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Optimal traffic light control for a single intersection: Addendum*

B. De Schutter and B. De Moor

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In this addendum we give some extra propositions, proofs and examples in connection with the model for the evolution of the queue lengths at the switching time instants that we have derived in Section 2 of the paper "Optimal traffic light control for a single intersection" (*European Journal of Control*, vol. 4, no. 3, pp. 260-276, 1998), and in connection with the design of optimal traffic light switching schemes for this model.

In Section A we derive necessary and sufficient conditions for stability of this system under a periodic traffic light switching policy. In Section B we consider an oversaturated intersection, i.e., an intersection for which the queue lengths never become equal to 0 (except possibly at the end of a green or an amber phase). In Section C we derive expressions that give the value of the objective functions J_1, J_2, J_3, J_4, J_5 as a function of the switching time instants and the queue lengths at the switching time instants. In Section D we discuss the convexity and the concavity of the objective functions J_1, J_2, J_3, J_4 and J_5. Finally, in Section E we discuss another approximation of J_1 and J_4 that is also strictly monotonous as a function of the queue length vector.

A Stability

Now we discuss the conditions under which the system is "stable", i.e., has queue lengths that remain bounded as k goes to infinity. We assume that after a finite number, say $2K$, of switching cycles the switching scheme reaches a periodic regime, i.e., $\delta_{2k} = \delta_\mathrm{e}$ and $\delta_{2k+1} = \delta_\mathrm{o}$ for all $k \geqslant K$.

If we consider lane L_1, then there arrive $\bar{\lambda}_1(\delta_\mathrm{e}+\delta_\mathrm{o})$ vehicles during one complete green-amber-red cycle and (at most) $\bar{\mu}_1(\delta_\mathrm{o} - \delta_\mathrm{amb}) + \bar{\kappa}_1\delta_\mathrm{amb}$ vehicles leave lane L_1. So in order to prevent an unlimited growth of the queue length the following condition should hold:

$$\bar{\mu}_1(\delta_\mathrm{o} - \delta_\mathrm{amb}) + \bar{\kappa}_1\delta_\mathrm{amb} \geqslant \bar{\lambda}_1(\delta_\mathrm{e} + \delta_\mathrm{o}) \ .$$

If we write down similar conditions for the other lanes, we obtain the following necessary and sufficient conditions for stability:

$$(\bar{\mu}_1 - \bar{\lambda}_1)\delta_\mathrm{o} - \bar{\lambda}_1\delta_\mathrm{e} \geqslant (\bar{\mu}_1 - \bar{\kappa}_1)\delta_\mathrm{amb} \tag{39}$$

$$-\bar{\lambda}_2\delta_\mathrm{o} + (\bar{\mu}_2 - \bar{\lambda}_2)\delta_\mathrm{e} \geqslant (\bar{\mu}_2 - \bar{\kappa}_2)\delta_\mathrm{amb} \tag{40}$$

$$(\bar{\mu}_3 - \bar{\lambda}_3)\delta_\mathrm{o} - \bar{\lambda}_3\delta_\mathrm{e} \geqslant (\bar{\mu}_3 - \bar{\kappa}_3)\delta_\mathrm{amb} \tag{41}$$

$$-\bar{\lambda}_4\delta_\mathrm{o} + (\bar{\mu}_4 - \bar{\lambda}_4)\delta_\mathrm{e} \geqslant (\bar{\mu}_4 - \bar{\kappa}_4)\delta_\mathrm{amb} \ . \tag{42}$$

Together with the conditions $\delta_\mathrm{min,green,1} \leqslant \delta_\mathrm{o} - \delta_\mathrm{amb} \leqslant \delta_\mathrm{max,green,1}$ and $\delta_\mathrm{min,green,2} \leqslant \delta_\mathrm{e} - \delta_\mathrm{amb} \leqslant \delta_\mathrm{max,green,2}$, the conditions $(39) - (42)$ define a convex region in the $\delta_\mathrm{o} - \delta_\mathrm{e}$ plane. Note that adding conditions of this form to the conditions $(14) - (18)$ of the optimal traffic light control problem still leads to an ELCP.

B   Non-saturated versus oversaturated intersections

If we consider an oversaturated network then the queue lengths never become 0 during the green cycle (except possibly at the end of a green or an amber phase). In that case the maximum operator in (4) and (5), and thus also in (6) – (7) is not needed any more, and then we get the following model:

$$\boldsymbol{x}_{2k+1} = \boldsymbol{x}_{2k} + \boldsymbol{b}_1 \delta_{2k} + \boldsymbol{b}_3 \tag{43}$$

$$\boldsymbol{x}_{2k+2} = \boldsymbol{x}_{2k+1} + \boldsymbol{b}_2 \delta_{2k+1} + \boldsymbol{b}_4 \tag{44}$$

for $k = 0, 1, 2, \ldots$ with the extra constraints

$$\boldsymbol{x}_{2k+1} \geqslant \boldsymbol{b}_5$$

$$\boldsymbol{x}_{2k+2} \geqslant \boldsymbol{b}_6$$

for $k = 0, 1, 2, \ldots$ to ensure that the queue lengths are nonnegative at the end of the green and the amber phase. It is easy to verify that (43) and (44) lead to

$$\boldsymbol{x}_{2k+1} = \boldsymbol{x}_0 + \sum_{j=0}^{k} \boldsymbol{b}_1 \delta_{2j} + \sum_{j=0}^{k-1} \boldsymbol{b}_2 \delta_{2j+1} + (k+1)\boldsymbol{b}_3 + k\boldsymbol{b}_4$$

$$\boldsymbol{x}_{2k+2} = \boldsymbol{x}_0 + \sum_{j=0}^{k} \boldsymbol{b}_1 \delta_{2j} + \sum_{j=0}^{k} \boldsymbol{b}_2 \delta_{2j+1} + (k+1)\boldsymbol{b}_3 + (k+1)\boldsymbol{b}_4 \ .$$

As a consequence, the optimal traffic light control problem now becomes

$$\text{minimize } J$$

subject to

$$\delta_{\min,\text{green},1} \leqslant \delta_{2k+1} - \delta_{\text{amb}} \leqslant \delta_{\max,\text{green},1} \qquad \text{for } k = 0, 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor - 1,$$

$$\delta_{\min,\text{green},2} \leqslant \delta_{2k} - \delta_{\text{amb}} \leqslant \delta_{\max,\text{green},2} \qquad \text{for } k = 0, 1, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor,$$

$$\boldsymbol{b}_5 \leqslant \boldsymbol{x}_0 + \sum_{j=0}^{k} \boldsymbol{b}_1 \delta_{2j} + \sum_{j=0}^{k-1} \boldsymbol{b}_2 \delta_{2j+1} +$$

$$(k+1)\boldsymbol{b}_3 + k\boldsymbol{b}_4 \leqslant \boldsymbol{x}_{\max} \qquad \text{for } k = 0, 1, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor,$$

$$\boldsymbol{b}_6 \leqslant \boldsymbol{x}_0 + \sum_{j=0}^{k} \boldsymbol{b}_1 \delta_{2j} + \sum_{j=0}^{k} \boldsymbol{b}_2 \delta_{2j+1} +$$

$$(k+1)\boldsymbol{b}_3 + (k+1)\boldsymbol{b}_4 \leqslant \boldsymbol{x}_{\max} \qquad \text{for } k = 0, 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor - 1.$$

$\tilde{\boldsymbol{\delta}}^*, \tilde{\boldsymbol{x}}^*$	$J_1(\tilde{\boldsymbol{\delta}}^*)$	$\tilde{J}_1(\tilde{\boldsymbol{x}}^*, \tilde{\boldsymbol{\delta}}^*)$
$\boldsymbol{\delta}_{\mathrm{ELCP}}^*, \boldsymbol{x}_{\mathrm{ELCP}}^*$	60.657	64.267
$\tilde{\boldsymbol{\delta}}^*, \tilde{\boldsymbol{x}}^*$	60.659	64.264
$\boldsymbol{\delta}_{\mathrm{lin}}^*, \boldsymbol{x}_{\mathrm{lin}}^*$	64.551	67.905
$\boldsymbol{\delta}_{\mathrm{os}}^*, \boldsymbol{x}_{\mathrm{os}}^*$	72.658	74.452

Table 3: The values of the objective functions J_1 and $\tilde{J}_1$ (up to 3 decimal places) for the traffic light switching sequences defined by the switching interval vectors $\boldsymbol{\delta}_{\mathrm{ELCP}}^*$, $\tilde{\boldsymbol{\delta}}^*$, $\boldsymbol{\delta}_{\mathrm{lin}}^*$ of Example 5.1 and the switching interval vector $\boldsymbol{\delta}_{\mathrm{os}}^*$ of Example B.1. The queue length vectors $\boldsymbol{x}^*$ are compatible with the switching interval vectors $\boldsymbol{\delta}^*$ for $\boldsymbol{x}_0$.

The feasible region of this optimization problem is convex, which implies that it can be solved more efficiently than the optimization problem $(13)-(18)$. However, the following example shows that in general applying the oversaturated model $(43)-(44)$ to non-saturated intersections does not lead to an optimal solution. Therefore we could say that the use of the maximum operator will in the first instance lead to complex optimization problems — but for which some approximations lead to good suboptimal solutions that can be computed very efficiently as has been shown in Section 3.2 — whereas omitting the maximum operator initially leads to simpler models but finally results in control schemes that for non-saturated intersections have an inferior performance.

