} \ .$$

If $A \in \mathbb{R}^{m \times n}$, then there exist an orthogonal matrix $Q \in \mathbb{R}^{m \times m}$ and an upper triangular matrix $R \in \mathbb{R}^{m \times n}$ such that $A = QR$. We say that QR is a *QR decomposition* (QRD) of A.

Let $A \in \mathbb{R}^{m \times n}$ and let $r = \min(m, n)$. Then there exist a diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ and two orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that $A = U \Sigma V^T$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$ where $\sigma_i = (\Sigma)_{ii}$ for $i = 1, 2, \ldots, r$. This factorization is called a *singular value decomposition* (SVD) of A.

For more information on the QRD and the SVD the interested reader is referred to, e.g., [6].

Definition 2.1 (Asymptotic equivalence)
Let f and g be real functions such that ∞ is an accumulation point of dom f *and* dom g.

If there does not exist a real number K such that g is identically zero in $[K, \infty)$ then we say that f is asymptotically equivalent to g in the neighborhood of ∞, denoted by $f(x) \sim g(x)$, $x \to \infty$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

We say that $f(x) \sim 0$, $x \to \infty$ if there exists a real number L such that $f(x) = 0$ for all $x \geqslant L$.

Let $\tilde{F}$ and $\tilde{G}$ be real m by n matrix-valued functions such that ∞ is an accumulation point of dom $\tilde{F}$ *and* dom $\tilde{G}$. *Then $\tilde{F}(x) \sim \tilde{G}(x)$, $x \to \infty$ if $\tilde{f}_{ij}(x) \sim \tilde{g}_{ij}(x)$, $x \to \infty$ for all i, j.*

3 The max-plus algebra and the symmetrized max-plus algebra

In this section we give a short introduction to the max-plus algebra and the symmetrized max-plus algebra. An extensive treatment of the (symmetrized) max-plus algebra can be found in [1, 2, 5].

3.1 The max-plus algebra

The basic max-plus-algebraic operations are defined as follows:

$$\begin{aligned} x \oplus y &= \max(x, y) \\ x \otimes y &= x + y \end{aligned}$$

for $x, y \in \mathbb{R} \cup \{-\infty\}$. There is a remarkable analogy between $\oplus$ and addition, and between $\otimes$ and multiplication: many concepts and properties from conventional linear algebra can be translated to the (symmetrized) max-plus algebra by replacing $+$ by $\oplus$ and $\times$ by $\otimes$. The algebraic structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the *max-plus algebra*. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$. The zero element for $\oplus$ in $\mathbb{R}_\varepsilon$ is represented by $\varepsilon \stackrel{\text{def}}{=} -\infty$.

The matrix E_n is the n by n max-plus-algebraic identity matrix: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The off-diagonal entries of a max-plus-algebraic diagonal matrix $D \in \mathbb{R}_\varepsilon^{m\times n}$ are equal to ε. A matrix $R \in \mathbb{R}_\varepsilon^{m\times n}$ is a max-plus-algebraic upper triangular matrix if $r_{ij} = \varepsilon$ for all i, j with $i > j$.

If $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ then $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ for all i, j. If $A \in \mathbb{R}_\varepsilon^{m\times p}$ and $B \in \mathbb{R}_\varepsilon^{p\times n}$ then $(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$ for all i, j.

3.2 The symmetrized max-plus algebra

One of the major differences between conventional algebra and the max-plus algebra is that there exist no inverse elements w.r.t. $\oplus$ in $\mathbb{R}_\varepsilon$. Therefore, we now introduce $\mathbb{S}_{\max}$ [1, 5], which is a kind of symmetrization of the max-plus algebra. We shall restrict ourselves to a short introduction to the most important features of $\mathbb{S}_{\max}$.

