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Optimal traffic light control for a single intersection

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Abstract— We consider a single intersection of two two-way streets with controllable traffic lights on each corner. We construct a model that describes the evolution of the queue lengths in each lane as a function of time, and we discuss how (sub)optimal traffic light switching schemes for this system can be determined.

I. Introduction

In this paper we study the optimal traffic light control problem for a single intersection. Given the arrival rates and the maximal departure rates of vehicles at the intersection we compute traffic light switching schemes that minimize criteria such as average queue length, worst case queue length, average waiting time, ..., thereby augmenting the flow of traffic and diminishing the effects of traffic congestion. We show that for a special class of objective functions (i.e., objective functions that depend strictly monotonously on the queue lengths at the traffic light switching time instants), the optimal traffic light switching scheme can be computed very efficiently. Moreover, if the objective function is linear, the problem reduces to a linear programming problem.

II. The set-up and the model of the system

We consider an intersection of two two-way streets (see Figure 1). There are four lanes L_1, L_2, L_3 and L_4, and on each corner of the intersection there is a traffic light (T_1, T_2, T_3 and T_4). For sake of simplicity we assume that the traffic lights can be either green or red. The average arrival rate of cars in lane L_i is λ_i. When the traffic light is green, the average departure rate in lane L_i is μ_i. Let t_0, t_1, t_2, ... be the time instants at which the traffic lights switch from green to red or vice versa. The traffic light switching scheme is shown in Table 1. Let $l_i(t)$ be the queue length (i.e., the number of cars waiting) in lane L_i at time instant t. Normally, $l_i(t)$ can only take on integer values, but we use continuous approximations for the queue lengths. Note that although in general this will not lead to an optimal traffic light switching scheme, the traffic light switching scheme that we shall obtain will be a good approximation to the optimal scheme. Furthermore, in practice there will also be uncertainty due to inaccurate measurements and the time-varying nature of the arrival and departure rates. Define $\delta_k = t_{k+1} - t_k$ for $k \in \mathbb{N}$.

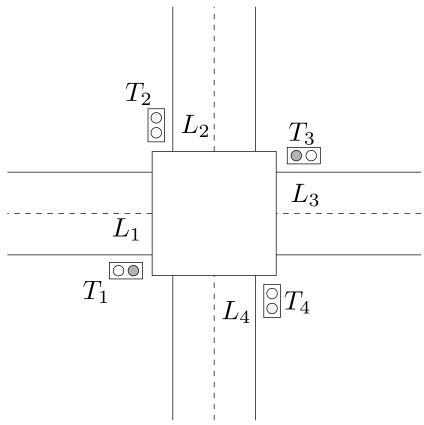

Figure 1: A traffic light controlled intersection of two two-way streets.

Period	T_1	T_2	T_3	T_4
$t_0 - t_1$	red	green	red	green
$t_1 - t_2$	green	red	green	red
$t_2 - t_3$	red	green	red	green
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: The traffic light switching scheme.

Let us now write down the equations that describe the relation between the switching time instants and the queue lengths.

When the traffic light T_1 is red, there are arrivals at lane L_1 and no departures. Hence,

$$\frac{d\,l_1(t)}{dt} = \lambda_1$$

for $t \in (t_{2k}, t_{2k+1})$ with $k \in \mathbb{N}$, and

$$l_1(t_{2k+1}) = l_1(t_{2k}) + \lambda_1 \delta_{2k}$$

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for $k = 0, 1, 2, \ldots$ When the traffic light T_1 is green, there are arrivals and departures at lane L_1. Since the net arrival rate is $\lambda_1 - \mu_1$ and since the queue length $l_1(t)$ cannot be negative, we have:

$$\frac{d\, l_1(t)}{dt} = \begin{cases} \lambda_1 - \mu_1 & \text{if } l_1(t) > 0 \\ 0 & \text{if } l_1(t) = 0 \end{cases}$$