Example B.1 Consider the intersection of Figure 1 with the same data as in Example 5.1: $\bar{\lambda}_1 = 0.25$, $\bar{\lambda}_2 = 0.12$, $\bar{\lambda}_3 = 0.20$, $\bar{\lambda}_4 = 0.1$, $\bar{\mu}_1 = \bar{\mu}_3 = 0.5$, $\bar{\mu}_2 = \bar{\mu}_4 = 0.4$, $\bar{\kappa}_1 = \bar{\kappa}_3 = 0.05$, $\bar{\kappa}_2 = \bar{\kappa}_4 = 0.03$, $\boldsymbol{x}_0 = [\,20\ \ 19\ \ 14\ \ 12\,]^T$, $\delta_{\mathrm{amb}} = 3$, $\delta_{\mathrm{min,green,1}} = \delta_{\mathrm{min,green,2}} = 6$, $\delta_{\mathrm{max,green,1}} = \delta_{\mathrm{max,green,2}} = 60$, $\boldsymbol{x}_{\mathrm{max}} = [\,25\ \ 20\ \ 25\ \ 20\,]^T$ and $\boldsymbol{w} = [\,2\ \ 1\ \ 2\ \ 1\,]^T$. Suppose that we want to compute a traffic light switching sequence $t_0, t_1, \ldots, t_7$ that minimizes J_1.
In Example 5.1 we have computed several suboptimal switching interval vectors based on the model $(6)-(7)$. Let us now compute a minimum $\boldsymbol{\delta}_{\mathrm{os}}^*$ of the objective function J_1 based on the model $(43)-(44)$ for oversaturated intersections using the e04ucf routine of the NAG library. This results in[1]

$$\boldsymbol{\delta}_{\mathrm{os}}^* = [\,20.000\ \ 45.750\ \ 18.600\ \ 34.150\ \ 38.433\ \ 30.122\ \ 13.741\,]^T\ .$$

In Table 3 we have listed the values of the objective functions for the switching interval vectors $\boldsymbol{\delta}_{\mathrm{ELCP}}^*$ (the optimal solution of the original problem using the ELCP approach), $\tilde{\boldsymbol{\delta}}^*$ (the optimal solution of the relaxed problem), $\boldsymbol{\delta}_{\mathrm{lin}}^*$ (the optimal solution of the linear programming problem) and $\boldsymbol{\delta}_{\mathrm{os}}^*$. The evolution of the queue lengths for the various control strategies is represented in Figures 6 and 8.
Clearly, the suboptimal solutions based on the model $(6)-(7)$ correspond to much lower values of the objective function J_1 than the optimal solution based on the model $(43)-(44)$ for oversaturated intersections. □

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[1]We have listed the best solution over 20 runs with random initial points. The mean of the objective values of the local minima returned by the minimization routine was 73.717 with a standard deviation of 1.336.
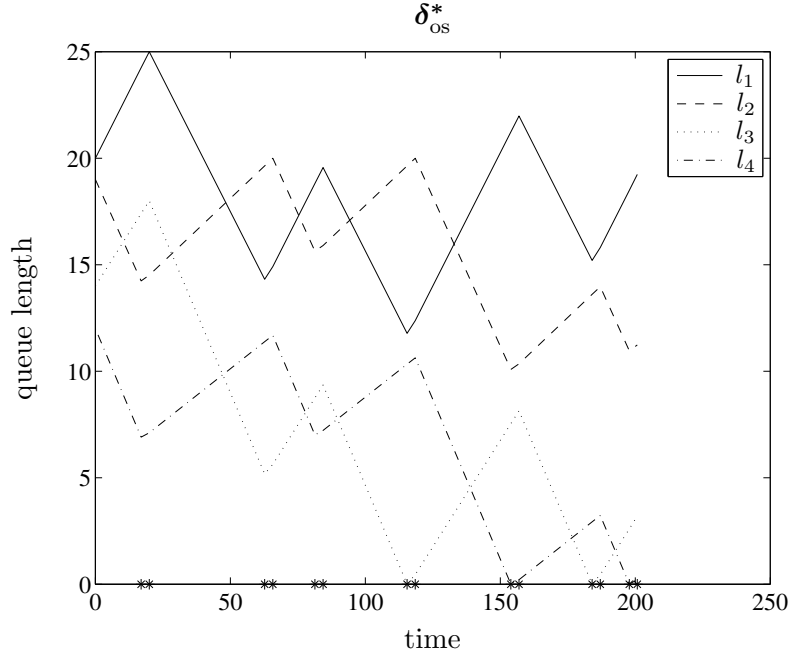
Figure 8: The queue lengths in the various lanes as a function of time for the traffic light switching sequence that corresponds to the switching interval vector $\boldsymbol{\delta}^*_{\mathrm{os}}$ of Example B.1. The $*$ signs on the time axis correspond to the switching time instants.

C   Evaluation of the objective functions

Let $\boldsymbol{x}_0 \in (\mathbb{R}^+)^4$, $\boldsymbol{\delta}^* \in (\mathbb{R}^+_0)^{4N}$ and let $\{\delta_k\}_{k=0}^{N-1}$ be the sequence of switching time intervals that corresponds to $\boldsymbol{\delta}^*$. First we derive a formula that expresses $J_3(\boldsymbol{\delta}^*)$ as a function of the sequence of queue length vectors $\{\boldsymbol{x}_k\}_{k=0}^{N}$ that is compatible with $\{\delta_k\}_{k=0}^{N-1}$ for a given $\boldsymbol{x}_0$.

We have assumed that $\bar{\kappa}_i \leqslant \bar{\mu}_i$ for all i. Recall that this implies that a situation such as in the left plot of Figure 3 where $\bar{\lambda}_i - \bar{\mu}_i > 0$ and $\bar{\lambda} - i - \bar{\kappa}_i < 0$ is not possible. So if $k \in \mathcal{G}_i$, then the maximum value of l_i in the interval (t_k, t_{k+1}) where T_i is first green and then amber will be reached in t_k or in t_{k+1}. Furthermore, in an interval (t_{k+1}, t_{k+2}) where T_i is red the maximum value of l_i will be reached in t_{k+1} or in t_{k+2}. Since $l_i(\cdot, \boldsymbol{\delta}^*)$ is a piecewise-linear function, this implies that l_i reaches its maximum over the interval $[t_0, t_N]$ in one of the switching time instants t_k. As a consequence, we have

$$ J_3(\boldsymbol{\delta}^*) = \max_{i,\, k} \left(w_i (\boldsymbol{x}_k)_i \right) \ . $$

Now we derive a formula for the evaluation of

$$ \int_{t_0}^{t_N} l_1(t)\, dt = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} l_1(t)\, dt \ . $$

Define $y_k = (\boldsymbol{x}_k)_1 = l_1(t_k)$ for $k = 0, 1, \ldots, N$ and $\tilde{y}_{2k+2} = l_1(t_{2k+2} - \delta_{\mathrm{amb}})$ for $k = 0, 1, \ldots, \left\lfloor \dfrac{N}{2} \right\rfloor - 1$. Note that

$$ \tilde{y}_{2k+2} = \max(y_{2k+1} + (\bar{\lambda}_1 - \bar{\mu}_1)(\delta_{2k+1} - \delta_{\mathrm{amb}}), 0) \ . $$
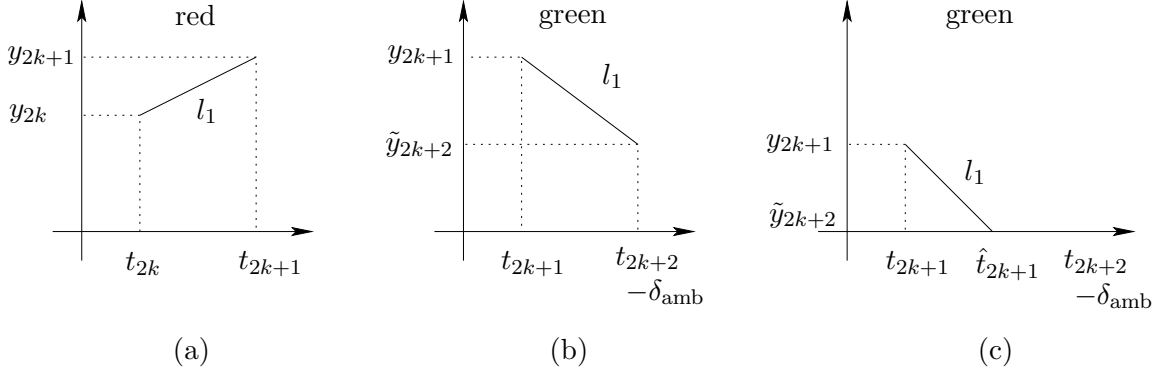
iv

Figure 9: Three possible basic cases for the evolution of the queue length l_1 as a function of time t in an interval (t_{2k}, t_{2k+1}) where the traffic light is red, or in an interval $(t_{2k+1}, t_{2k+2} - \delta_{\text{amb}})$ where the traffic light is green.

If we want to evaluate $\int_{t_{\text{b}}}^{t_{\text{e}}} l_1(t)\, dt$ where t_{b} and t_{e} are respectively the beginning and the end of a red or a green phase, there are three possible basic cases (see Figure 9).

- In an interval of the form (t_{2k}, t_{2k+1}) the light is red, which means that l_1 is a nondecreasing function. In this case $\int_{t_{2k}}^{t_{2k+1}} l_1(t)\, dt$ is equal to the surface of the trapezium defined by the points $(t_{2k}, 0)$, (t_{2k}, y_{2k}), (t_{2k+1}, y_{2k+1}) and $(t_{2k+1}, 0)$. Hence,

$$\int_{t_{2k}}^{t_{2k+1}} l_1(t)\, dt \;=\; \frac{y_{2k} + y_{2k+1}}{2}\, \delta_{2k} \;. \tag{45}$$

- In an interval of the form $(t_{2k+1}, t_{2k+2} - \delta_{\text{amb}})$ the light is green. If the queue length l_1 is identically 0 in $(t_{2k+1}, t_{2k+2} - \delta_{\text{amb}})$ or if l_1 never becomes 0 in $(t_{2k+1}, t_{2k+2} - \delta_{\text{amb}})$, $\int_{t_{2k+1}}^{t_{2k+2} - \delta_{\text{amb}}} l_1(t)\, dt$ is equal the surface of the trapezium defined by the points $(t_{2k+1}, 0)$, (t_{2k+1}, y_{2k+1}), $(t_{2k+2} - \delta_{\text{amb}}, \tilde{y}_{2k+2})$ and $(t_{2k+2} - \delta_{\text{amb}}, 0)$. So,

$$\int_{t_{2k+1}}^{t_{2k+2} - \delta_{\text{amb}}} l_1(t)\, dt \;=\; \frac{y_{2k+1} + \tilde{y}_{2k+2}}{2}\, (\delta_{2k+1} - \delta_{\text{amb}}) \qquad \text{if } \tilde{y}_{2k+2} \neq 0\,. \tag{46}$$

- If the queue length l_1 becomes 0 in the interval $(t_{2k+1}, t_{2k+2} - \delta_{\text{amb}})$, $\int_{t_{2k+1}}^{t_{2k+2} - \delta_{\text{amb}}} l_1(t)\, dt$ is equal to the surface of the triangle defined by the points $(t_{2k+1}, 0)$, (t_{2k+1}, y_{2k+1}), $(\hat{t}_{2k+1}, 0)$ where $\hat{t}_{2k+1}$ is the smallest value of $t \in (t_{2k+1}, t_{2k+2} - \delta_{\text{amb}})$ for which $l_1(t) = 0$ (see Figure 9(c)). Since in this case the absolute value of the slope of $l_1(t)$ is equal to $\bar{\mu}_1 - \bar{\lambda}_1$ in $(t_{2k+1}, t_{2k+2} - \delta_{\text{amb}})$, we have $y_{2k+1} = (\bar{\mu}_1 - \bar{\lambda}_1)(\hat{t}_{2k+1} - t_{2k+1})$ and thus $\hat{t}_{2k+1} - t_{2k+1} = \dfrac{y_{2k+1}}{\bar{\mu}_1 - \bar{\lambda}_1}$. As consequence, we have

$$\int_{t_{2k+1}}^{t_{2k+2} - \delta_{\text{amb}}} l_1(t)\, dt \;=\; \frac{y_{2k+1}^2}{2(\bar{\mu}_1 - \bar{\lambda}_1)} \qquad \text{if } \tilde{y}_{2k+2} = 0\,. \tag{47}$$

v

Before we consider the amber phase, we shall show that the expressions (46) and (47) for the green phase also cover expression (45) for the red phase if the appropriate changes of variables are made, i.e., we shall prove that

$$
\int_{t_{2k}}^{t_{2k+1}} l_1(t)\, dt =
\begin{cases}
\dfrac{y_{2k} + y_{2k+1}}{2}\, \delta_{2k} & \text{if } y_{2k+1} \neq 0\,, \\[2ex]
\dfrac{y_{2k+1}^2}{2\bar{\lambda}_1} & \text{if } y_{2k+1} = 0\,.
\end{cases}
\tag{48}
$$

If the queue length y_{2k+1} at the end of the red phase is different from 0, then (45) corresponds to the first case of (48). On the other hand, if y_{2k+1} is equal to 0, then y_{2k} will also be zero since l_1 is nondecreasing in (t_{2k}, t_{2k+1}). So then both (45) and the second case of (48) yield 0.