For each $x \in \mathbb{R}_\varepsilon$ we introduce two new elements: $\ominus x$ and $x^\bullet$. This leads to an extension $\mathbb{S}$ of $\mathbb{R}_\varepsilon$ that contains three classes of elements:

- the max-positive or zero elements: $\mathbb{S}^\oplus \equiv \mathbb{R}_\varepsilon$,
- the max-negative or zero elements: $\mathbb{S}^\ominus = \{ \ominus a \mid a \in \mathbb{R}_\varepsilon \}$,
- the balanced elements: $\mathbb{S}^\bullet = \{ a^\bullet \mid a \in \mathbb{R}_\varepsilon \}$,

with $\mathbb{S} = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$. By definition we have $\varepsilon = \ominus\varepsilon = \varepsilon^\bullet$. The elements of $\mathbb{S}^\oplus$ and $\mathbb{S}^\ominus$ are called *signed*.

The $\oplus$ operation between an element of $\mathbb{S}^\oplus$ and an element of $\mathbb{S}^\ominus$ is defined as follows:

$$\begin{aligned} a \oplus (\ominus b) &= a && \text{if } a > b \\ a \oplus (\ominus b) &= \ominus b && \text{if } a < b \\ a \oplus (\ominus a) &= a^\bullet && \end{aligned}$$

where a, $b \in \mathbb{R}_\varepsilon$. The $\ominus$ operator can be considered as the analogue of the $-$ operator of conventional algebra. The structure $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$ is called the *symmetrized max-plus algebra*.

Let $a \in \mathbb{S}$. The max-positive part $a^\oplus$ and the max-negative part $a^\ominus$ of a are defined as follows:

- if $a \in \mathbb{R}_\varepsilon$ then $a^\oplus = a$ and $a^\ominus = \varepsilon$,
- if $a \in \mathbb{S}^\ominus$ then $a = \ominus b$ for some $b \in \mathbb{R}_\varepsilon$ and then we have $a^\oplus = \varepsilon$ and $a^\ominus = b$,
- if $a \in \mathbb{S}^\bullet$ then $a = c^\bullet$ for some $c \in \mathbb{R}_\varepsilon$ and then we have $a^\oplus = a^\ominus = c$.

So $a^\oplus, a^\ominus \in \mathbb{R}_\varepsilon$ and $a = a^\oplus \oplus (\ominus a^\ominus)$. The max-absolute value of $a \in \mathbb{S}$ is defined by $|a|_\oplus = a^\oplus \oplus a^\ominus$.

Definition 3.1 (Balance relation) *Consider $a, b \in \mathbb{S}$. We say that a balances b, denoted by $a \,\nabla\, b$, if $a^\oplus \oplus b^\ominus = b^\oplus \oplus a^\ominus$.*

We could say that the balance relation is the $\mathbb{S}_{\max}$ counterpart of the equality relation. However, the balance relation is not an equivalence relation, since it is not transitive.

An element with an $\ominus$ sign in a balance can be moved to the other side as follows:

Proposition 3.2 $\forall a, b, c \in \mathbb{S} : a \oplus (\ominus b) \,\nabla\, c$ *if and only if* $a \,\nabla\, b \oplus c$.

If both sides of a balance are signed, we can replace the balance by an equality:

Proposition 3.3 $\forall a, b \in \mathbb{S}^\oplus \cup \mathbb{S}^\ominus : a \,\nabla\, b \Rightarrow a = b$.

$$a \oplus b = c \quad \Rightarrow \quad \begin{cases} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \qquad \mathcal{F}(a, \mu_a, s) + \mathcal{F}(b, \mu_b, s) \sim \mathcal{F}(c, \mu_c, s),\ s \to \infty \end{cases} \tag{1}$$

$$\left.\begin{array}{l} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \qquad \mathcal{F}(a, \mu_a, s) + \mathcal{F}(b, \mu_b, s) \sim \mathcal{F}(c, \mu_c, s),\ s \to \infty \end{array}\right\} \quad \Rightarrow \quad a \oplus b \,\nabla\, c \tag{2}$$

$$a \otimes b = c \quad \Rightarrow \quad \begin{cases} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \qquad \mathcal{F}(a, \mu_a, s) \cdot \mathcal{F}(b, \mu_b, s) = \mathcal{F}(c, \mu_c, s) \text{ for all } s \in \mathbb{R}_0^+ \end{cases} \tag{3}$$

$$\left.\begin{array}{l} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \qquad \mathcal{F}(a, \mu_a, s) \cdot \mathcal{F}(b, \mu_b, s) = \mathcal{F}(c, \mu_c, s) \text{ for all } s \in \mathbb{R}_0^+ \end{array}\right\} \quad \Rightarrow \quad a \otimes b \,\nabla\, c \tag{4}$$