for $t \in (t_{2k+1}, t_{2k+2})$ with $k \in \mathbb{N}$. So

$$l_1(t_{2k+2}) = \max\big(l_1(t_{2k+1}) + (\lambda_1 - \mu_1)\delta_{2k+1},\, 0\big)$$

for $k = 0, 1, 2, \ldots$ Note that we also have

$$l_1(t_{2k+1}) = \max\big(l_1(t_{2k}) + \lambda_1 \delta_{2k},\, 0\big)$$

for $k = 0, 1, 2, \ldots$, since $l_1(t) \geqslant 0$ for all t.
We can write down similar equations for $l_2(t_k)$, $l_3(t_k)$ and $l_4(t_k)$. So if we define

$$\begin{aligned} x_k &= \begin{bmatrix} l_1(t_k) & l_2(t_k) & l_3(t_k) & l_4(t_k) \end{bmatrix}^T \\ b_1 &= \begin{bmatrix} \lambda_1 & \lambda_2 - \mu_2 & \lambda_3 & \lambda_4 - \mu_4 \end{bmatrix}^T \\ b_2 &= \begin{bmatrix} \lambda_1 - \mu_1 & \lambda_2 & \lambda_3 - \mu_3 & \lambda_4 \end{bmatrix}^T \quad , \end{aligned}$$

then we have

$$x_{2k+1} = \max\left(x_{2k} + b_1 \delta_{2k},\, 0\right) \tag{1}$$
$$x_{2k+2} = \max\left(x_{2k+1} + b_2 \delta_{2k+1},\, 0\right) \tag{2}$$

for $k = 0, 1, 2, \ldots$

The model we have derived above is different from the models used by most other researchers due to the fact that we consider red-green cycle lengths that may vary from cycle to cycle. Furthermore, we consider non-saturated intersections, i.e., we allow queue lengths to become equal to 0 during the green cycle. For more information on other models that describe the evolution of the queue lengths at a traffic-light-controlled intersection and on optimal traffic light control the interested reader is referred to [4, 5, 6, 7, 9, 10] and the references given therein.

III. Optimal traffic light control

From now on we assume that the arrival and departure rates are known. We want to compute an optimal sequence $t_0,\, t_1,\, \ldots t_N$ of switching time instants that minimizes a given criterion J. Possible objective functions are:

- (weighted) average queue length over all queues:

$$J_1 = \sum_{i=1}^{4} w_i \, \frac{1}{t_N} \int_0^{t_N} l_i(t)\, dt \;\; , \tag{3}$$

- (weighted) average queue length over the worst queue:

$$J_2 = \max_i \left(w_i \, \frac{1}{t_N} \int_0^{t_N} l_i(t)\, dt \right) \;\; , \tag{4}$$

- (weighted) worst case queue length:

$$J_3 = \max_{i,\, t} \big(w_i \, l_i(t) \big) \;\; , \tag{5}$$

- (weighted) average waiting time over all queues:

$$J_4 = \sum_{i=1}^{4} w_i \, \frac{1}{\lambda_i t_N} \int_0^{t_N} l_i(t)\, dt \;\; , \tag{6}$$

- (weighted) average waiting time over the worst queue:

$$J_5 = \max_i \left(w_i \, \frac{1}{\lambda_i t_N} \int_0^{t_N} l_i(t)\, dt \right) \;\; , \tag{7}$$

where $w_i > 0$ for all i. Note that J_1 and J_4 are in fact equivalent in the sense that for any weight vector w for J_1 there exists a weight vector $\tilde{w}$ for J_4 such that J_1 and J_4 are equal. This also holds for J_2 and J_5.