When the traffic light is amber (i.e., in an interval of the form $(t_{2k+1} - \delta_{\text{amb}}, t_{2k+2})$), we either have the situation of case (a) if the queue length is nondecreasing in the given interval (i.e., if $\bar{\lambda}_i - \bar{\kappa}_i \geqslant 0$), or case (b) or (c) if the queue length is decreasing in the given interval (i.e., if $\bar{\lambda}_i - \bar{\kappa}_i < 0$). Since expressions (46) and (47) also cover expression (45) if the appropriate changes of variables are made, this implies that

$$
\int_{t_{2k+2}-\delta_{\text{amb}}}^{t_{2k+2}} l_1(t)\, dt =
\begin{cases}
\dfrac{\tilde{y}_{2k+2} + y_{2k+2}}{2}\, \delta_{\text{amb}} & \text{if } y_{2k+2} \neq 0\,, \\[2ex]
\dfrac{\tilde{y}_{2k+2}^2}{2(\bar{\kappa}_1 - \bar{\lambda}_1)} & \text{if } y_{2k+2} = 0\,.
\end{cases}
$$

So

$$
\int_{t_0}^{t_N} l_1(t)\, dt = \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{y_{2k} + y_{2k+1}}{2}\, \delta_{2k} + \sum_{\substack{k=0 \\ \tilde{y}_{2k+2} \neq 0}}^{\lfloor \frac{N}{2} \rfloor - 1} \frac{y_{2k+1} + \tilde{y}_{2k+2}}{2}\, (\delta_{2k+1} - \delta_{\text{amb}}) +
$$

$$
\sum_{\substack{k=0 \\ \tilde{y}_{2k+2} = 0}}^{\lfloor \frac{N}{2} \rfloor - 1} \frac{y_{2k+1}^2}{2(\bar{\mu}_1 - \bar{\lambda}_1)} + \sum_{\substack{k=0 \\ y_{2k+2} \neq 0}}^{\lfloor \frac{N}{2} \rfloor - 1} \frac{\tilde{y}_{2k+2} + y_{2k+2}}{2}\, \delta_{\text{amb}} +
$$

$$
\sum_{\substack{k=0 \\ y_{2k+2} = 0}}^{\lfloor \frac{N}{2} \rfloor - 1} \frac{\tilde{y}_{2k+2}^2}{2(\bar{\kappa}_1 - \bar{\lambda}_1)} \quad.
$$

We can write down similar expressions for $\int_{t_0}^{t_N} l_i(t)\, dt$ for $i = 2, 3, 4$.

D Convexity or concavity of the objective functions

First we show that J_3 is convex as a function of $\boldsymbol{\delta}^*$, which implies that problem $(13)-(18)$ with $J = J_3$ can be solved efficiently (if there is no upper bound on the queue lengths or if we introduce a convex penalty term if one or more components of $\boldsymbol{x}_{\max}$ are finite).

Proposition D.1 *For given $\boldsymbol{x}_0$, δ_{amb}, $\bar{\lambda}_i$'s, $\bar{\mu}_i$'s and $\bar{\kappa}_i$'s the function J_3 is convex as a function of $\boldsymbol{\delta}^*$.*

Proof: In Section C we have already shown that

$$J_3(\boldsymbol{\delta}^*) = \max_{i,\,k} \big(w_i(\boldsymbol{x}_k)_i\big) \tag{49}$$

if $\boldsymbol{x}^*$ and $\boldsymbol{\delta}^*$ are compatible for a given $\boldsymbol{x}_0$.

Let $\boldsymbol{x}_0 \in (\mathbb{R}^+)^4$ and consider $\boldsymbol{\delta}^*, \boldsymbol{\eta}^* \in (\mathbb{R}_0^+)^N$. Let $\{\delta_k\}_{k=0}^{N-1}$ and $\{\eta_k\}_{k=0}^{N-1}$ be the sequences of switching time intervals that correspond to $\boldsymbol{\delta}^*$ and $\boldsymbol{\eta}^*$ respectively. Define $\boldsymbol{y}_0 = \boldsymbol{z}_0 = \boldsymbol{x}_0$. Let the sequences $\{\boldsymbol{x}_k\}_{k=0}^{N}$ and $\{\boldsymbol{y}_k\}_{k=0}^{N}$ be compatible with $\{\delta_k\}_{k=0}^{N-1}$ and $\{\eta_k\}_{k=0}^{N-1}$ respectively. Consider an arbitrary number $u \in [0,1]$ and let the sequence $\{\boldsymbol{z}_k\}_{k=0}^{N}$ be compatible with the sequence $\{u\delta_k + (1-u)\eta_k\}_{k=0}^{N-1}$.

Note that the sequences $\{\boldsymbol{x}_k\}_{k=0}^{N}$, $\{\boldsymbol{y}_k\}_{k=0}^{N}$ and $\{\boldsymbol{z}_k\}_{k=0}^{N}$ all satisfy recurrence equations of the form $(6)-(7)$.

Let us now show by induction that

$$\boldsymbol{z}_k \leqslant u\boldsymbol{x}_k + (1-u)\boldsymbol{y}_k \qquad \text{for } k = 0, 1, \ldots, N \ . \tag{50}$$

We have $\boldsymbol{z}_0 = u\boldsymbol{x}_0 + (1-u)\boldsymbol{y}_0 = u\boldsymbol{x}_0 + (1-u)\boldsymbol{x}_0 = \boldsymbol{x}_0$.

Now we assume that $\boldsymbol{z}_k \leqslant u\boldsymbol{x}_k + (1-u)\boldsymbol{y}_k$ for $k = 0, 1, \ldots, K$ with $K < N$ and we show that $\boldsymbol{z}_{K+1} \leqslant u\boldsymbol{x}_{K+1} + (1-u)\boldsymbol{y}_{K+1}$.

Suppose that K is even. Hence, $K = 2l$ for some integer l. Now we have

$$u\boldsymbol{x}_{2l+1} + (1-u)\boldsymbol{y}_{2l+1}$$

$$= u \max\left(\boldsymbol{x}_{2l} + \boldsymbol{b}_1\delta_{2l} + \boldsymbol{b}_3,\, \boldsymbol{b}_5\right) +$$

$$\qquad (1-u) \max\left(\boldsymbol{y}_{2l} + \boldsymbol{b}_1\eta_{2l} + \boldsymbol{b}_3,\, \boldsymbol{b}_5\right) \qquad \text{(by (6))}$$

$$= \max\big(u(\boldsymbol{x}_{2l} + \boldsymbol{b}_1\delta_{2l} + \boldsymbol{b}_3) + (1-u)(\boldsymbol{y}_{2l} + \boldsymbol{b}_1\eta_{2l} + \boldsymbol{b}_3),$$

$$\qquad u(\boldsymbol{x}_{2l} + \boldsymbol{b}_1\delta_{2l} + \boldsymbol{b}_3) + (1-u)\boldsymbol{b}_5,$$

$$\qquad u\boldsymbol{b}_5 + (1-u)(\boldsymbol{y}_{2l} + \boldsymbol{b}_1\eta_{2l} + \boldsymbol{b}_3),\, u\boldsymbol{b}_5 + (1-u)\boldsymbol{b}_5\big)$$

$$\geqslant \max\big(u(\boldsymbol{x}_{2l} + \boldsymbol{b}_1\delta_{2l} + \boldsymbol{b}_3) + (1-u)(\boldsymbol{y}_{2l} + \boldsymbol{b}_1\eta_{2l} + \boldsymbol{b}_3),\, \boldsymbol{b}_5\big)$$

$$\geqslant \max\Big(u\boldsymbol{x}_{2l} + (1-u)\boldsymbol{y}_{2l}+$$

$$\qquad \boldsymbol{b}_1\big(u\delta_{2l} + (1-u)\eta_{2l}\big) + \boldsymbol{b}_3,\, \boldsymbol{b}_5\Big)$$

$$\geqslant \max\Big(\boldsymbol{z}_{2l} + \boldsymbol{b}_1\big(u\delta_{2l} + (1-u)\eta_{2l}\big) + \boldsymbol{b}_3,\, \boldsymbol{b}_5\Big) \qquad \text{(by the induction hypothesis)}$$

$$\geqslant \boldsymbol{z}_{2l+1} \ .$$

If $K = 2l+1$ is odd, then we can show in a similar way that $u\boldsymbol{x}_{2l+2} + (1-u)\boldsymbol{y}_{2l+2} \geqslant \boldsymbol{z}_{2l+2}$. As a consequence, we have

$$J_3(u\boldsymbol{\delta}^* + (1-u)\boldsymbol{\eta}^*) = \max_{i,\,k}\big(w_i(\boldsymbol{z}_k)_i\big) \qquad \text{(by (49))}$$

$$\leqslant \max_{i,\,k}\big(w_i(u\boldsymbol{x}_k + (1-u)\boldsymbol{y}_k)_i\big) \qquad \text{(by (50))}$$

$$\leqslant u \max_{i,\,k}\big(w_i(\boldsymbol{x}_k)_i\big) + (1-u)\max_{i,\,k}\big(w_i(\boldsymbol{y}_k)_i\big)$$

$$\leqslant u J_3(\boldsymbol{\delta}^*) + (1-u)J_3(\boldsymbol{\eta}^*) \qquad \text{(by (49))},$$

which implies that J_3 is convex as a function of $\boldsymbol{\delta}^*$. $\qquad\qquad \square$