Table 1: A link between the operations $+$ and $\times$ in $\mathcal{R}_\mathrm{e}$ and the operations $\oplus$ and $\otimes$ in $\mathbb{S}_{\max}$ (with $a, b, c \in \mathbb{S}$).

The balance relation is extended to matrices in the usual way: if $A, B \in \mathbb{S}^{m \times n}$ then $A \nabla B$ if $a_{ij} \nabla b_{ij}$ for all i, j.

Definition 3.4 (Max-plus-algebraic norm)
The max-plus-algebraic norm of the matrix $A \in \mathbb{S}^{m \times n}$ is defined by $\|A\|_\oplus = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_\oplus$.

4 A link between conventional algebra and the symmetrized max-plus algebra

In [9] Olsder and Roos have used a kind of link between conventional algebra and the max-plus algebra that was based on asymptotic equivalences to prove a max-plus-algebraic version of Cramer's rule and of the Cayley-Hamilton theorem. In [4] we have extended and formalized this link. Now we recapitulate the reasoning of [4] but in a slightly different form that is mathematically more rigorous.

Let $\mathcal{R}_\mathrm{e}^+$ be the set of functions f with $\operatorname{dom} f = \mathbb{R}_0^+$ and with $f(s) = \sum_{i=0}^{n} \mu_i e^{x_i s}$ for some $n \in \mathbb{N}$ and with $\mu_i \in \mathbb{R}_0^+$ and $x_i \in \mathbb{R}_\varepsilon$ for all i. Let $\mathcal{R}_\mathrm{e}$ be defined in a similar way but with $\mu_i \in \mathbb{R}_0$ for all i.

For all $x, y, z \in \mathbb{R}_\varepsilon$ we have

$$\begin{aligned} x \oplus y = z \quad &\Leftrightarrow \quad e^{xs} + e^{ys} \sim c e^{zs}\ ,\ s \to \infty \\ x \otimes y = z \quad &\Leftrightarrow \quad e^{xs} \cdot e^{ys} = e^{zs} \quad \text{for all } s \in \mathbb{R}_0^+ \end{aligned}$$

where $c = 1$ if $x \neq y$ and $c = 2$ if $x = y$. These relations show that there exists a connection between the operations $\oplus$ and $\otimes$ performed on the real numbers and $-\infty$, and the operations $+$ and $\times$ performed on exponentials. Now we extend this link between $(\mathcal{R}_\mathrm{e}^+, +, \times)$ and $\mathbb{R}_{\max}$ to $\mathbb{S}_{\max}$.

We define a mapping $\mathcal{F}$ with domain of definition $\mathbb{S} \times \mathbb{R}_0 \times \mathbb{R}_0^+$ and with

$$\begin{aligned} \mathcal{F}(a, \mu, s) &= |\mu| e^{as} && \text{if } a \in \mathbb{S}^\oplus \\ \mathcal{F}(a, \mu, s) &= -|\mu| e^{|a|_\oplus s} && \text{if } a \in \mathbb{S}^\ominus \\ \mathcal{F}(a, \mu, s) &= \mu e^{|a|_\oplus s} && \text{if } a \in \mathbb{S}^\bullet \end{aligned}$$

where $a \in \mathbb{S}$, $\mu \in \mathbb{R}_0$ and $s \in \mathbb{R}_0^+$.
In the remainder of this paper the first two arguments of $\mathcal{F}$ will most of the time be fixed and we shall only consider $\mathcal{F}$ in function of the third argument, i.e., for a given $a \in \mathbb{S}$ and $\mu \in \mathbb{R}_0$ we consider the function $\mathcal{F}(a, \mu, \cdot)$. Note that if $x \in \mathbb{R}_\varepsilon$ and $\mu \in \mathbb{R}_0$ then we have

