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# The minimal realization problem in the max-plus algebra: An overview

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**Abstract**

In this overview report we present known results and open problems in connection with the minimal state space realization problem in the max-plus algebra, which is a framework that can be used to model a class of discrete event systems.

## 1 Description of the problem

Given an $n \times n$ matrix $A$, an $n \times 1$ vector $b$ and a $1 \times n$ vector $c$ one can construct the sequence $g_i,\ i = 1, 2, \ldots$, where $g_i$ is defined by

$$g_i = cA^{i-1}b \ . \tag{1}$$

If instead of the starting point of given $A$, $b$ and $c$, the starting point is an arbitrary sequence $g_i,\ i = 1, 2, \ldots$, then necessary and sufficiency conditions are known under which appropriate $A$, $b$ and $c$ exist such that (1) is valid for $i = 1, 2, \ldots$. An additional requirement is that $n$, which determines the sizes of $A$, $b$ and $c$, must be as small as possible. Efficient algorithms to calculate such $A$, $b$ and $c$ are known.

The problem of this chapter is to reformulate these necessary and sufficiency conditions when the underlying algebra is the so-called max-plus algebra rather than the conventional algebra tacitly used above. One obtains the max-plus algebra from the conventional algebra

*Senior research assistant with the F.W.O. (Fund for Scientific Research-Flanders) by replacing addition by maximization and multiplication by addition. These operations are indicated by $\oplus$ (maximization) and $\otimes$ (addition). In the max-plus algebra one for instance has

$$\begin{pmatrix} 1 & 4 \\ -3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \otimes 5 \oplus 4 \otimes 1 \\ -3 \otimes 5 \oplus 0 \otimes 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} .$$

Next to the real numbers, one also uses the 'number' $-\infty$, being the neutral element of maximization, in the max-plus algebra. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$.

## 2 Motivation

The quantities $g_i,\ i = 1, 2, \ldots$ arise as the Markov parameters, also called the impulse response values, of the conventional linear, finite dimensional, discrete-time, time-invariant SISO[1] state space description

$$x(k+1) = Ax(k) + Bu(k) \ , \quad y(k) = Cx(k) \ , \tag{2}$$

where $x \in \mathbb{R}^n,\ u, y \in \mathbb{R}$.

**Theorem 2.1** *To the sequence $g_1, g_2, \ldots$ corresponds a finite-dimensional realization of the form (2) and of order $n$ (i.e., the state space is $\mathbb{R}^n$) if and only if*

$$\det\ H(n+i, n+i) = 0 \quad \textit{for} \quad i = 1, 2, \ldots.$$

*If moreover* $\det\ H(n,n) \neq 0$ *then $n$ is the order of the minimal realization of the sequence $\{g_i\}_{i=1}^{\infty}$.*

The so-called Hankel matrix $H(\alpha, \beta)$ of size $\alpha \times \beta$ which appears in this theorem is defined as

$$H(\alpha, \beta) = \begin{pmatrix} g_1 & g_2 & g_3 & \cdots & g_\beta \\ g_2 & g_3 & g_4 & \cdots & g_{\beta+1} \\ g_3 & g_4 & g_5 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_\alpha & g_{\alpha+1} & g_{\alpha+2} & \cdots & g_{\alpha+\beta-1} \end{pmatrix} . \tag{3}$$

The proof of this theorem can for instance be found in [Kailath, 1980], where also algorithms are given for the construction of the $A$, $B$ and $C$-matrices by means of which the realization is characterized. Generalizations to MIMO[2] systems exist, but will not be emphasized here.

The problem stated is to reformulate Theorem 2.1, or its MIMO equivalent, in terms of max-plus linear systems, i.e. systems of the form

$$x(k+1) = A \otimes x(k) \oplus Bu(k) \ , \quad y(k) = C \otimes x(k) \ . \tag{4}$$

In spite of its misleading simple formulation, the problem has met with formidable difficulties.