Proposition D.2 *Let the set of feasible solutions of the ELCP that corresponds to a given optimal traffic light switching problem be characterized by the set of vertices* $V = \left\{ \begin{bmatrix} \boldsymbol{x}_i^* \\ \boldsymbol{\delta}_i^* \end{bmatrix} \middle| \, i = 1, 2, \ldots, r \right\}$ *and the set of index sets* Λ. *Let* $\phi_j = \{i_1, i_2, \ldots, i_s\} \in \Lambda$. *Then the function* $J_3 \left(\sum_{j=1}^{s} \nu_j \boldsymbol{\delta}_{i_j}^* \right)$ *with* $\nu_j \geqslant 0$ *for all* j *and* $\sum_{j=1}^{s} \nu_j = 1$ *is a convex function of the* ν_j*'s.*

Proof: Note that we may assume without loss of generality that $\phi_j = \{1, 2, \ldots, s\}$. Let $\boldsymbol{\nu} = \begin{bmatrix} \nu_1 & \nu_2 & \ldots & \nu_s \end{bmatrix}^T$ with $\nu_j \geqslant 0$ for all j and $\sum_{j=1}^{s} \nu_j = 1$. Let $\boldsymbol{\delta}^*(\boldsymbol{\nu}) = \sum_{j=1}^{s} \nu_j \boldsymbol{\delta}_j^*$ and $\boldsymbol{x}^*(\boldsymbol{\nu}) = \sum_{j=1}^{s} \nu_j \boldsymbol{x}_j^*$. Let $\{\boldsymbol{x}_{j,k}\}_{k=1}^{N}$ be the sequence of 4-component queue length vectors that corresponds to the $4N$-component queue length vector $\boldsymbol{x}_j^*$ for $j = 1, 2, \ldots, s$.

Define $I_3(\boldsymbol{\nu}) = J_3(\boldsymbol{\delta}^*(\boldsymbol{\nu})) = J_3 \left(\sum_{j=1}^{s} \nu_j \boldsymbol{\delta}_j^* \right)$. Now we prove that I_3 is a convex function.

Since $\begin{bmatrix} \boldsymbol{x}^*(\boldsymbol{\nu}) \\ \boldsymbol{\delta}^*(\boldsymbol{\nu}) \end{bmatrix}$ is a convex combination of $\begin{bmatrix} \boldsymbol{x}_1^* \\ \boldsymbol{\delta}_1^* \end{bmatrix}$, $\begin{bmatrix} \boldsymbol{x}_2^* \\ \boldsymbol{\delta}_2^* \end{bmatrix}$, $\ldots$, $\begin{bmatrix} \boldsymbol{x}_s^* \\ \boldsymbol{\delta}_s^* \end{bmatrix}$, it is also a solution of the ELCP that corresponds to the given optimal traffic light switching problem. As a consequence, $\boldsymbol{x}^*(\boldsymbol{\nu})$ and $\boldsymbol{\delta}^*(\boldsymbol{\nu})$ are compatible for the given $\boldsymbol{x}_0$. This implies that

$$I_3(\boldsymbol{\nu}) = \max_{i,\,k} \left(w_i \left(\sum_{j=1}^{s} \nu_j (\boldsymbol{x}_{j,k})_i \right), \, w_i(\boldsymbol{x}_0)_i \right) \qquad \text{(by (49))}.$$

Let $\boldsymbol{\eta} = \begin{bmatrix} \eta_1 & \eta_2 & \ldots & \eta_s \end{bmatrix}^T$ with $\eta_j \geqslant 0$ for all j and $\sum_{j=1}^{s} \eta_j = 1$. Let $u \in [0, 1]$. Note that $u\nu_j + (1-u)\eta_j \geqslant 0$ for all j and that $\sum_{j=1}^{s} u\nu_j + (1-u)\eta_j = u + (1-u) = 1$. As a consequence, we have

$$I_3(u\boldsymbol{\nu} + (1-u)\boldsymbol{\eta}) = \max_{i,\,k} \left(w_i \left(\sum_{j=1}^{s} (u\nu_j + (1-u)\eta_j)(\boldsymbol{x}_{j,k})_i \right), \, w_i(\boldsymbol{x}_0)_i \right)$$

$$= \max_{i,\,k} \left(u \left(w_i \sum_{j=1}^{s} \nu_j (\boldsymbol{x}_{j,k})_i \right) + (1-u) \left(w_i \sum_{j=1}^{s} \eta_j (\boldsymbol{x}_{j,k})_i \right), \right.$$

$$\left. u w_i(\boldsymbol{x}_0)_i + (1-u) w_i(\boldsymbol{x}_0)_i \right)$$

$$\leqslant u \max_{i,\,k} \left(w_i \sum_{j=1}^{s} \nu_j (\boldsymbol{x}_{j,k})_i, \, w_i(\boldsymbol{x}_0)_i \right) +$$

$$(1 - u) \max_{i, k} \left(w_i \sum_{j=1}^{s} \eta_j(\boldsymbol{x}_{j,k})_i, \; w_i(\boldsymbol{x}_0)_i \right)$$

$$\leqslant u I_3(\boldsymbol{\nu}) + (1 - u) I_3(\boldsymbol{\eta}) \; .$$

So I_3 is convex. Hence, J_3 is a convex function of the ν_i's. $\qquad\square$

Note that the objective functions J_1, J_2, J_4 and J_5 do not depend directly on $\boldsymbol{x}^*$ since for given $\bar{\lambda}_i$'s, $\bar{\mu}_i$'s, $\bar{\kappa}_i$'s, $\boldsymbol{x}_0$ and δ_{amb} the switching interval vector $\boldsymbol{\delta}^*$ uniquely determines $\boldsymbol{x}^*$. The following example shows that J_1, J_2, J_4 and J_5 are in general neither convex nor concave as a function of $\boldsymbol{\delta}^*$. Recall that we use the notation $l_i(\cdot, \boldsymbol{\delta}^*)$ to indicate that the queue length function $l_i(\cdot)$ corresponds to the switching interval vector $\boldsymbol{\delta}^*$.

Example D.3 Let $\delta_{\mathrm{amb}} = 3$ and $\bar{\lambda}_i = 0.25$, $\bar{\mu}_i = 0.5$, $\bar{\kappa}_i = 0$ for $i, 1, 2, 3, 4$. Let

$$\boldsymbol{x}_0 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{\delta}_1^* = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad \boldsymbol{\delta}_2^* = \begin{bmatrix} 10 \\ 30 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\delta}_3^* = \begin{bmatrix} 10 \\ 20 \end{bmatrix} \; .$$

In Figure 10 we have plotted the evolution of l_1 as a function of time for the switching sequences defined by $\boldsymbol{\delta}_1^*$, $\boldsymbol{\delta}_2^*$, $\boldsymbol{\delta}_3^* = \dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}$ and $\boldsymbol{\delta}_4^* = \dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_3^*}{2}$.

Define

$$f(\boldsymbol{\delta}^*) = \frac{\displaystyle\int_{t_0}^{t_N} l_1(t, \boldsymbol{\delta}^*) \, dt}{t_N - t_0} \; .$$

We have $f(\boldsymbol{\delta}_1^*) \approx 3.363$, $f(\boldsymbol{\delta}_2^*) \approx 1.853$,

$$f\!\left(\frac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}\right) \approx 2.492 \quad \text{and} \quad \frac{f(\boldsymbol{\delta}_1^*) + f(\boldsymbol{\delta}_2^*)}{2} \approx 2.608 \; .$$

So

$$f\!\left(\frac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}\right) < \frac{f(\boldsymbol{\delta}_1^*) + f(\boldsymbol{\delta}_2^*)}{2} \; ,$$

which implies that f is not concave.

On the other hand, we have $f(\boldsymbol{\delta}_1^*) \approx 3.363$, $f(\boldsymbol{\delta}_3^*) \approx 2.492$,

$$f\!\left(\frac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_3^*}{2}\right) = 2.965 \quad \text{and} \quad \frac{f(\boldsymbol{\delta}_1^*) + f(\boldsymbol{\delta}_3^*)}{2} \approx 2.927 \; .$$

So

$$f\!\left(\frac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_3^*}{2}\right) > \frac{f(\boldsymbol{\delta}_1^*) + f(\boldsymbol{\delta}_3^*)}{2} \; ,$$

which implies that f is not convex.

As a consequence, the objective functions J_1, J_2, J_4 and J_5 are in general neither convex nor concave.

Indeed, we have

$$J_l\!\left(\frac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}\right) < \frac{J_l(\boldsymbol{\delta}_1^*) + J_l(\boldsymbol{\delta}_2^*)}{2}$$

and

$$J_l\!\left(\frac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_3^*}{2}\right) > \frac{J_l(\boldsymbol{\delta}_1^*) + J_l(\boldsymbol{\delta}_3^*)}{2}$$
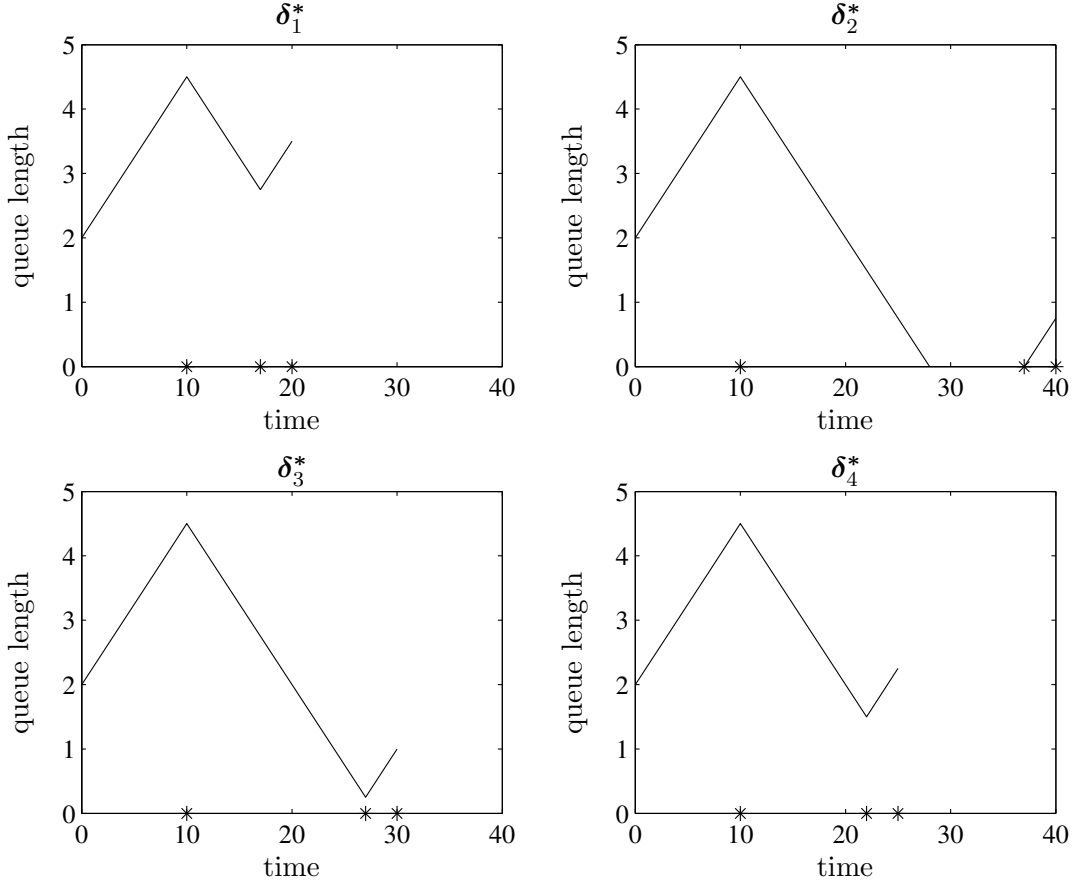
for $l = 1, 2, 4, 5$ (see Table 4). $\qquad\square$