$$\begin{aligned} \mathcal{F}(x, \mu, s) &= |\mu| e^{xs} \\ \mathcal{F}(\ominus x, \mu, s) &= -|\mu| e^{xs} \\ \mathcal{F}(x^\bullet, \mu, s) &= \mu e^{xs} \end{aligned}$$

for all $s \in \mathbb{R}_0^+$. By definition we have $e^{\varepsilon s} = 0$ for all $s \in \mathbb{R}_0^+$. Hence, $\mathcal{F}(\varepsilon, \mu, \cdot) = 0$ for all $\mu \in \mathbb{R}_0$.

For a given $\mu \in \mathbb{R}_0$ the number $a \in \mathbb{S}$ will be mapped by $\mathcal{F}$ to an exponential function $s \mapsto \nu e^{|a|_\oplus s}$ where $\nu = |\mu|$, $\nu = -|\mu|$ or $\nu = \mu$ depending on the max-plus-algebraic sign of a. In order to reverse this process, we define the mapping $\mathcal{R}$, which we shall call the *reverse mapping* and which will map a function that is asymptotically equivalent to an exponential function $s \mapsto \nu e^{|a|_\oplus s}$ in the neighborhood of ∞ to the number $|a|_\oplus$ or $\ominus |a|_\oplus$ depending on the sign of ν. More specifically, if f is a real function, if $x \in \mathbb{R}_\varepsilon$ and if $\mu \in \mathbb{R}_0$ then we have

$$\begin{aligned} f(s) \sim |\mu| e^{xs}\ ,\ s \to \infty \quad &\Rightarrow \quad \mathcal{R}(f) = x \\ f(s) \sim -|\mu| e^{xs}\ ,\ s \to \infty \quad &\Rightarrow \quad \mathcal{R}(f) = \ominus x\ . \end{aligned}$$

Note that $\mathcal{R}$ will always map a function that is asymptotically equivalent to exponential function in the neighborhood of ∞ to a signed number and never to a balanced number that is different from ε.

Consider the implications listed in Table 1. As a consequence, we could say that the mapping $\mathcal{F}$ provides a link between the structure $(\mathcal{R}_\mathrm{e}^+, +, \times)$ and $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$, and a link between the structure $(\mathcal{R}_\mathrm{e}, +, \times)$ and $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$.

The balance in (2) results from the fact that we can have cancellation of equal terms with opposite sign in

$$A \oplus B = C \quad \Rightarrow \quad \begin{cases} \exists M_A, M_B, M_C \text{ such that} \\ \quad \mathcal{F}(A, M_A, s) + \mathcal{F}(B, M_B, s) \;\sim\; \mathcal{F}(C, M_C, s) \;,\; s \to \infty \end{cases} \tag{5}$$

$$\left.\begin{array}{l} \exists M_A, M_B, M_C \text{ such that} \\ \quad \mathcal{F}(A, M_A, s) + \mathcal{F}(B, M_B, s) \;\sim\; \mathcal{F}(C, M_C, s) \;,\; s \to \infty \end{array}\right\} \quad \Rightarrow \quad A \oplus B \;\nabla\; C \tag{6}$$

$$A \otimes B = C \quad \Rightarrow \quad \begin{cases} \exists M_A, M_B, M_C \text{ such that} \\ \quad \mathcal{F}(A, M_A, s) \cdot \mathcal{F}(B, M_B, s) \;\sim\; \mathcal{F}(C, M_C, s) \;,\; s \to \infty \end{cases} \tag{7}$$

$$\left.\begin{array}{l} \exists M_A, M_B, M_C \text{ such that} \\ \quad \mathcal{F}(A, M_A, s) \cdot \mathcal{F}(B, M_B, s) \;\sim\; \mathcal{F}(C, M_C, s) \;,\; s \to \infty \end{array}\right\} \quad \Rightarrow \quad A \otimes B \;\nabla\; C \tag{8}$$

Table 2: A link between the operations $+$ and $\times$ for matrix-valued functions with entries in $\mathcal{R}_{\mathrm{e}}$ and the operations $\oplus$ and $\otimes$ for matrices with entries in $\mathbb{S}$ (A, B and C are matrices with entries in $\mathbb{S}$).

