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**On the boolean minimal realization
problem in the max-plus algebra:
Addendum***

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On the boolean minimal realization problem in the max-plus algebra: Addendum

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In this addendum we present an upper bound for the minimal system order of a max-linear time-invariant DES that can be computed very efficiently, and we give some lemmas that characterize the *ultimate* behavior of the sequence $\{A^{\otimes k}\}_{k=0}^{\infty}$ for a matrix $A \in \mathbb{R}_{\varepsilon}^{n \times n}$.

A Upper bounds for the minimal system order

Definition A.1 (Ultimately geometric impulse response [12, A4])

Let $\{G_k\}_{k=0}^{\infty}$ be the impulse response of a max-linear time-invariant DES. If

$$\exists k_0 \in \mathbb{N}, \exists c \in \mathbb{N}_0, \exists \lambda \in \mathbb{R}_{\varepsilon} \text{ such that } \forall k \geq k_0 : G_{k+c} = \lambda^{\otimes c} \otimes G_k, \quad (\text{A.1})$$

then we say that the impulse response $\{G_k\}_{k=0}^{\infty}$ is ultimately geometric.

Note that an ultimately geometric sequence $G = \{G_k\}_{k=0}^{\infty}$ is also ultimately periodic. Furthermore, the smallest integers c and k_0 for which (A.1) holds, correspond to respectively the period of G and the length of the transient part of G .

Suppose that we have a DES that can be characterized by a triple (A, B, C) . A sufficient but not necessary condition for the impulse response of this DES to be ultimately geometric is that A is irreducible (cf. Theorem 2.4). This will, e.g., be the case for a DES without separate independent subsystems, and with a cyclic behavior or with feedback from the output to the input (such as, e.g., a flexible production system in which the parts are carried around on a limited number of pallets that circulate in the system [3]).

Definition A.2 (Max-plus-algebraic weak column rank [11, 12]) Let $A \in \mathbb{R}_{\varepsilon}^{m \times n}$. If $A \neq \mathcal{E}_{m \times n}$ then the max-plus-algebraic weak column rank of A is defined by

$$\text{rank}_{\oplus, \text{wc}}(A) = \min \left\{ \#I \mid I \subseteq \{1, 2, \dots, n\} \text{ and } \forall k \in \{1, 2, \dots, n\}, \right. \\ \left. \exists l \in \mathbb{N}_0, \exists i_1, i_2, \dots, i_l \in I, \exists \alpha_1, \alpha_2, \dots, \alpha_l \in \mathbb{R}_{\varepsilon} \right. \\ \left. \text{such that } A_{\cdot, k} = \bigoplus_{j=1}^l \alpha_j A_{\cdot, i_j} \right\}.$$

By definition we have $\text{rank}_{\oplus, \text{wc}}(\mathcal{E}) = 0$.

Efficient methods to compute the max-plus-algebraic weak column rank of a matrix are described in [4, 11, A2]. It is easy to verify that for any matrix $A \in \mathbb{R}_{\varepsilon}^{m \times n}$ we have $\text{rank}_{\oplus, \text{Schein}}(A) \leq \text{rank}_{\oplus, \text{wc}}(A)$.

Lemma A.3 Let G be an ultimately geometric sequence with period c . Let k_0 be the length of the transient part of G . Then we have

$$\text{rank}_{\oplus, \text{wc}} H(G) = \text{rank}_{\oplus, \text{wc}} (H(G))_{\{1, 2, \dots, k\}, \{1, 2, \dots, k\}} \quad \text{for all } k \geq k_0 + c. \quad (\text{A.2})$$

Proof: We shall prove this lemma for a sequence of numbers $g = \{g_k\}_{k=0}^\infty$. The extension of this proof to a sequence of matrices is straightforward.

Define $H_1 = (H(g))_{.,\{1,2,\dots,k_0+c\}}$ and $H_2 = (H(g))_{\{1,2,\dots,k_0+c\},\{1,2,\dots,k_0+c\}}$.