$\boldsymbol{\delta}_1^*$

$\boldsymbol{\delta}_2^*$

$\boldsymbol{\delta}_3^*$

$\boldsymbol{\delta}_4^*$

queue length

time

Figure 10: The evolution of the queue length in lane L_1 as a function of time for the switching interval vectors $\boldsymbol{\delta}_1^*$, $\boldsymbol{\delta}_2^*$, $\boldsymbol{\delta}_3^* = \dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}$ and $\boldsymbol{\delta}_4^* = \dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_3^*}{2}$ of Example D.3. The $*$ signs on the time axis correspond to the switching time instants.

Let us now look at the convexity or concavity of the objective functions over the faces that constitute the solution set of the ELCP defined by $(26)-(29)$.

In the next proposition we shall make the following extra assumption:

- in each lane, the average departure rate when the light is amber is less than the average arrival rate of vehicles, i.e., $\bar{\kappa}_i < \bar{\lambda}_i$.

Note that this is again a reasonable assumption if we take into account that designing optimal traffic light switching schemes is only useful if the arrival rates $\bar{\lambda}_i$ are high and that under normal circumstances $\bar{\kappa}_i$ is very small. This assumption implies that the net queue growth rate during the amber phase $\bar{\lambda}_i - \bar{\kappa}_i$ is positive. Therefore, the queue length at the end of a green phase is given by: $l_i(t_{k+1} - \delta_{\mathrm{amb}}) = l_i(t_{k+1}) - (\bar{\lambda}_i - \bar{\kappa}_i)\delta_{\mathrm{amb}}$ where $k \in \mathcal{G}_i(N)$.

Proposition D.4 *Consider an optimal traffic light switching problem with $\bar{\kappa}_i < \bar{\lambda}_i$ for all i. Let the set of feasible solutions of the ELCP that corresponds to the given problem be characterized by the set of vertices $V = \left\{ \begin{bmatrix} \boldsymbol{x}_i^* \\ \boldsymbol{\delta}_i^* \end{bmatrix} \,\middle|\, i = 1, 2, \ldots, r \right\}$ and the set of index sets Λ.*

l	$J_l(\boldsymbol{\delta}_1^*)$	$J_l(\boldsymbol{\delta}_2^*)$	$J_l\!\left(\dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}\right)$	$\dfrac{J_l(\boldsymbol{\delta}_1^*) + J_l(\boldsymbol{\delta}_2^*)}{2}$
1	8.838	10.513	9.392	9.675
2	3.363	3.403	2.492	3.383
4	35.350	42.050	37.567	38.700
5	13.450	13.613	9.967	13.531

l	$J_l(\boldsymbol{\delta}_1^*)$	$J_l(\boldsymbol{\delta}_3^*)$	$J_l\!\left(\dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_3^*}{2}\right)$	$\dfrac{J_l(\boldsymbol{\delta}_1^*) + J_l(\boldsymbol{\delta}_3^*)}{2}$
1	8.838	9.392	9.170	9.115
2	3.363	2.492	2.965	2.927
4	35.350	37.567	36.680	36.458
5	13.450	9.967	11.860	11.708

Table 4: The values of the objective functions J_1, J_2, J_4 and J_5 (up to 3 decimal places) for the switching time vectors $\boldsymbol{\delta}_1^*$, $\boldsymbol{\delta}_2^*$, $\boldsymbol{\delta}_3^* = \dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}$ and $\boldsymbol{\delta}_4^* = \dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_3^*}{2}$ of Example D.3.

Let $\phi_j = \{i_1, i_2, \ldots, i_s\} \in \Lambda$. Consider a vector $\boldsymbol{\nu} \in \mathbb{R}^s$ with $\nu_j \geqslant 0$ for all j and $\sum_{j=1}^s \nu_j = 1$. Define $\boldsymbol{\delta}^*(\boldsymbol{\nu}) = \sum_{j=1}^s \nu_j \boldsymbol{\delta}_{i_j}^*$. Then the function I_i defined by $I_i(\boldsymbol{\nu}) = \int_{t_0}^{t_N} l_i\big(t, \boldsymbol{\delta}^*(\boldsymbol{\nu})\big)\, dt$ is a convex function of $\boldsymbol{\nu}$.

Proof: We may assume without loss of generality that $i = 1$ and $\phi_j = \{1, 2, \ldots, s\}$. Define $\boldsymbol{x}^*(\boldsymbol{\nu}) = \sum_{j=1}^s \nu_j \boldsymbol{x}_j^*$. Since $\begin{bmatrix} \boldsymbol{x}^*(\boldsymbol{\nu}) \\ \boldsymbol{\delta}^*(\boldsymbol{\nu}) \end{bmatrix}$ is a convex combination of the vertices of V that are indexed by ϕ_j, it is a solution of the ELCP that corresponds to the given optimal traffic light switching problem. So $\boldsymbol{x}^*(\boldsymbol{\nu})$ and $\boldsymbol{\delta}^*(\boldsymbol{\nu})$ are compatible for the given $\boldsymbol{x}_0$. For each $\boldsymbol{x}_i^*$ with $i \in \phi_j$ we define a sequence $y_{i,0}, y_{i,1}, \ldots, y_{i,N}$ that contains the components of $\boldsymbol{x}_0$ and $\boldsymbol{x}_i^*$ that correspond to the queue length in lane L_1: $y_{i,0} = (\boldsymbol{x}_0)_1$ and $y_{i,j} = (\boldsymbol{x}_i^*)_{4(j-1)+1}$ for $j = 1, 2, \ldots, N$. Let $y_0 = (\boldsymbol{x_0})_1$ and $y_j = (\boldsymbol{x}^*)_1$ for $j = 1, 2, \ldots, N$. Define $\delta_{i,k} = (\boldsymbol{\delta}_i^*)_{k+1}$ for $i = 1, 2, \ldots, s$ and $k = 0, 1, \ldots, N-1$. Note that $\sum_{j=1}^s \nu_j y_{j,0} = y_0 = (\boldsymbol{x}_0)_1$.

Since $\bar{\kappa}_1 < \bar{\lambda}_1$ we have

$$\tilde{y}_{2k+2} \overset{\text{def}}{=} l_1(t_{2k+2} - \delta_{\text{amb}}) = y_{2k+2} - (\bar{\lambda}_1 - \bar{\kappa}_1)\delta_{\text{amb}} \ .$$

If we define

$$\tilde{y}_{j,2k+2} = y_{j,2k+2} - (\bar{\lambda}_1 - \bar{\kappa}_1)\delta_{\text{amb}}$$

for all j, k, then we have

$$\sum_{j=1}^{s} \nu_j \tilde{y}_{j,2k+2} = \sum_{j=1}^{s} \nu_j \big(y_{j,2k+2} - (\bar{\lambda}_1 - \bar{\kappa}_1)\delta_{\mathrm{amb}}\big)$$

$$= \sum_{j=1}^{s} \nu_j y_{j,2k+2} - \Big(\sum_{j=1}^{s} \nu_j\Big)(\bar{\lambda}_1 - \bar{\kappa}_1)\delta_{\mathrm{amb}}$$

$$= y_{2k+2} - (\bar{\lambda}_1 - \bar{\kappa}_1)\delta_{\mathrm{amb}}$$

$$= \tilde{y}_{2k+2} \ .$$

Furthermore, the assumption $\bar{\kappa}_1 < \bar{\lambda}_1$ also implies that $y_{2k+2} \neq 0$ and $y_{i,2k+2} \neq 0$ for all i, k. Recall that $\boldsymbol{x}^*(\boldsymbol{\nu})$ and $\boldsymbol{\delta}^*(\boldsymbol{\nu})$ are compatible for $\boldsymbol{x}_0$. As a consequence, we have (cf. Section C)

$$I_1(\boldsymbol{\nu}) \ = \ S_1(\boldsymbol{\nu}) + S_{2,1}(\boldsymbol{\nu}) - S_{2,2}(\boldsymbol{\nu}) + S_3(\boldsymbol{\nu}) + S_4(\boldsymbol{\nu})$$

with

$$S_1(\boldsymbol{\nu}) \ = \ \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{\Big(\sum_{j=1}^{s} \nu_j\, y_{j,2k}\Big) + \Big(\sum_{j=1}^{s} \nu_j\, y_{j,2k+1}\Big)}{2} \Big(\sum_{j=1}^{s} \nu_j\, \delta_{j,2k}\Big)$$

$$S_{2,1}(\boldsymbol{\nu}) \ = \ \sum_{\substack{k=0 \\ \sum_{j=1}^{s} \nu_j \tilde{y}_{j,2k+2} \neq 0}}^{\lfloor \frac{N}{2} \rfloor - 1} \frac{\Big(\sum_{j=1}^{s} \nu_j\, y_{j,2k+1}\Big) + \Big(\sum_{j=1}^{s} \nu_j\, \tilde{y}_{j,2k+2}\Big)}{2} \Big(\sum_{j=1}^{s} \nu_j\, \delta_{j,2k+1}\Big)$$

$$S_{2,2}(\boldsymbol{\nu}) \ = \ \sum_{\substack{k=0 \\ \sum_{j=1}^{s} \nu_j \tilde{y}_{j,2k+2} \neq 0}}^{\lfloor \frac{N}{2} \rfloor - 1} \frac{\Big(\sum_{j=1}^{s} \nu_j\, y_{j,2k+1}\Big) + \Big(\sum_{j=1}^{s} \nu_j\, \tilde{y}_{j,2k+2}\Big)}{2} \delta_{\mathrm{amb}}$$

$$S_3(\boldsymbol{\nu}) \ = \ \sum_{\substack{k=0 \\ \sum_{j=1}^{s} \nu_j \tilde{y}_{j,2k+2} = 0}}^{\lfloor \frac{N}{2} \rfloor - 1} \frac{\Big(\sum_{j=1}^{s} \nu_j\, y_{j,2k+1}\Big)^2}{2\,(\bar{\mu}_1 - \bar{\lambda}_1)}$$

$$S_4(\boldsymbol{\nu}) \ = \ \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor - 1} \frac{\Big(\sum_{j=1}^{s} \nu_j\, \tilde{y}_{j,2k+2}\Big) + \Big(\sum_{j=1}^{s} \nu_j\, y_{j,2k+2}\Big)}{2} \delta_{\mathrm{amb}} \ .$$