$(\mathcal{R}_{\mathrm{e}}^+, +, \times)$ whereas this is in general not possible in $\mathbb{S}_{\max}$ since we have $a \oplus (\ominus a) \neq \varepsilon$ for all $a \in \mathbb{S} \setminus \{\varepsilon\}$.

We extend the mapping $\mathcal{F}$ to matrices as follows. If $A \in \mathbb{S}^{m \times n}$ and if $M \in \mathbb{R}_0^{m \times n}$ then $\tilde{A} = \mathcal{F}(A, M, \cdot)$ is a real m by n matrix-valued function with domain of definition $\mathbb{R}_0^+$ and with $\tilde{a}_{ij}(s) = \mathcal{F}(a_{ij}, m_{ij}, s)$ for all i, j. The reverse mapping $\mathcal{R}$ is extended to matrices in a similar way. Note that the mappings $\mathcal{F}$ and $\mathcal{R}$ are performed entrywise. This leads to the implications listed in Table 2.

5 The QRD and the SVD in the symmetrized max-plus algebra

In [4] we have used the mapping from $\mathbb{S}_{\max}$ to $(\mathcal{R}_{\mathrm{e}}, +, \times)$ and the reverse mapping to prove the existence of a kind of SVD in $\mathbb{S}_{\max}$. The proof of [4] made use of the concept "analytic SVD". In this paper we present an alternative method to prove the existence of the max-plus-algebraic SVD (and the max-plus-algebraic QRD). This proof technique consists in transforming a matrix with entries in $\mathbb{S}$ to a matrix-valued function with exponential entries (using the mapping $\mathcal{F}$), applying an algorithm from linear algebra and transforming the result back to the symmetrized max-plus algebra (using the mapping $\mathcal{R}$).

The entries of the matrices that appear in the existence proofs for the max-plus-algebraic QRD and the max-plus-algebraic SVD that will be presented in this section are sums or series of exponentials.

Definition 5.1 *Let $\mathcal{S}_{\mathrm{e}}$ be the set of real functions that are analytic and that can be written as a (possibly infinite, but absolutely convergent) sum of exponentials of the form $\sum_{i=0}^{n} \alpha_i e^{a_i x}$ in some neighborhood of ∞ with $\alpha_i \in \mathbb{R}_0$ and $a_i \in \mathbb{R}_\varepsilon$ for all i.*

The behavior of the functions in $\mathcal{S}_{\mathrm{e}}$ in the neighborhood of ∞ is given by the following property:

Lemma 5.2 *If $f \in \mathcal{S}_{\mathrm{e}}$ then there exist numbers $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}_\varepsilon$ such that $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$.*

Proof: See [3]. □

The set $\mathcal{S}_{\mathrm{e}}$ is closed under elementary operations such as additions, multiplications, subtractions, divisions, square roots and absolute values:

Proposition 5.3 *If f and g belong to $\mathcal{S}_{\mathrm{e}}$ then ρf, $f + g$, $f - g$, fg, f^l and $|f|$ also belong to $\mathcal{S}_{\mathrm{e}}$ for any $\rho \in \mathbb{R}$ and any $l \in \mathbb{N}$.*
Furthermore, if there exists a real number P such that $f(x) \neq 0$ for all $x \geqslant P$ then the functions $\dfrac{1}{f}$ and $\dfrac{g}{f}$ restricted to $[P, \infty)$ also belong to $\mathcal{S}_{\mathrm{e}}$.
If there exists a real number Q such that $f(x) > 0$ for all $x \geqslant Q$ then the function $\sqrt{f}$ restricted to $[Q, \infty)$ also belongs to $\mathcal{S}_{\mathrm{e}}$.