First we show that $\text{rank}_{\oplus,\text{wc}} H(g) = \text{rank}_{\oplus,\text{wc}} H_1$.

Let $k \in \mathbb{N}$. We have

$$(H(G))_{.,k_0+k+1} = \begin{bmatrix} g_{k_0+k} \\ g_{k_0+k+1} \\ g_{k_0+k+2} \\ \vdots \end{bmatrix} .$$

Since g is ultimately geometric, there exists a number $\lambda \in \mathbb{R}_\varepsilon$ such that $g_{k_0+c+k} = \lambda^{\otimes c} \otimes g_{k_0+k}$ for all $k \in \mathbb{N}$. Hence, $g_{k_0+rc+k} = \lambda^{\otimes rc} \otimes g_{k_0+k}$ for all $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$, and thus also

$$(H(G))_{.,k_0+rc+k+1} = \lambda^{\otimes rc} \otimes (H(G))_{.,k_0+k+1} \quad \text{for all } r \in \mathbb{N}_0 \text{ and } k \in \mathbb{N} .$$

This implies that any column $(H(G))_{.,k_0+c+l}$ with $l \in \mathbb{N}_0$ can be written as $\alpha \otimes (H(G))_{.,k_0+s}$ for some $s \in \{1, 2, \dots, c\}$ and some $\alpha \in \mathbb{R}_\varepsilon$. As a consequence, we have

$$\text{rank}_{\oplus,\text{wc}} H(G) = \text{rank}_{\oplus,\text{wc}} (H(G))_{.,\{1,2,\dots,k_0+c\}} = \text{rank}_{\oplus,\text{wc}} H_1 .$$

Using a similar reasoning as the one that has been used above, it can be shown that any row $(H_1)_{k_0+c+l, .}$ with $l \in \mathbb{N}_0$ can be written as $\alpha \otimes (H_1)_{k_0+s, .}$ for some $s \in \{1, 2, \dots, c\}$ and some $\alpha \in \mathbb{R}_\varepsilon$. So if we have

$$(H_2)_{.,k} = \bigoplus_{j=1}^l \alpha_j (H_2)_{.,i_j}$$

for some $l, k, i_1, i_2, \dots, i_l \in \{1, 2, \dots, k_0 + c\}$ and $\alpha_1, \alpha_2, \dots, \alpha_l \in \mathbb{R}_\varepsilon$, then we also have

$$(H_1)_{.,k} = \bigoplus_{j=1}^l \alpha_j (H_1)_{.,i_j} .$$

This implies that $\text{rank}_{\oplus} H_1 = \text{rank}_{\oplus,\text{wc}} (H_1)_{\{1,2,\dots,k_0+c\}, .} = \text{rank}_{\oplus,\text{wc}} H_2$.

Hence, $\text{rank}_{\oplus,\text{wc}} H(G) = \text{rank}_{\oplus,\text{wc}} H_2$. As a consequence, (A.2) holds. \square

Remark A.4 Note that Lemma A.3 implies that if G is an ultimately geometric sequence then $\text{rank}_{\oplus,\text{wc}} H(G)$ is finite and can be determined using a finite number of elementary operations.

The max-plus-algebraic sum of sequences is defined as follows. If $G = \{G_k\}_{k=0}^\infty$ and $H = \{H_k\}_{k=0}^\infty$ with $G_k, H_k \in \mathbb{R}_\varepsilon^{l \times m}$ for all $k \in \mathbb{N}$, then $G \oplus H$ is a sequence with $(G \oplus H)_k = G_k \oplus H_k$ for all $k \in \mathbb{N}$.

From Theorem 3.1 it follows that the impulse response of a max-linear time-invariant DES can always be considered as the max-plus-algebraic sum of a finite number of ultimately geometric impulse responses (see also [1, 11, 12]).