It easy to verify that S_3 is a convex, quadratic function of $\boldsymbol{\nu}$. Furthermore, $S_{2,2}$ and S_4 are linear functions of $\boldsymbol{\nu}$, which implies that they are also convex. So from now on, we only consider S_1 and $S_{2,1}$. Since S_1 and $S_{2,1}$ are continuous functions of the ν_j's, it is sufficient to prove that S_1 and $S_{2,1}$ are convex functions in the relative interior of the feasible region, i.e., for positive ν_j's. As a consequence, we have $\sum_{j=1}^{s} \nu_j \tilde{y}_{j,2k+2} \neq 0$ if and only if there exists an index $i \in \{1, 2,, \ldots, s\}$ such that $\tilde{y}_{i,2k+2} \neq 0$. Note that is condition is independent of the values of the ν_j's (provided that they are positive). If we define

$$z_{j,2k} = y_{j,2k} + y_{j,2k+1}$$

$$z_{j,2k+1} = \begin{cases} y_{j,2k+1} + \tilde{y}_{j,2k+2} & \text{if } \exists i \in \{1, 2,, \ldots, s\} \text{ such that } \tilde{y}_{i,2k+2} \neq 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

for all j, k, then we have

$$S_1(\boldsymbol{\nu}) + S_{2,1}(\boldsymbol{\nu}) = \sum_{k=0}^{N} \frac{\left(\sum_{j=1}^{s} \nu_j z_{j,k} \right) \left(\sum_{j=1}^{s} \nu_j \delta_{j,k} \right)}{2}$$

$$= \sum_{k=0}^{N} \frac{1}{2} \boldsymbol{\nu}^T \boldsymbol{Q}_k \boldsymbol{\nu} \ .$$

with

$$\boldsymbol{Q}_k = \begin{bmatrix} z_{1,k} \\ z_{2,k} \\ \vdots \\ z_{s,k} \end{bmatrix} \begin{bmatrix} \delta_{1,k} & \delta_{2,k} & \ldots & \delta_{s,k} \end{bmatrix} \ .$$

So $\boldsymbol{Q}_k$ is a matrix of rank 1 with nonnegative entries. This implies that $\boldsymbol{Q}_k$ is a positive semi-definite matrix and that $S_1 + S_{2,1}$ is a convex, quadratic function.

Hence, $I_1 = S_1 + S_{2,1} - S_{2,2} + S_3 + S_4$ is a convex, quadratic function of $\boldsymbol{\nu}$. $\qquad\square$

Note that in order to obtain J_1, J_2, J_4 or J_5 we have to divide $\int_{t_0}^{t_N} l_1(t, \boldsymbol{\delta}^*(\boldsymbol{\nu})) \, dt$ by $t_N - t_0 = \sum_{k=0}^{N-1} \sum_{j=1}^{s} \nu_j \delta_{i_j,k}$ which is a linear function of $\boldsymbol{\nu}$. As a consequence, J_1, J_2, J_4 and J_5 are in general not convex functions of $\boldsymbol{\nu}$. However, computational experiments have shown that in most cases J_1, J_2, J_4 and J_5 are very smooth functions of $\boldsymbol{\nu}$ (for many faces they are even almost linear or almost convex). This means that finding the combination ν_1, ν_2, $\ldots$, ν_s for which $J_l \left(\sum_{j=1}^{s} \nu_j \boldsymbol{\delta}_{i_j}^* \right)$ with $l = 1, 2, 4$ or 5 reaches a global minimum is a well-behaved problem in the sense that for almost all initial starting points the same numerical solution (within a certain tolerance) will be obtained.

Since there exist very efficient algorithms to minimize convex objective functions over a convex feasible sets, we now examine whether the approximate objective functions $\tilde{J}_1$ and

$\tilde{J}_4$ are convex over the feasible set of the relaxed problem $\tilde{\mathcal{P}}$. Note that it makes only sense to determine convexity (or concavity) for objective functions that are strictly monotonous functions of $\boldsymbol{x}^*$, since only for these objective functions we can use Proposition 3.2 to minimize over the convex feasible set of the relaxed problem $\tilde{\mathcal{P}}$ instead of over the feasible set of the original problem $\mathcal{P}$.

The following example shows that the approximate objective functions $\tilde{J}_1$ and $\tilde{J}_4$ are in general neither convex nor concave as a function of $\boldsymbol{x}^*$ and $\boldsymbol{\delta}^*$.

Example D.5 Let $\delta_{\mathrm{amb}} = 4$ and $\bar{\lambda}_i = 0.25$, $\bar{\mu}_i = 0.5$, $\bar{\kappa}_i = 0$ for $i = 1, 2, 3, 4$. Let

$$
\boldsymbol{x}_0 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \ \boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \boldsymbol{\delta}_1^* = \begin{bmatrix} 12 \\ 12 \end{bmatrix}, \ \boldsymbol{\delta}_2^* = \begin{bmatrix} 12 \\ 32 \end{bmatrix}, \ \boldsymbol{\delta}_3^* = \begin{bmatrix} 12 \\ 40 \end{bmatrix},
$$

$$
\boldsymbol{y}_1^* = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \ \text{and} \ \boldsymbol{y}_2^* = \boldsymbol{y}_3^* = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \ .
$$

Note that the queue length sequence that corresponds to $\boldsymbol{y}_1^*$ is compatible with $\boldsymbol{\delta}_1^*$ for $l_1(0) = y_0 \stackrel{\text{def}}{=} (\boldsymbol{x}_0)_1$. This also holds for $\boldsymbol{y}_2^*$ and $\boldsymbol{\delta}_2^*$, and for $\boldsymbol{y}_3^*$ and $\boldsymbol{\delta}_3^*$. In Figure 11 we have plotted the evolution of l_1 as a function of time for the switching sequences defined by $\boldsymbol{\delta}_1^*$, $\boldsymbol{\delta}_2^*$, $\boldsymbol{\delta}_3^*$, $\boldsymbol{\delta}_4^* = \dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}$ and $\boldsymbol{\delta}_5^* = \dfrac{\boldsymbol{\delta}_2^* + \boldsymbol{\delta}_3^*}{2}$.

If $y_0 \in \mathbb{R}^+$, $\boldsymbol{y}^* \in (\mathbb{R}^+)^N$ and $\boldsymbol{\delta}^* \in (\mathbb{R}_0^+)^N$, then $\tilde{l}_1(\cdot, \boldsymbol{y}^*, \boldsymbol{\delta}^*)$ is the piecewise-linear function that interpolates in the points (t_0, y_0), (t_1, y_1^*), $\ldots$, (t_N, y_N^*). Define

$$
\tilde{f}(\boldsymbol{y}^*, \boldsymbol{\delta}^*) = \frac{\displaystyle\int_{t_0}^{t_N} \tilde{l}_1(t, \boldsymbol{y}^*, \boldsymbol{\delta}^*) \, dt}{t_N - t_0} \ .
$$

We have $\tilde{f}(\boldsymbol{y}_1^*, \boldsymbol{\delta}_1^*) = 6$, $\tilde{f}(\boldsymbol{y}_2^*, \boldsymbol{\delta}_2^*) \approx 4.409$,

$$
\tilde{f}\left(\frac{\boldsymbol{y}_1^* + \boldsymbol{y}_2^*}{2}, \frac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2} \right) \approx 5.338 \quad \text{and} \quad \frac{\tilde{f}(\boldsymbol{y}_1^*, \boldsymbol{\delta}_1^*) + \tilde{f}(\boldsymbol{y}_2^*, \boldsymbol{\delta}_2^*)}{2} \approx 5.205 \ .
$$

So

$$
\tilde{f}\left(\frac{\boldsymbol{y}_1^* + \boldsymbol{y}_2^*}{2}, \frac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2} \right) > \frac{\tilde{f}(\boldsymbol{y}_1^*, \boldsymbol{\delta}_1^*) + \tilde{f}(\boldsymbol{y}_2^*, \boldsymbol{\delta}_2^*)}{2} \ ,
$$

which implies that $\tilde{f}$ is not convex.

On the other hand, we have $\tilde{f}(\boldsymbol{y}_2^*, \boldsymbol{\delta}_2^*) \approx 4.409$, $\tilde{f}(\boldsymbol{y}_3^*, \boldsymbol{\delta}_3^*) \approx 4.346$,

$$
\tilde{f}\left(\frac{\boldsymbol{y}_2^* + \boldsymbol{y}^*3}{2}, \frac{\boldsymbol{\delta}_2^* + \boldsymbol{\delta}_3^*}{2} \right) = 4.375 \quad \text{and} \quad \frac{\tilde{f}(\boldsymbol{y}_2^*, \boldsymbol{\delta}_2^*) + \tilde{f}(\boldsymbol{y}_3^*, \boldsymbol{\delta}_3^*)}{2} \approx 4.378 \ .
$$

So

$$
\tilde{f}\left(\frac{\boldsymbol{y}_2^* + \boldsymbol{y}_3^*}{2}, \frac{\boldsymbol{\delta}_2^* + \boldsymbol{\delta}_3^*}{2} \right) < \frac{\tilde{f}(\boldsymbol{y}_2^*, \boldsymbol{\delta}_2^*) + \tilde{f}(\boldsymbol{y}_3^*, \boldsymbol{\delta}_3^*)}{2} \ ,
$$

which implies that $\tilde{f}$ is not concave.