Proof: See [3]. □

Let $\tilde{A}$ and $\tilde{R}$ be real m by n matrix-valued functions and let $\tilde{Q}$ be a real m by m matrix-valued function. Suppose that $\tilde{A}$, $\tilde{Q}$ and $\tilde{R}$ are defined in $J \subseteq \mathbb{R}$. If $\tilde{Q}(s)\,\tilde{R}(s) = \tilde{A}(s)$, $\tilde{Q}^T(s)\,\tilde{Q}(s) = I_m$ and $\tilde{R}(s)$ is an upper triangular matrix for all $s \in J$, then we say that $\tilde{Q}\tilde{R}$ is a *path of QRDs* of $\tilde{A}$ on J. A path of SVDs is defined in a similar way.

Proposition 5.4 *If $\tilde{A} \in \mathcal{S}_{\mathrm{e}}^{m \times n}$ then there exists a path of QRDs $\tilde{Q}\tilde{R}$ of $\tilde{A}$ for which the entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_{\mathrm{e}}$.*

Proof: To compute the QRD of a matrix with real entries we can use the Givens QR algorithm (See, e.g., [6]). The operations used in this algorithm are additions, multiplications, subtractions, divisions and square roots. Furthermore, the number of operations used in this algorithm is finite. So if we apply this algorithm to a matrix-valued function $\tilde{A}$ with entries in $\mathcal{S}_{\mathrm{e}}$ then the entries of the resulting matrix-valued functions $\tilde{Q}$ and $\tilde{R}$ will also belong to $\mathcal{S}_{\mathrm{e}}$ by Proposition 5.3. □

Theorem 5.5 (Max-plus-algebraic QRD)
If $A \in \mathbb{S}^{m \times n}$ then there exist a matrix $Q \in (\mathbb{S}^{\vee})^{m \times m}$ and a max-plus-algebraic upper triangular matrix $R \in (\mathbb{S}^{\vee})^{m \times n}$ such that

$$A \nabla Q \otimes R \tag{9}$$

with $Q^T \otimes Q \nabla E_m$ and $\|R\|_{\oplus} = \|A\|_{\oplus}$.
Every decomposition of the form (9) that satisfies the above conditions is called a max-plus-algebraic QRD of A.

Proof: If $a, b \in \mathbb{S}$ then $a \nabla b$ implies that $a^{\bullet} \nabla b$. Therefore, it is sufficient to prove this theorem for signed matrices A. So from now on we assume that A is signed. We construct $\tilde{A} = \mathcal{F}(A, M, \cdot)$ where $M \in \mathbb{R}^{m \times n}$ with $m_{ij} = 1$ for all i, j. Note that the entries of $\tilde{A}$ belong to $\mathcal{S}_{\mathrm{e}}$. By Proposition 5.4 there exists a path of QRDs $\tilde{Q}\tilde{R}$ of $\tilde{A}$ for which the entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_{\mathrm{e}}$. So there exist a positive real number L and matrix-valued functions $\tilde{Q}$ and $\tilde{R}$ with entries in $\mathcal{S}_{\mathrm{e}}$ such that

$$\tilde{A}(s) = \tilde{Q}(s)\,\tilde{R}(s) \qquad \text{for all } s \geqslant L \tag{10}$$
$$\tilde{Q}^T(s)\,\tilde{Q}(s) = I_m \qquad \text{for all } s \geqslant L \tag{11}$$
$$\|\tilde{R}(s)\|_{\mathrm{F}} = \|\tilde{A}(s)\|_{\mathrm{F}} \qquad \text{for all } s \geqslant L\ . \tag{12}$$

Since the entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_{\mathrm{e}}$, they are asymptotically equivalent to an exponential in the neighborhood of ∞ by Lemma 5.2. If we define $Q = \mathcal{R}(\tilde{Q})$ and $R = \mathcal{R}(\tilde{R})$, then Q and R have signed entries. If we apply the reverse mapping $\mathcal{R}$ to (10) – (12), we get $A \nabla Q \otimes R$, $Q^T \otimes Q \nabla E_m$ and $\|R\|_{\oplus} = \|A\|_{\oplus}$. □