Theorem A.5 *Let g be the impulse response of a max-linear time-invariant SISO DES with $g \neq \{\varepsilon\}_{k=0}^\infty$. Let g_1, g_2, \dots, g_s be ultimately geometric sequences such that $g = g_1 \oplus g_2 \oplus \dots \oplus g_s$.*

Then there exists a state space realization of g of order $\sum_{i=1}^s \text{rank}_{\oplus,\text{wc}} (H(g_i))$.

Proof: See [11, 12]. □

Proposition A.6 *For any ultimately periodic sequence G we can compute a finite upper bound for the minimal system order of the max-linear time-invariant DES the impulse response of which coincides with G using a finite number of elementary operations.*

Proof: This is a direct consequence of Lemma A.3 and Theorem A.5. □

B The ultimate behavior of the sequence of consecutive max-plus-algebraic matrix powers

If we permute the rows or the columns of the max-plus-algebraic identity matrix, we obtain a max-plus-algebraic permutation matrix. If $P \in \mathbb{R}_\varepsilon^{n \times n}$ is a max-plus-algebraic permutation matrix, then we have $P \otimes P^T = P^T \otimes P = E_n$. A matrix $R \in \mathbb{R}_\varepsilon^{m \times n}$ is a max-plus-algebraic upper triangular matrix if $r_{ij} = \varepsilon$ for all i, j with $i > j$.

Lemma B.1 *If $A \in \mathbb{R}_\varepsilon^{n \times n}$ then there exists a max-plus-algebraic permutation matrix $P \in \mathbb{R}_\varepsilon^{n \times n}$ such that the matrix $\hat{A} = P \otimes A \otimes P^T$ is a max-plus-algebraic block upper triangular matrix of the form*

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \dots & \hat{A}_{1l} \\ \varepsilon & \hat{A}_{22} & \dots & \hat{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & \hat{A}_{ll} \end{bmatrix} \quad (\text{A.3})$$

with $l \geq 1$ and where the matrices $\hat{A}_{11}, \hat{A}_{22}, \dots, \hat{A}_{ll}$ are square and irreducible. The matrices $\hat{A}_{11}, \hat{A}_{22}, \dots, \hat{A}_{ll}$ are uniquely determined to within simultaneous permutation of their rows and columns, but their ordering in (A.3) is not necessarily unique.

Proof: See, e.g., [1]. This lemma is also the max-plus-algebraic equivalent of a result of [A5]. A proof of the uniqueness assertion can be found in [A1] (Theorem 3.2.4¹). □

The form in (A.3) is called the max-plus-algebraic Frobenius normal form of the matrix A . Note that if A is irreducible then there is only one block in (A.3) and then A is a max-plus-algebraic Frobenius normal form of itself.

Let $A \in \mathbb{B}^{n \times n}$ (or $A \in \mathbb{R}_\varepsilon^{n \times n}$). If $\hat{A} = P \otimes A \otimes P^T$ is the max-plus-algebraic Frobenius normal form of A , then we have $A = P^T \otimes \hat{A} \otimes P$. Hence,

$$A^{\otimes k} = (P^T \otimes \hat{A} \otimes P)^{\otimes k} = P^T \otimes \hat{A}^{\otimes k} \otimes P$$

for all $k \in \mathbb{N}$. Therefore, we may consider without loss of generality the sequence $\{\hat{A}^{\otimes k}\}_{k=0}^\infty$ instead of the sequence $\{A^{\otimes k}\}_{k=0}^\infty$. Furthermore, since the transformation from A to \hat{A} corresponds to a simultaneous reordering of the rows and columns of A (or to a reordering of the vertices of $\mathcal{G}(A)$), we have $c(A) = c(\hat{A})$.