As a consequence, the objective functions $\tilde{J}_1$ and $\tilde{J}_4$ are in general neither convex nor concave.
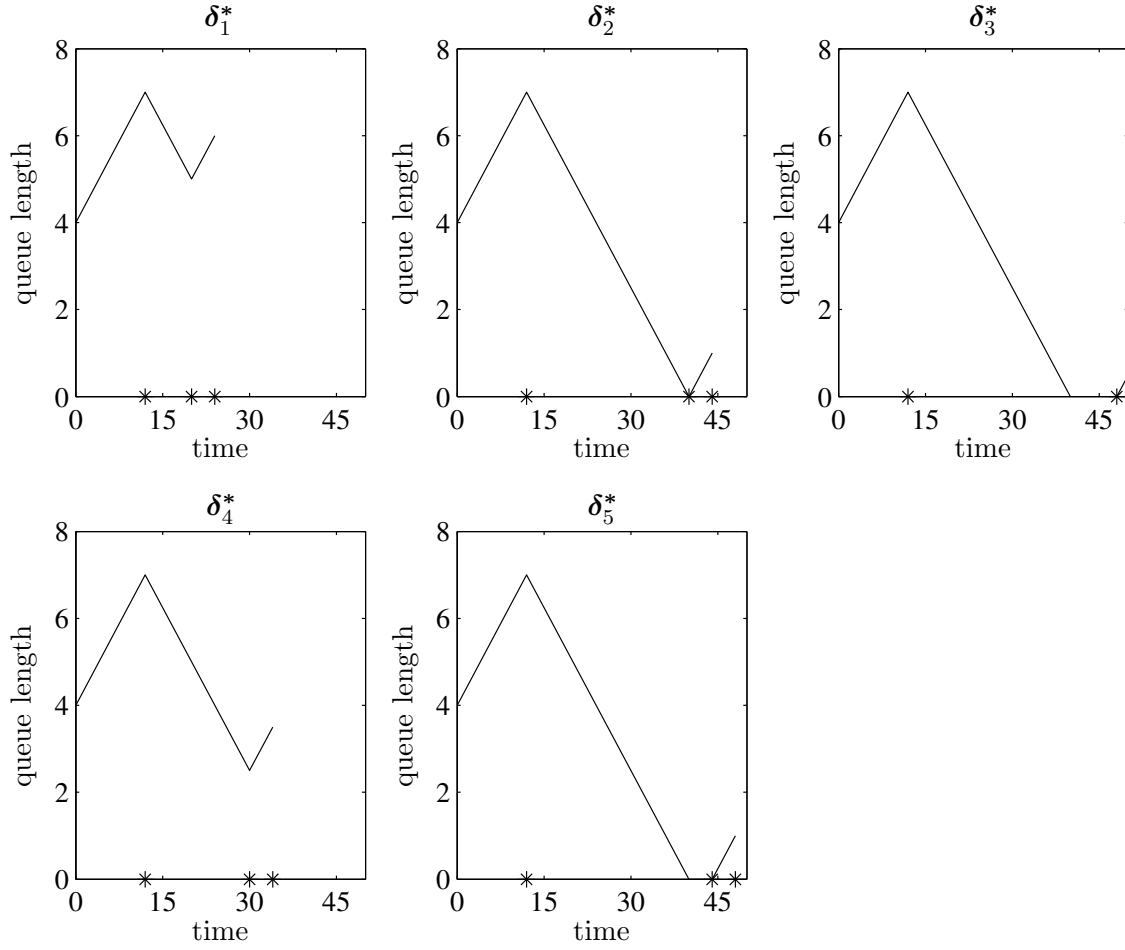
Figure 11: The evolution of the queue length in lane L_1 as a function of time t for the switching interval vectors $\boldsymbol{\delta}_1^*$, $\boldsymbol{\delta}_2^*$, $\boldsymbol{\delta}_3^*$, $\boldsymbol{\delta}_4^* = \dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}$ and $\boldsymbol{\delta}_5^* = \dfrac{\boldsymbol{\delta}_2^* + \boldsymbol{\delta}_3^*}{2}$ of Example D.5. The $*$ signs on the time axis correspond to the switching time instants.

Indeed, for the queue length vectors $\boldsymbol{x}_1^*$, $\boldsymbol{x}_2^*$ and $\boldsymbol{x}_3^*$ that are compatible with respectively $\boldsymbol{\delta}_1^*$, $\boldsymbol{\delta}_2^*$ and $\boldsymbol{\delta}_3^*$ for $\boldsymbol{x}_0$, we have

$$\tilde{J}_l\Big(\frac{\boldsymbol{x}_1^* + \boldsymbol{x}_2^*}{2}, \frac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}\Big) > \frac{\tilde{J}_l(\boldsymbol{x}_1^*, \boldsymbol{\delta}_1^*) + \tilde{J}_l(\boldsymbol{x}_2^*, \boldsymbol{\delta}_2^*)}{2}$$

and

$$\tilde{J}_l\Big(\frac{\boldsymbol{x}_2^* + \boldsymbol{x}_3^*}{2}, \frac{\boldsymbol{\delta}_2^* + \boldsymbol{\delta}_3^*}{2}\Big) < \frac{\tilde{J}_l(\boldsymbol{x}_2^*, \boldsymbol{\delta}_2^*) + \tilde{J}_l(\boldsymbol{x}_3^*, \boldsymbol{\delta}_3^*)}{2}$$

for $l = 1$ and 4 (see Table 5).   □

E   Another approximation for the objective functions J_1, J_2, J_3, J_4 and J_5

In this section we make again the following extra assumption:

l	$\tilde{J}_l(\boldsymbol{x}_1^*, \boldsymbol{\delta}_1^*)$	$\tilde{J}_l(\boldsymbol{x}_2^*, \boldsymbol{\delta}_2^*)$	$\tilde{J}_l\left(\dfrac{\boldsymbol{x}_1^* + \boldsymbol{x}_2^*}{2}, \dfrac{\boldsymbol{\delta}_1^* + \boldsymbol{\delta}_2^*}{2}\right)$	$\dfrac{\tilde{J}_l(\boldsymbol{x}_1^*, \boldsymbol{\delta}_1^*) + \tilde{J}_l(\boldsymbol{x}_2^*, \boldsymbol{\delta}_2^*)}{2}$
1	15.000	16.364	15.882	15.682
4	60.000	65.455	63.529	62.727

l	$\tilde{J}_l(\boldsymbol{x}_2^*, \boldsymbol{\delta}_2^*)$	$\tilde{J}_l(\boldsymbol{x}_3^*, \boldsymbol{\delta}_3^*)$	$\tilde{J}_l\left(\dfrac{\boldsymbol{x}_2^* + \boldsymbol{x}_3^*}{2}, \dfrac{\boldsymbol{\delta}_2^* + \boldsymbol{\delta}_3^*}{2}\right)$	$\dfrac{\tilde{J}_l(\boldsymbol{x}_2^*, \boldsymbol{\delta}_2^*) + \tilde{J}_l(\boldsymbol{x}_3^*, \boldsymbol{\delta}_3^*)}{2}$
1	16.364	18.154	17.250	17.259
4	65.455	72.615	69.000	69.035

Table 5: The values of the objective functions $\tilde{J}_1$ and $\tilde{J}_4$ (up to 3 decimal places) for the queue length vectors $\boldsymbol{x}_1^*$, $\boldsymbol{x}_2^*$ and $\boldsymbol{x}_3^*$ that are compatible with respectively the switching interval vectors $\boldsymbol{\delta}_1^*$, $\boldsymbol{\delta}_2^*$ and $\boldsymbol{\delta}_3^*$ of Example D.5.

- in each lane, the average departure rate when the light is amber is less than the average arrival rate of vehicles, i.e., $\bar{\kappa}_i < \bar{\lambda}_i$.

Recall that this assumption implies that the net queue growth rate during the amber phase $\bar{\lambda}_i - \bar{\kappa}_i$ is positive. As a consequence, the queue length at the end of the green phase $(t_k, t_{k+1} - \delta_{\mathrm{amb}})$ with $k \in \mathcal{G}_i(N)$ is given by: $l_i(t_{k+1} - \delta_{\mathrm{amb}}) = l_i(t_{k+1}) - (\bar{\lambda}_i - \bar{\kappa}_i)\delta_{\mathrm{amb}}$ and the queue length at the end of the subsequent amber phase $l_i(t_{k+1})$ is positive.

For a given $\boldsymbol{x}_0$ and t_0, we define the function $\check{l}_i(\cdot, \boldsymbol{x}^*, \boldsymbol{\delta}^*)$ — or $\check{l}_i(\cdot)$ for short — as the piecewise-linear function that interpolates in the points $(t_0, l_i(t_0))$, $(t_{k+1} - \delta_{\mathrm{amb}}, l_i(t_{k+1} - \delta_{\mathrm{amb}}))$ for $k \in \mathcal{G}_i(N)$ — i.e., the points at the beginning and the end of the green phase for T_i — and the point $(t_N, l_i(t_N))$. The approximate objective functions $\check{J}_l$ for $l = 1, 2, 3, 4, 5$, are defined as in $(8) - (12)$ but with l_i replaced by $\check{l}_i$.

The values of J_3 and $\check{J}_3$ always coincide. Now let $l \in \{1, 2, 4, 5\}$. Recall that the value of J_l and $\check{J}_l$ is determined by the surface under the functions l_i and $\check{l}_i$ respectively. If the queue lengths never become zero during the green phases and if no vehicles depart when the traffic light is amber (i.e., $\bar{\kappa}_i = 0$ for all i), then the functions l_i and $\check{l}_i$ and the values of J_l and $\check{J}_l$ coincide (cf. Figure 12). In practice, the departure rate during the amber phase will be small. Moreover, the length of the amber phase will also be small compared to the length of the green or the red phase. Furthermore, if we have an optimal traffic light switching scheme, then the periods during which the queue length in some lane is equal to 0 are in general short. As a consequence, for traffic light switching schemes in the neighborhood of the optimal scheme $\check{J}_l$ will be a good approximation of J_l. It is easy to verify that under normal circumstances $\check{J}_l$ will be a better approximation of J_l than $\tilde{J}_l$ (cf. Figures 2 and 12). However, in general (i.e., if we allow large values for δ_{amb}) we cannot impose a relative order on $\tilde{J}_l$ and $\check{J}_l$.

