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The minimal realization problem in the max-plus algebra: An overview

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Abstract

In this overview report we present known results and open problems in connection with the minimal state space realization problem in the max-plus algebra, which is a framework that can be used to model a class of discrete event systems.

1 Description of the problem

Given an $n \times n$ matrix A , an $n \times 1$ vector b and a $1 \times n$ vector c one can construct the sequence g_i , $i = 1, 2, \dots$, where g_i is defined by

$$g_i = cA^{i-1}b . \quad (1)$$

If instead of the starting point of given A , b and c , the starting point is an arbitrary sequence g_i , $i = 1, 2, \dots$, then necessary and sufficiency conditions are known under which appropriate A , b and c exist such that (1) is valid for $i = 1, 2, \dots$. An additional requirement is that n , which determines the sizes of A , b and c , must be as small as possible. Efficient algorithms to calculate such A , b and c are known.

The problem of this chapter is to reformulate these necessary and sufficiency conditions when the underlying algebra is the so-called max-plus algebra rather than the conventional algebra tacitly used above. One obtains the max-plus algebra from the conventional algebra by replacing addition by maximization and multiplication by addition. These operations are

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indicated by \oplus (maximization) and \otimes (addition). In the max-plus algebra one for instance has

$$\begin{pmatrix} 1 & 4 \\ -3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \otimes 5 \oplus 4 \otimes 1 \\ -3 \otimes 5 \oplus 0 \otimes 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} .$$

Next to the real numbers, one also uses the ‘number’ $-\infty$, being the neutral element of maximization, in the max-plus algebra. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$.

2 Motivation

The quantities g_i , $i = 1, 2, \dots$ arise as the Markov parameters, also called the impulse response values, of the conventional linear, finite dimensional, discrete-time, time-invariant SISO¹ state space description

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k), \quad (2)$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$.

Theorem 2.1 *To the sequence g_1, g_2, \dots corresponds a finite-dimensional realization of the form (2) and of order n (i.e., the state space is \mathbb{R}^n) if and only if*

$$\det H(n+i, n+i) = 0 \quad \text{for } i = 1, 2, \dots$$

If moreover $\det H(n, n) \neq 0$ then n is the order of the minimal realization of the sequence $\{g_i\}_{i=1}^\infty$.

The so-called Hankel matrix $H(\alpha, \beta)$ of size $\alpha \times \beta$ which appears in this theorem is defined as

$$H(\alpha, \beta) = \begin{pmatrix} g_1 & g_2 & g_3 & \cdots & g_\beta \\ g_2 & g_3 & g_4 & \cdots & g_{\beta+1} \\ g_3 & g_4 & g_5 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_\alpha & g_{\alpha+1} & g_{\alpha+2} & \cdots & g_{\alpha+\beta-1} \end{pmatrix} . \quad (3)$$

The proof of this theorem can for instance be found in [Kailath, 1980], where also algorithms are given for the construction of the A , B and C -matrices by means of which the realization is characterized. Generalizations to MIMO² systems exist, but will not be emphasized here.

The problem stated is to reformulate Theorem 2.1, or its MIMO equivalent, in terms of max-plus linear systems, i.e. systems of the form

$$x(k+1) = A \otimes x(k) \oplus Bu(k), \quad y(k) = C \otimes x(k). \quad (4)$$

In spite of its misleading simple formulation, the problem has met with formidable difficulties.

¹SISO: single-input single output

²MIMO: multiple-input multiple-output

3 History and partial results

3.1 Characterization of max-plus-algebraic impulse responses

A necessary and sufficient condition for a sequence $\{g_i\}_{i=1}^{\infty}$ to be the impulse response of a system that can be described by a model of the form (2) is that the sequence is *ultimately periodic* [10, 11], i.e.,

$$\begin{aligned} \exists c, \lambda_0, \dots, \lambda_{c-1}, k_0 \text{ such that } \forall k \geq k_0 : \\ g_{kc+c+s} = \lambda_s^{\otimes c} \otimes g_{k+c} \quad \text{for } s = 0, 1, \dots, c-1 . \end{aligned}$$

where $\lambda^{\otimes c} = \lambda \cdot c$.

If we consider systems that can be modeled by timed event graphs then this condition reduces to the fact that the sequence $\{g_i\}_{i=1}^{\infty}$ should be ultimately geometric, i.e.,

$$\exists c, \lambda, k_0 \text{ such that } \forall k \geq k_0 : g_{k+c} = \lambda^{\otimes c} \otimes g_k .$$

3.2 The minimal system order

In conventional system theory the minimal system order is given by the rank of the Hankel matrix $H(\infty, \infty)$. However, in contrast to linear algebra the different notions of rank (like column rank, row rank, minor rank, ...) are in general not equivalent in the max-plus algebra. An overview of the relations between the different ranks in the max-plus algebra can be found on p. 122 of [10]. Related work can be found in [3, 18].

Let $H = H(\infty, \infty)$. It can be shown [10] that the minimal system order is equal to the smallest integer r for which there exist matrices $U \in \mathbb{R}_\varepsilon^{\infty \times r}$, $V \in \mathbb{R}_\varepsilon^{r \times \infty}$ and $A \in \mathbb{R}_\varepsilon^{r \times r}$ such that

$$\begin{aligned} H_\infty &= U \otimes V \\ U \otimes A &= \bar{U} \end{aligned}$$

where \bar{U} is the matrix obtained by removing the first row of U .

The different notions of matrix rank in the max-plus-algebra can be used to obtain lower and upper bounds for the minimal system order.