Define the following sets:

$$\begin{aligned} \alpha(N) &= \left\{ 0, 1, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor \right\} \\ \beta(N) &= \left\{ 0, 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor - 1 \right\} \quad . \end{aligned}$$

We can impose some extra conditions such as minimum and maximum durations for the red and green time, maximum queue lengths, and so on. This leads to the following problem:

$$\text{minimize } J \tag{8}$$

subject to

$$\delta_{\min,\mathrm{r}} \leqslant \delta_{2k} \leqslant \delta_{\max,\mathrm{r}} \quad \text{for } k \in \alpha(N) \tag{9}$$
$$\delta_{\min,\mathrm{g}} \leqslant \delta_{2k+1} \leqslant \delta_{\max,\mathrm{g}} \quad \text{for } k \in \beta(N) \tag{10}$$
$$x_k \leqslant x_{\max} \quad \text{for } k = 1, \ldots, N \tag{11}$$
$$x_{2k+1} = \max(x_{2k} + b_1 \delta_{2k},\, 0) \quad \text{for } k \in \alpha(N) \tag{12}$$
$$x_{2k+2} = \max(x_{2k+1} + b_2 \delta_{2k+1},\, 0) \quad \text{for } k \in \beta(N)\,. \tag{13}$$

Now we discuss some methods to solve problem (8)–(13).
Consider (12) for an arbitrary index k. It is easy to verify that this equation is equivalent to:

$$\begin{gathered} x_{2k+1} - x_{2k} - b_1 \delta_{2k} \geqslant 0 \\ x_{2k+1} \geqslant 0 \\ \sum_{i=1}^{4} \left(x_{2k+1} - x_{2k} - b_1 \delta_{2k} \right)_i \left(x_{2k+1} \right)_i = 0 \end{gathered}$$

(See also [3]). We can repeat this reasoning for (13) and for each k.

If we define $x^* = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$ and $\delta^* = \begin{bmatrix} \delta_0 \\ \vdots \\ \delta_{N-1} \end{bmatrix}$, we finally get a problem of the form

$$\text{minimize } J \tag{14}$$

subject to

$$Ax^* + B\delta^* + c \geqslant 0 \tag{15}$$

$$x^* \geqslant 0 \tag{16}$$

$$Ex^* + D\delta^* + f \geqslant 0 \tag{17}$$

$$(Ax^* + B\delta^* + c)^T x^* = 0 \ . \tag{18}$$

The system (15) – (18) is a special case of an Extended Linear Complementarity Problem (ELCP) [1, 2]. In [1, 2] we have developed an algorithm to compute a parametric description of the complete solution set of an ELCP. Once this description has been obtained, we can determine the values of the parameters for which J reaches a global minimum over the solution set of the ELCP. It can be shown that J_3 is a convex function of the parameters [3]. Furthermore, our computational experiments have shown that the determination of the minimum value of J_1, J_2, J_4 or J_5 over the solution set of the ELCP is a well-behaved problem in the sense that using a local minimization routine (that uses, e.g., sequential quadratic programming) starting from different initial points always yields the same numerical result (within a certain tolerance).

However, the general ELCP is an NP-hard problem [1, 2]. Furthermore, the size of the parametric description of the solution set of an ELCP with p (non-redundant) inequality constraints in an n-dimensional space is $O\left(p^{\lfloor \frac{n}{2} \rfloor}\right)$ if $p \gg n \gg 1$. This implies that the approach sketched above is not feasible if the number of switching cycles N is large. In [3] we have developed a method to compute suboptimal solutions of problem (8) – (13) by considering several ELCPs of a smaller size.

Note that if there is no upper bound on x_k, then problem (8) – (13) reduces to a constrained optimization problem in δ^*, which could be solved using a constrained minimization procedure. The major disadvantage of this approach is that in general the resulting objective function is neither convex nor concave so that the minimization routine will only return a local minimum. However, in [3] we have shown that J_3 is convex as a function of δ^*. This implies that problem (8) – (13) with $J = J_3$ can be solved efficiently (if there is no upper bound on the queue lengths or if we introduce a convex penalty term if one or more components of $x_{\max}$ are finite).

Now we define some objective functions that can be considered as approximations to the objective functions J_i ($i = 1, \ldots, 5$). Although these approximations will lead to suboptimal solutions with respect to J_i, the corresponding optimization problems can be solved much more efficiently†.