Using proofs that are similar to those of Propositions 3.1 and 3.3 it can be shown that the following two propositions hold:

Proposition E.1 *Let $\boldsymbol{x}_0 \in (\mathbb{R}^+)^4$, $\boldsymbol{x}^* \in (\mathbb{R}^+)^{4N}$ and $\boldsymbol{\delta}^* \in (\mathbb{R}_0^+)^N$. If $\boldsymbol{x}^*$ and $\boldsymbol{\delta}^*$ are compatible for $\boldsymbol{x}_0$ then we have $J_3(\boldsymbol{\delta}^*) = \check{J}_3(\boldsymbol{x}^*, \boldsymbol{\delta}^*)$, and $J_l(\boldsymbol{\delta}^*) \leqslant \check{J}_l(\boldsymbol{x}^*, \boldsymbol{\delta}^*)$ for $l = 1, 2, 4, 5$.*
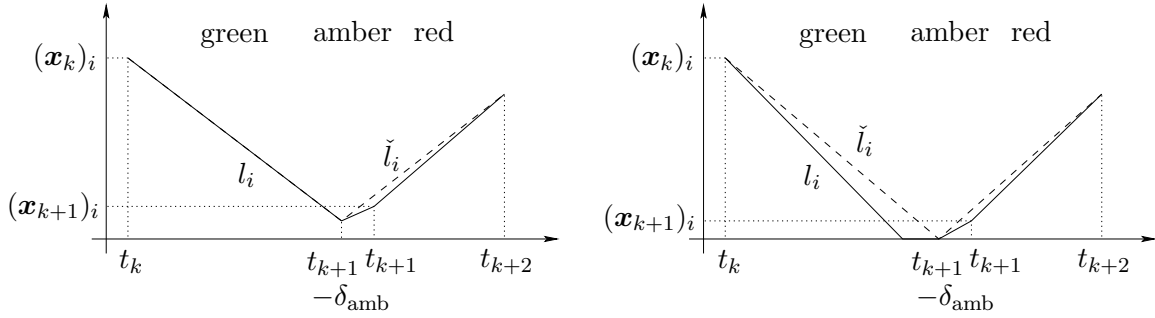
Figure 12: The functions l_i (full line) and $\check{l}_i$ (dashed line). The left plot shows a situation in which the queue length does not become 0 during the green phase and then l_i and $\check{l}_i$ coincide during the green phase. The right plot shows a situation where the queue length becomes 0 during the green phase.

Proposition E.2 *For given $\boldsymbol{x}_0$, δ_{amb}, $\bar{\lambda}_i$'s, $\bar{\mu}_i$'s, $\bar{\kappa}_i$'s and a given $\boldsymbol{\delta^*}$ the functions $\check{J}_1$ and $\check{J}_4$ are strictly monotonous functions of $\boldsymbol{x^*}$.*

So for $\check{J}_1$ and $\check{J}_4$ we can compute optimal traffic light switching schemes using the relaxed problem $\tilde{\mathcal{P}}$ instead of the original problem $\mathcal{P}$.

Let us now derive a formula for the evaluation of

$$\int_{t_0}^{t_N} \check{l}_1(t, \boldsymbol{x^*}, \boldsymbol{\delta^*})\, dt = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \check{l}_1(t, \boldsymbol{x^*}, \boldsymbol{\delta^*})\, dt \ .$$

Define $y_k = (\boldsymbol{x}_k)_1 = l_1(t_k)$ for $k = 0, 1, \ldots, N$ and $\tilde{y}_{2k+2} = l_1(t_{2k+2} - \delta_{\mathrm{amb}})$ for $k = 0, 1, \ldots, \left\lfloor \dfrac{N}{2} \right\rfloor - 1$. Let the function *even* be defined by

$$even(n) = \begin{cases} 1 & \text{if } n \text{ is an even integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Now it is easy to verify that

$$\int_{t_0}^{t_N} \check{l}_1(t, \boldsymbol{x^*}, \boldsymbol{\delta^*})\, dt = \frac{y_0 + y_1}{2} \delta_0 \ + \ \sum_{k=0}^{\left\lfloor \frac{N}{2} \right\rfloor - 1} \frac{y_{2k+1} + \tilde{y}_{2k+2}}{2} \left(\delta_{2k+1} - \delta_{\mathrm{amb}}\right) +$$

$$\sum_{k=0}^{\left\lfloor \frac{N-3}{2} \right\rfloor} \frac{\tilde{y}_{2k+2} + y_{2k+3}}{2} \left(\delta_{2k+2} + \delta_{\mathrm{amb}}\right) \ + \ even(n) \frac{\tilde{y}_N + y_N}{2} \delta_{\mathrm{amb}} \ .$$

Note that in contrast to $\tilde{J}_1$ and $\tilde{J}_4$ making the assumption $\delta_k \approx \dfrac{t_N - t_0}{N}$ does not lead a linear objective function for $\check{J}_1$ and $\check{J}_4$.

Let us now compute a suboptimal traffic light switching scheme based on the objective function $\check{J}_1$ for the set-up of Example 5.1.

$\boldsymbol{\delta}^*, \boldsymbol{x}^*$	$J_1(\boldsymbol{\delta}^*)$	$\check{J}_1(\boldsymbol{x}^*, \boldsymbol{\delta}^*)$	$\tilde{J}_1(\boldsymbol{x}^*, \boldsymbol{\delta}^*)$	$\hat{J}_1(\boldsymbol{x}^*, \boldsymbol{\delta}^*)$	$J_{\mathrm{lin}}(\boldsymbol{x}^*)$	CPU time
$\check{\boldsymbol{\delta}}^*, \check{\boldsymbol{x}}^*$	60.669	62.768	64.268	69.036	433.751	4.01
$\boldsymbol{\delta}^*_{\mathrm{ELCP}}, \boldsymbol{x}^*_{\mathrm{ELCP}}$	60.657	62.780	64.267	69.190	434.827	404.83
$\boldsymbol{\delta}^*_{\mathrm{pen}}, \boldsymbol{x}^*_{\mathrm{pen}}$	61.150	63.258	64.740	69.916	439.909	78.69
$\boldsymbol{\delta}^*_{\mathrm{mul}}, \boldsymbol{x}^*_{\mathrm{mul}}$	61.613	63.513	65.118	67.881	425.664	13.83
$\tilde{\boldsymbol{\delta}}^*, \tilde{\boldsymbol{x}}^*$	60.659	62.772	64.264	69.117	434.319	2.49
$\boldsymbol{\delta}^*_{\mathrm{lin}}, \boldsymbol{x}^*_{\mathrm{lin}}$	64.551	66.239	67.905	67.199	420.895	0.94
$\boldsymbol{\delta}^*_{\mathrm{con}}, \boldsymbol{x}^*_{\mathrm{con}}$	63.101	64.741	66.363	67.565	423.455	96.40

Table 6: The values of the objective functions J_1, $\check{J}_1$, $\tilde{J}_1$, $\hat{J}_1$ and J_{lin} (up to 3 decimal places) and the CPU time (up to 2 decimal places) needed to compute the suboptimal switching interval vector $\check{\boldsymbol{\delta}}^*$ of Example E.3 and the (sub)optimal switching interval vectors $\boldsymbol{\delta}^*_{\mathrm{ELCP}}$, $\boldsymbol{\delta}^*_{\mathrm{pen}}$, $\boldsymbol{\delta}^*_{\mathrm{mul}}$, $\tilde{\boldsymbol{\delta}}^*$, $\boldsymbol{\delta}^*_{\mathrm{lin}}$ and $\boldsymbol{\delta}^*_{\mathrm{con}}$ of Example 5.1. The queue length vectors $\boldsymbol{x}^*$ are compatible with the switching interval vectors $\boldsymbol{\delta}^*$ for $\boldsymbol{x}_0$.

Example E.3 Consider the intersection of Figure 1 with the same data as in Example 5.1. Suppose that we want to compute a traffic light switching sequence t_0, t_1, ..., t_7 that minimizes J_1. We use the `e04ucf` routine of the NAG library to compute a solution $\check{\boldsymbol{x}}^*$, $\check{\boldsymbol{\delta}}^*$ that minimizes the approximate objective function $\check{J}_1$ (using the relaxed problem $\tilde{\mathcal{P}}$). This yields[2]:

$$\check{\boldsymbol{\delta}}^* = \begin{bmatrix} 20.000 & 45.750 & 30.964 & 63.000 & 30.964 & 63.000 & 55.509 \end{bmatrix}^T \ .$$

In Table 6 we have listed the values of the various objective functions for the switching interval vector $\check{\boldsymbol{\delta}}^*$ and for the switching interval vectors of Example 5.1. The evolution of the queue lengths for the traffic light control strategy that corresponds to $\check{\boldsymbol{\delta}}^*$ is represented in Figure 13. Clearly, the $\tilde{\boldsymbol{\delta}}^*$ solution also offers a good trade-off between optimality and efficiency. Note that for all the switching interval vectors of Table 6 the value of the objective function $\check{J}_1$ is lower than the value of the objective function $\tilde{J}_1$. $\square$

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[2]In this case using different starting points always leads to more or less the same numerical value of the optimal objective function: in an experiment with 20 random starting points the first 12 decimal places of the final objective function always had the same value. Therefore, we have only performed one run with an arbitrary random initial point here.
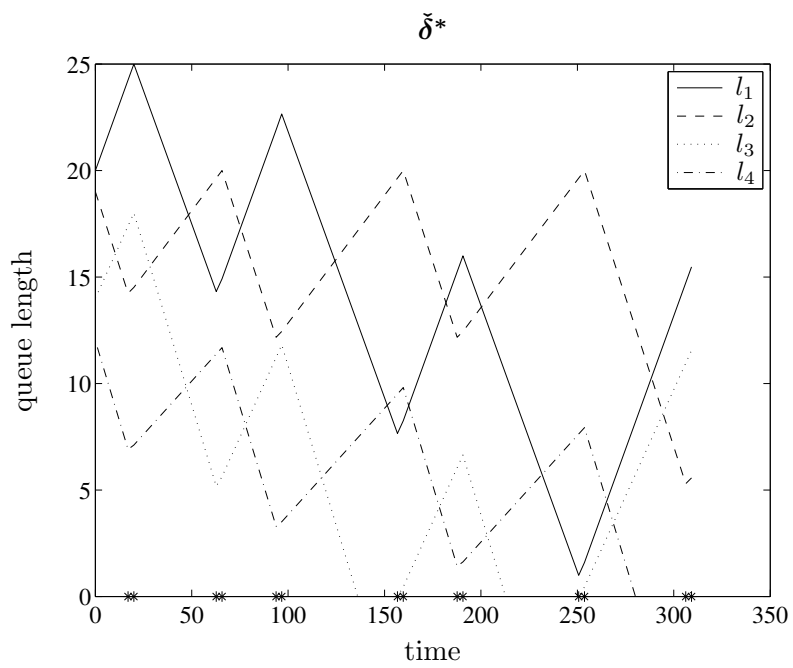
Figure 13: The queue lengths in the various lanes as a function of time for the traffic light switching sequence that corresponds to the switching interval vector $\check{\boldsymbol{\delta}}^*$ of Example E.3. The * signs on the time axis correspond to the switching time instants.