[1]SISO: single-input single output
[2]MIMO: multiple-input multiple-output

# 3 History and partial results

## 3.1 Characterization of max-plus-algebraic impulse responses

A necessary and sufficient condition for a sequence $\{g_i\}_{i=1}^{\infty}$ to be the impulse response of a system that can be described by a model of the form (2) is that the sequence is *ultimately periodic* [10, 11], i.e.,

$$\exists c, \lambda_0, \ldots, \lambda_{c-1}, k_0 \text{ such that } \forall k \geq k_0 : \quad g_{kc+c+s} = {\lambda_s}^{\otimes^c} \otimes g_{kc+s} \quad \text{for } s = 0, 1, \ldots, c-1 \ .$$

where $\lambda^{\otimes^c} = \lambda \cdot c$.

If we consider systems that can be modeled by timed event graphs then this condition reduces to the fact that the sequence $\{g_i\}_{i=1}^{\infty}$ should be ultimately geometric, i.e.,

$$\exists c, \lambda, k_0 \text{ such that } \forall k \geq k_0 : g_{k+c} = \lambda^{\otimes^c} \otimes g_k \ .$$

## 3.2 The minimal system order

In conventional system theory the minimal system order is given by the rank of the Hankel matrix $H(\infty, \infty)$. However, in contrast to linear algebra the different notions of rank (like column rank, row rank, minor rank, ...) are in general not equivalent in the max-plus algebra. An overview of the relations between the different ranks in the max-plus algebra can found on p. 122 of [10]. Related work can be found in [3, 18].

Let $H = H(\infty, \infty)$. It can be shown [10] that the minimal system order is equal to the smallest integer $r$ for which there exist matrices $U \in \mathbb{R}_{\varepsilon}^{\infty \times r}$, $V \in \mathbb{R}_{\varepsilon}^{r \times \infty}$ and $A \in \mathbb{R}_{\varepsilon}^{r \times r}$ such that

$$\begin{aligned} H_{\infty} &= U \otimes V \\ U \otimes A &= \overline{U} \end{aligned}$$

where $\overline{U}$ is the matrix obtained by removing the first row of $U$.

The different notions of matrix rank in the max-plus-algebra can be used to obtain lower and upper bounds for the minimal system order.

### 3.2.1 Lower bounds

- the minor rank of $H$ or of $H(N, N)$ for some $N$ in $\mathbb{N}$ [10, 11]:
  The minor rank of a matrix $H$ is equal to the dimension of the largest square submatrix $S$ of $H$ such that

  $$\bigoplus_{\sigma \in \mathcal{P}_{\text{even},\dim S}} \bigotimes_i s_{i\sigma(i)} \neq \bigoplus_{\sigma \in \mathcal{P}_{\text{odd},\dim S}} \bigotimes_i s_{i\sigma(i)}$$

  where $\mathcal{P}_{\text{even,k}}$ ($\mathcal{P}_{\text{odd,k}}$) is the set of even (odd) permutations of the first $k$ integers and $\dim S$ is the number of rows or columns of $S$.

- Schein rank of $H$ or $H(N, N)$ [11]:
  The Schein rank of an $m \times n$ matrix $H$ is equal to the smallest integer $r$ for which there exist matrices $U \in \mathbb{R}_{\varepsilon}^{m \times r}$ and $V \in \mathbb{R}_{\varepsilon}^{r \times n}$ such that $H = U \otimes V$.

At present, there are no efficient (i.e., polynomial time) algorithms to compute the max-plus-algebraic minor rank or the Schein rank of a matrix.

#### 3.2.2 Upper bounds

- weak column rank:
  Informally, the weak column rank of a matrix is defined as cardinality of the smallest set $I = \{i_1, \ldots, i_l\}$ such that every column of $H$ can be written as max-linear combination of columns indexed by $I$, i.e.,

  $$\forall k : \exists \alpha_1, \ldots, \alpha_l \text{ such that } H_{.,k} = \bigoplus_j \alpha_j \otimes H_{.,i_j}$$

  where $H_{.,k}$ is the $k$th column of $H$.
  If the sequence $\{g_i\}_{i=1}^{\infty}$ is ultimately geometric then the weak column rank of $H$ is an upper bound for the minimal system order [10].
  In general, an ultimately periodic sequence $g$ can be written as the merge of ultimately geometric sequences $g^0$, $g^1$, $\ldots$, $g^s$, and then an upper bound for the minimal system order is given by the sum of the weak column ranks of the Hankel matrices corresponding to $g^0$, $g^1$, $\ldots$, $g^s$ [10, 11].