3.2.1 Lower bounds

- the minor rank of H or of $H(N, N)$ for some N in \mathbb{N} [10, 11]:
The minor rank of a matrix H is equal to the dimension of the largest square submatrix S of H such that

$$\bigoplus_{\sigma \in \mathcal{P}_{\text{even}, \dim S}} \bigotimes_i s_{i\sigma(i)} \neq \bigoplus_{\sigma \in \mathcal{P}_{\text{odd}, \dim S}} \bigotimes_i s_{i\sigma(i)}$$

where $\mathcal{P}_{\text{even}, k}$ ($\mathcal{P}_{\text{odd}, k}$) is the set of even (odd) permutations of the first k integers and $\dim S$ is the number of rows or columns of S .

- Schein rank of H or $H(N, N)$ [11]:
The Schein rank of an $m \times n$ matrix H is equal to the smallest integer r for which there exist matrices $U \in \mathbb{R}_\varepsilon^{m \times r}$ and $V \in \mathbb{R}_\varepsilon^{r \times n}$ such that $H = U \otimes V$.

At present, there are no efficient (i.e., polynomial time) algorithms to compute the max-plus-algebraic minor rank or the Schein rank of a matrix.

3.2.2 Upper bounds

- weak column rank:

Informally, the weak column rank of a matrix is defined as cardinality of the smallest set $I = \{i_1, \dots, i_l\}$ such that every column of H can be written as max-linear combination of columns indexed by I , i.e.,

$$\forall k : \exists \alpha_1, \dots, \alpha_l \text{ such that } H_{.,k} = \bigoplus_j \alpha_j \otimes H_{.,i_j}$$

where $H_{.,k}$ is the k th column of H .

If the sequence $\{g_i\}_{i=1}^\infty$ is ultimately geometric then the weak column rank of H is an upper bound for the minimal system order [10].

In general, an ultimately periodic sequence g can be written as the merge of ultimately geometric sequences g^0, g^1, \dots, g^s , and then an upper bound for the minimal system order is given by the sum of the weak column ranks of the Hankel matrices corresponding to g^0, g^1, \dots, g^s [10, 11].

A more formal definition of the max-algebraic weak column rank of a matrix can be found in [10, 11]. Efficient methods to compute the max-algebraic weak column rank of a matrix are described in [3, 4, 10].

3.3 Minimal state space realization: partial results

3.3.1 Transformation to conventional algebra

There exists a transformation from the max-plus algebra to linear algebra that is based on the following equivalences:

$$x \oplus y = z \quad \Leftrightarrow \quad e^{xs} + e^{ys} \sim c e^{zs}, \quad s \rightarrow \infty \quad (5)$$

$$x \otimes y = z \quad \Leftrightarrow \quad e^{xs} \cdot e^{ys} = e^{zs} \quad \text{for all } s > 0 \quad (6)$$

with $x, y, z \in \mathbb{R}_\varepsilon$, and $c = 2$ if $x = y$ and $c = 1$ otherwise.

Using this transformation the minimal realization problem in the max-plus algebra can be mapped to a minimal realization problem for matrices with exponentials as entries and with conventional addition and multiplication as basic operations [13, 14, 15]. This implies that we can use the techniques from conventional realization theory to obtain a minimal realization afterwards (try to) transform the results back to the max-plus algebra. However, only realizations with positive coefficients for the leading exponentials can be mapped back to the max-plus algebra, and it is not always obvious how and whether such a realization can be constructed. In general the minimal system order obtained using the procedure above is a lower bound for the minimal system order.

3.3.2 Partial state space realization

Let us now consider the partial minimal realization problem: given a finite sequence g_1, g_2, \dots, g_N , find matrices A, B and C such that (1) holds for $i = 1, 2, \dots, N$. It can be shown that this leads to a system of so-called max-plus-algebraic polynomial equations and that such a system can be recast as a mathematical programming problem that is called the Extended Linear Complementarity Problem (ELCP) [6, 7, 8]. This procedure can also be used for MIMO systems. This enables us to solve the partial minimal realization problem and by applying some limit arguments this results in a realization of the entire impulse response. However, it can be shown that the general ELCP is NP-hard [7].

3.3.3 Special sequences of Markov parameters

For some special cases there exist methods to efficiently compute minimal state space realizations:

- if the sequence $\{g_i\}_{i=1}^{\infty}$ exhibits uniformly up-terrace behavior [19, 20, 21], i.e., if it consists of a concatenation of, say, m subsequences with rates c_1, c_2, \dots, c_M , where in the k th subsequence we have $g_{i+1} = g_i + c_k$ and $c_1 < c_2 < \dots < c_M$.
- if the sequence $\{g_i\}_{i=1}^{\infty}$ exhibits a convex transient behavior and an ultimately geometric behavior with period 1 [5, 12]:

$$\begin{aligned} g_{k+1} - g_k &\geq g_k - g_{k-1} && \text{for } k = 2, \dots, k_0, \\ g_{k+1} &= \lambda \otimes g_k && \text{for } k \geq k_0. \end{aligned}$$

Related results can be found in [4, 9, 16, 22, 23].

4 Related fields

Based on the relations (5) and (6) it is easy to verify that there exists a connection between the minimal realization problem in the max-plus algebra and the minimal realization problem for nonnegative systems. Indeed, some of the results obtained in system theory for nonnegative systems also hold in the max-plus algebra (see, e.g., [9]).

For more information on the minimal realization problem for nonnegative systems the reader is referred to [1, 2, 17].

5 Conclusions

The minimal realization problem in linear system theory can be solved very efficiently. Furthermore, there exist remarkable analogies between conventional algebra and max-plus algebra (based on the analogies between the operations $+$ and \times on the one hand and \oplus and \otimes on the other hand). Nevertheless, there still do not exist efficient algorithms to solve to minimal state space realization problem in the max-plus algebra.

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