If QR is a QRD of a matrix $A \in \mathbb{R}^{m \times n}$ then we always have $\|R\|_{\mathrm{F}} = \|A\|_{\mathrm{F}}$ since Q is an orthogonal matrix. However, in [3] we have shown that $A \nabla Q \otimes R$ and $Q^T \otimes Q \nabla E_m$ do not always imply that $\|R\|_{\oplus} = \|A\|_{\oplus}$. Therefore we have included the condition $\|R\|_{\oplus} = \|A\|_{\oplus}$ explicitly in the definition of the max-plus-algebraic QRD.

Theorem 5.6 (Max-plus-algebraic SVD)
Let $A \in \mathbb{S}^{m \times n}$ and let $r = \min(m, n)$. Then there exist a max-plus-algebraic diagonal matrix $\Sigma \in \mathbb{R}_{\varepsilon}^{m \times n}$ and matrices $U \in (\mathbb{S}^{\vee})^{m \times m}$ and $V \in (\mathbb{S}^{\vee})^{n \times n}$ such that

$$A \nabla U \otimes \Sigma \otimes V^T \tag{13}$$

with $U^T \otimes U \nabla E_m$, $V^T \otimes V \nabla E_n$ and $\|A\|_{\oplus} = \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r$ where $\sigma_i = (\Sigma)_{ii}$ for $i = 1, 2, \ldots, r$.
Every decomposition of the form (13) that satisfies the above conditions is called a max-plus-algebraic SVD of A.

Proof: This proof is similar to that of Theorem 5.5, but now we use Kogbetliantz's SVD algorithm [7] to construct a path of SVDs of the matrix-valued function $\tilde{A}$. Afterwards we apply the reverse mapping $\mathcal{R}$ to obtain a max-plus-algebraic SVD of A. See [3] for the details of this proof. □

Note that for the max-plus-algebraic SVD we also have an extra condition ($\|A\|_{\oplus} = \sigma_1$) that does not appear in the definition of the SVD in conventional linear algebra. For properties of the max-plus-algebraic SVD the interested reader is referred to [3, 4].

In analogy with the definition of rank in linear algebra we can also define a rank based on the max-plus-algebraic SVD. In [3, 4] we have indicated how this max-plus-algebraic SVD rank could be used to obtain an estimate of the minimal system order of a max-linear discrete event system.

The proof technique that has been used in this section essentially consists in applying an algorithm from linear algebra to a matrix with entries in $\mathcal{S}_{\mathrm{e}}$. As a consequence, this proof technique can also be used to prove the existence of many other max-plus-algebraic matrix decompositions such as a max-plus-algebraic eigenvalue decomposition for symmetric matrices, a max-plus-algebraic LU decomposition, a max-plus-algebraic Schur decomposition, a max-plus-algebraic Hessenberg decomposition and so on.

6 A worked example of the max-plus-algebraic QRD

Let us compute a max-plus-algebraic QRD of

$$A = \begin{bmatrix} 1 & \varepsilon & \ominus(-5) \\ \ominus 2 & 3^{\bullet} & 0 \end{bmatrix} .$$

Define $\tilde{A} = \mathcal{F}(A, M, \cdot)$ with $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Hence,

$$\tilde{A}(s) = \begin{bmatrix} e^{s} & 0 & -e^{-5s} \\ -e^{2s} & e^{3s} & 1 \end{bmatrix} \quad \text{for all } s \in \mathbb{R}_0^+ .$$

If we use the Givens QR algorithm, we get a path of QRDs $\tilde{Q}\tilde{R}$ of $\tilde{A}$ with

$$\tilde{Q}(s) \sim \begin{bmatrix} e^{-s} & 1 \\ -1 & e^{-s} \end{bmatrix},\ s \to \infty$$
$$\tilde{R}(s) \sim \begin{bmatrix} e^{2s} & -e^{3s} & -1 \\ 0 & e^{2s} & e^{-s} \end{bmatrix},\ s \to \infty\ .$$