Note that the objective functions J_i do not depend explicitly on x^* since for given x_0, λ_i's and μ_i's, the components of x^* (i.e., the x_k's) are uniquely determined by δ^*. The approximate objective functions that we introduce now will depend explicitly on x^* and δ^*. For a given x_0, x^* and δ^* we define the function $\tilde{l}_i$ as the piecewise-linear function that interpolates in the points $(t_k,(x_k)_i)$ for $k = 0, \ldots, N$. The approximate objective functions $\tilde{J}_i$ ($i = 1, \ldots, 5$) are defined as in (3) – (7) but with l_i replaced by $\tilde{l}_i$.

We say that the vectors x^* and δ^* are *compatible* for a given x_0 if the corresponding sequences $\{x_k\}_{k=0}^{N}$ and $\{\delta_k\}_{k=0}^{N-1}$ satisfy the recurrence equations (1) and (2) for all k.

Proposition 1 *If x^* and δ^* are compatible for a given x_0 then we have $J_3(\delta) = \tilde{J}_3(x, \delta)$ and $J_l(\delta) \leqslant \tilde{J}_l(x, \delta)$ for $l = 1, 2, 4, 5$.*

Proof: See [3]. □

Let the queue length vector x^* and the switching interval vector δ^* be compatible for a given x_0. Note that if the queue lengths never stay 0 in a time interval then $J_l(\delta^*)$ and $\tilde{J}_l(x^*, \delta^*)$ are equal. In practice, an optimal traffic light switching scheme implies the absence of long periods in which no cars wait in one lane while in the other lanes the queue lengths increase, i.e., the periods during which the queue length in some lane is equal to 0 are in general short for optimal traffic light switching schemes. This implies that for traffic light switching schemes in the neighborhood of the optimal scheme $\tilde{J}_l$ will be a good approximation of J_l.

Now we show that the use of the approximate objective function $\tilde{J}_1$ or $\tilde{J}_4$ leads to an optimization problem that can be solved more efficiently than the original problem in which J_1 or J_4 is used.

Let $\mathcal{P}$ be the problem (8) – (13). We define the "relaxed" problem $\tilde{\mathcal{P}}$ corresponding to the original problem $\mathcal{P}$ as follows:

$$\text{minimize } \tilde{J} \tag{19}$$

subject to

$$\delta_{\text{min,r}} \leqslant \delta_{2k} \leqslant \delta_{\text{max,r}} \qquad \text{for } k \in \alpha(N) \tag{20}$$

†Recall that we have already introduced an approximation by considering continuous queue lengths. Furthermore, there is also some uncertainty and variation in time of the arrival and departure rates, which makes that in general computing the exact optimal traffic light switching scheme is not feasible. Moreover, in practice we are more interested in quickly obtaining a good approximation of the optimal traffic light switching scheme than in spending a large amount of time to obtain the exact optimal switching scheme.

$$\delta_{\min,\mathrm{g}} \leqslant \delta_{2k+1} \leqslant \delta_{\max,\mathrm{g}} \qquad \text{for } k \in \beta(N) \qquad (21)$$

$$0 \leqslant x_k \leqslant x_{\max} \qquad \text{for } k = 1, \ldots, N, \qquad (22)$$

$$x_{2k+1} \geqslant x_{2k} + b_1\delta_{2k} \qquad \text{for } k \in \alpha(N) \qquad (23)$$

$$x_{2k+2} \geqslant x_{2k+1} + b_2\delta_{2k+1} \qquad \text{for } k \in \beta(N)\ . \qquad (24)$$

Note that compared to the original problem we have replaced (12) – (13) by the relaxed equations $0 \leqslant x_k$, (23) and (24), i.e., we do not take the maximum condition into account.

Proposition 2 *If J is a strictly monotonous function of x^* — i.e., if for any δ^* with positive components and for all $\tilde{x}^*, \hat{x}^*$ with $0 \leqslant \tilde{x}^* \leqslant \hat{x}^*$ and with $\tilde{x}^*_j < \hat{x}^*_j$ for at least one index j, we have $J(\tilde{x}^*, \delta^*) < J(\hat{x}^*, \delta^*)$ — then any solution of the relaxed problem $\tilde{\mathcal{P}}$ is also a solution of the original problem $\mathcal{P}$.*

Proof: See [3]. □

In [3] we have shown that $\tilde{J}_1$ and $\tilde{J}_4$ are strictly monotonous functions of x^*, i.e., they satisfy the conditions of Proposition 2.