The following lemma is an extension of Theorem 2.4 and a corrected version of a lemma that can be found in [A6]:

¹Although this theorem is stated for $(0, 1)$ -matrices, there is a one-to-one correspondence between a max-plus-algebraic boolean matrix and a $(0, 1)$ -matrix if we let 0 and ε correspond with 1 and 0 respectively.

Lemma B.2 Let $\hat{A} \in \mathbb{R}_\varepsilon^{n \times n}$ be a matrix of the form (A.3) where the matrices $\hat{A}_{11}, \dots, \hat{A}_{ll}$ are square and irreducible. Let λ_i and c_i be respectively the max-plus-algebraic eigenvalue and the cyclicity of \hat{A}_{ii} for $i = 1, \dots, l$. Define sets $\alpha_1, \dots, \alpha_l$ such that $\hat{A}_{\alpha_i \alpha_j} = \hat{A}_{ij}$ for all i, j with $i \leq j$.

Define

$$S_{ij} = \{ \{i_0, \dots, i_s\} \subseteq \{1, \dots, l\} \mid i = i_0 < i_1 < \dots < i_s = j \text{ and} \\ \hat{A}_{i_r i_{r+1}} \neq \varepsilon \text{ for } r = 0, \dots, s-1 \}$$

$$\Gamma_{ij} = \bigcup_{\gamma \in S_{ij}} \gamma \\ \Lambda_{ij} = \begin{cases} \{\lambda_t \mid t \in \Gamma_{ij}\} & \text{if } \Gamma_{ij} \neq \emptyset, \\ \{\varepsilon\} & \text{if } \Gamma_{ij} = \emptyset, \end{cases} \\ c_{ij} = \begin{cases} \text{lcm}\{c_t \mid t \in \Gamma_{ij}\} & \text{if } \Gamma_{ij} \neq \emptyset \text{ and } c_t \neq 0 \text{ for some } t \in \Gamma_{ij}, \\ 1 & \text{otherwise,} \end{cases}$$

for all i, j with $i < j$. We have

$$\forall i, j \in \{1, \dots, l\} \text{ with } i > j : \left(\hat{A}^{\otimes k} \right)_{\alpha_i \alpha_j} = \varepsilon_{n_i \times n_j} \quad \text{for all } k \in \mathbb{N}. \quad (\text{A.4})$$

Moreover, there exists an integer $K \in \mathbb{N}$ such that

$$\forall i \in \{1, \dots, l\} : \left(\hat{A}^{\otimes k+c_i} \right)_{\alpha_i \alpha_i} = \lambda_i^{\otimes c_i} \otimes \left(\hat{A}^{\otimes k} \right)_{\alpha_i \alpha_i} \quad \text{for all } k \geq K \quad (\text{A.5})$$

and

$$\forall i, j \in \{1, \dots, l\} \text{ with } i < j, \forall p \in \alpha_i, \forall q \in \alpha_j, \exists \gamma_0, \dots, \gamma_{c_{ij}-1} \in \Lambda_{ij} \text{ such that} \\ \left(\hat{A}^{\otimes kc_{ij}+c_{ij}+s} \right)_{pq} = \gamma_s^{\otimes c_{ij}} \otimes \left(\hat{A}^{\otimes kc_{ij}+s} \right)_{pq} \quad \text{for all } k \geq K \text{ and for } s = 0, \dots, c_{ij} - 1. \quad (\text{A.6})$$

Furthermore, for each combination i, j, p, q with $i < j$, $p \in \alpha_i$ and $q \in \alpha_j$, there exists at least one index $s \in \{0, \dots, c_{ij} - 1\}$ such that the smallest γ_s for which (A.6) holds is equal to $\max \Lambda_{ij}$.

Proof: See [A3]. □

If $G = \{G_k\}_{k=0}^\infty$ is the impulse response of a max-linear time-invariant DES and if the triple (A, B, C) is a state space realization of the DES, then it follows from Lemmas B.1 and B.2 that the period of G is a divisor of the cyclicity $c(A)$ of the system matrix A .

Additional references

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