If we define $Q = \mathcal{R}(\tilde{Q})$ and $R = \mathcal{R}(\tilde{R})$, we obtain

$$Q = \begin{bmatrix} -1 & 0 \\ \ominus 0 & -1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 2 & \ominus 3 & \ominus 0 \\ \varepsilon & 2 & -1 \end{bmatrix}.$$

We have

$$Q \otimes R = \begin{bmatrix} 1 & 2^{\bullet} & (-1)^{\bullet} \\ \ominus 2 & 3 & 0 \end{bmatrix} \nabla A$$
$$Q^T \otimes Q = \begin{bmatrix} 0 & (-1)^{\bullet} \\ (-1)^{\bullet} & 0 \end{bmatrix} \nabla E_2$$

and $\|R\|_{\oplus} = 3 = \|A\|_{\oplus}$.
An example of the computation of a max-plus-algebraic SVD can be found in [3, 4].

7 Conclusions and future research

In this paper we have tried to fill one of the gaps in the theory of the (symmetrized) max-plus algebra by showing that there exist max-plus-algebraic analogues of fundamental matrix decompositions from linear algebra such as the QR decomposition and the singular value decomposition. The proof technique that has been used to prove the existence of these max-plus-algebraic matrix decompositions can also be used to prove the existence of max-plus-algebraic analogues of other matrix decompositions from linear algebra such as, e.g., the LU decomposition, the Hessenberg decomposition, the eigenvalue decomposition (for symmetric matrices), the Schur decomposition and so on.

Topics for future research are: further investigation of the properties of the max-plus-algebraic matrix decompositions that have been introduced in this paper, development of efficient algorithms to compute these decompositions, and application of these decompositions in the system theory for max-linear discrete event systems.

Acknowledgments

Bart De Schutter is a senior research assistant with the F.W.O. (Fund for Scientific Research – Flanders). Bart De Moor is a research associate with the F.W.O.
This research was sponsored by the Concerted Action Project of the Flemish Community, entitled "Model-based Information Processing Systems", by the Belgian program on interuniversity attraction poles (IUAP-02 and IUAP-24) and by the ALAPEDES project of the European Community Training and Mobility of Researchers Program.

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat, *Synchronization and Linearity.* New York: John Wiley & Sons, 1992.

[2] R.A. Cuninghame-Green, *Minimax Algebra*, vol. 166 of *Lecture Notes in Economics and Mathematical Systems.* Berlin, Germany: Springer-Verlag, 1979.

[3] B. De Schutter, *Max-Algebraic System Theory for Discrete Event Systems.* PhD thesis, Faculty of Applied Sciences, K.U.Leuven, Leuven, Belgium, Feb. 1996.

[4] B. De Schutter and B. De Moor, "The singular value decomposition in the extended max algebra," *Linear Algebra and Its Applications*, vol. 250, pp. 143–176, Jan. 1997.

[5] S. Gaubert, *Théorie des Systèmes Linéaires dans les Dioïdes.* PhD thesis, Ecole Nationale Supérieure des Mines de Paris, France, July 1992.

[6] G.H. Golub and C.F. Van Loan, *Matrix Computations.* Baltimore, Maryland: The John Hopkins University Press, 1989.

[7] E.G. Kogbetliantz, "Solution of linear equations by diagonalization of coefficients matrix," *Quarterly of Applied Mathematics*, vol. 13, no. 2, pp. 123–132, 1955.

[8] L. Ljung, *System Identification: Theory for the User.* Englewood Cliffs, New Jersey: Prentice-Hall, 1987.

[9] G.J. Olsder and C. Roos, "Cramer and Cayley-Hamilton in the max algebra," *Linear Algebra and Its Applications*, vol. 101, pp. 87–108, 1988.

[10] P. Van Overschee and B. De Moor, "Subspace algorithms for the stochastic identification problem," *Automatica*, vol. 29, no. 3, pp. 649–660, May 1993.