Note that in general it is easier to solve the relaxed problem $\tilde{\mathcal{P}}$ since the set of feasible solutions of $\tilde{\mathcal{P}}$ is a convex set, whereas the set of feasible solutions of $\mathcal{P}$ is in general not convex since it consists of the union of a subset of the set of faces of the polyhedron defined by the system of inequalities (20) – (24) (See [1, 2]).

In [3] we have also shown that if all the δ_k's are equal then $\tilde{J}_1$ and $\tilde{J}_4$ are linear, strictly monotonous functions of x^*, which implies that problem (8) – (13) then reduces to a linear programming problem, which can be solved efficiently using (variants of the) simplex method or using an interior point method (see, e.g., [8]).

For a worked-out example of the computation of (sub)optimal traffic light schemes for the set-up that has been considered in this paper the interested reader is referred to [3].

IV. Conclusions and further research

We have derived a model that describes the evolution of the queue lengths at an intersection of two two-way streets with controllable traffic lights on each corner. We have indicated how an optimal traffic light switching scheme for the given system can be determined. We have shown that for objective functions that depend strictly monotonously on the queue lengths at the traffic light switching time instants, the optimal traffic light switching scheme can be computed very efficiently. We have discussed some approximate objective functions for which this property holds. Moreover, if the objective function is linear, the problem reduces to a linear programming problem.

Topics for further research include: inclusion of an all-red phase and/or an amber phase, extension of the results to traffic networks or clusters of intersections, extension to models with integer queue lengths, development of efficient algorithms to compute optimal traffic light switching schemes for the objective functions discussed in this paper and for other more general objective functions.

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References

[1] B. De Schutter, *Max-Algebraic System Theory for Discrete Event Systems*. PhD thesis, Faculty of Applied Sciences, K.U.Leuven, Leuven, Belgium, 1996.

[2] B. De Schutter and B. De Moor, "The extended linear complementarity problem," *Mathematical Programming*, vol. 71, no. 3, pp. 289–325, Dec. 1995.

[3] B. De Schutter and B. De Moor, "Optimal traffic signal control for a single intersection," Tech. rep. 96-90, ESAT-SISTA, K.U.Leuven, Leuven, Belgium, Dec. 1996. Submitted for publication.

[4] N.H. Gartner, J.D.C. Little, and H. Gabbay, "Simultaneous optimization of offsets, splits, and cycle time," *Transportation Research Record*, vol. 596, pp. 6–15, 1976.

[5] H.R. Kashani and G.N. Saridis, "Intelligent control for urban traffic systems," *Automatica*, vol. 19, no. 2, pp. 191–197, Mar. 1983.

[6] J.H. Lim, S.H. Hwang, I.H. Suh, and Z. Bien, "Hierarchical optimal control of oversaturated urban traffic networks," *International Journal of Control*, vol. 33, no. 4, pp. 727–737, Apr. 1981.

[7] D. Lin, F. Ulrich, L.L. Kinney, and K.S.P. Kumar, "Hierarchical techniques in traffic control," in *Proceedings of the IFAC/IFIP/IFORS 3rd International Symposium*, pp. 163–171, 1976.

[8] Y. Nesterov and A. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming*. Philadelphia: SIAM, 1994.

[9] E.S. Park, J.H. Lim, I.H. Suh, and Z. Bien, "Hierarchical optimal control of urban traffic networks," *International Journal of Control*, vol. 40, no. 4, pp. 813–829, Oct. 1984.

[10] M.G. Singh and H. Tamura, "Modelling and hierarchical optimization for oversaturated urban road traffic networks," *International Journal of Control*, vol. 20, no. 6, pp. 913–934, 1974.