A more formal definition of the max-algebraic weak column rank of a matrix can be found in [10, 11]. Efficient methods to compute the max-algebraic weak column rank of a matrix are described in [3, 4, 10].

### 3.3 Minimal state space realization: partial results

#### 3.3.1 Transformation to conventional algebra

There exists a transformation from the max-plus algebra to linear algebra that is based on the following equivalences:

$$x \oplus y = z \quad \Leftrightarrow \quad e^{xs} + e^{ys} \sim c\, e^{zs}\ ,\ s \to \infty \tag{5}$$

$$x \otimes y = z \quad \Leftrightarrow \quad e^{xs} \cdot e^{ys} = e^{zs} \quad \text{for all } s > 0 \tag{6}$$

with $x, y, z \in \mathbb{R}_\varepsilon$, and $c = 2$ if $x = y$ and $c = 1$ otherwise.

Using this transformation the minimal realization problem in the max-plus algebra can be mapped to a minimal realization problem for matrices with exponentials as entries and with conventional addition and multiplication as basic operations [13, 14, 15]. This implies that we can use the techniques from conventional realization theory to obtain a minimal realization afterwards (try to) transform the results back to the max-plus algebra. However, only realizations with positive coefficients for the leading exponentials can be mapped back to the max-plus algebra, and it is not always obvious how and whether such a realization can be constructed. In general the minimal system order obtained using the procedure above is a lower bound for the minimal system order.

### 3.3.2 Partial state space realization

Let us now consider the partial minimal realization problem: given a finite sequence $g_1$, $g_2$, $\ldots$, $g_N$, find matrices $A$, $B$ and $C$ such that (1) holds for $i = 1, 2, \ldots, N$. It can be shown that this leads to a system of so-called max-plus-algebraic polynomial equations and that such a system can be recast as a mathematical programming problem that is called the Extended Linear Complementarity Problem (ELCP) [6, 7, 8]. This procedure can also be used for MIMO systems. This enables us to solve the partial minimal realization problem and by applying some limit arguments this results in a realization of the entire impulse response. However, it can be shown that the general ELCP is NP-hard [7].

### 3.3.3 Special sequences of Markov parameters

For some special cases there exist methods to efficiently compute minimal state space realizations:

- if the sequence $\{g_i\}_{i=1}^{\infty}$ exhibits uniformly up-terrace behavior [19, 20, 21], i.e., if it consists of a concatenation of, say, $m$ subsequences with rates $c_1$, $c_2$, $\ldots$, $c_M$, where in the $k$th subsequence we have $g_{i+1} = g_i + c_k$ and $c_1 < c_2 < \cdots < c_m$.
- if the sequence $\{g_i\}_{i=1}^{\infty}$ exhibits a convex transient behavior and an ultimately geometric behavior with period 1 [5, 12]:

$$\begin{aligned} g_{k+1} - g_k \geq g_k - g_{k-1} & \qquad \text{for } k = 2, \ldots, k_0 \, , \\ g_{k+1} = \lambda \otimes g_k & \qquad \text{for } k \geq k_0 \ \ . \end{aligned}$$

Related results can be found in [4, 9, 16, 22, 23].

# 4 Related fields

Based on the relations (5) and (6) it is easy to verify that there exists a connection between the minimal realization problem in the max-plus algebra and the minimal realization problem for nonnegative systems. Indeed, some of the results obtained in system theory for nonnegative systems also hold in the max-plus algebra (see, e.g., [9]).

For more information on the minimal realization problem for nonnegative systems the reader is referred to [1, 2, 17].

# 5 Conclusions

The minimal realization problem in linear system theory can be solved very efficiently. Furthermore, there exist remarkable analogies between conventional algebra and max-plus algebra (based on the analogies between the operations $+$ and $\times$ on the one hand and $\oplus$ and $\otimes$ on the other hand). Nevertheless, there still do not exist efficient algorithms to solve to minimal state space realization problem in the max-plus algebra.

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