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On the boolean minimal realization problem in the max-plus algebra: Addendum

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In this addendum we present an upper bound for the minimal system order of a max-linear time-invariant DES that can be computed very efficiently, and we give some lemmas that characterize the ultimate behavior of the sequence \( \{A^k\} \) for a matrix \( A \in \mathbb{R}^{n \times n} \).

### A Upper bounds for the minimal system order

#### Definition A.1 (Ultimately geometric impulse response [12, A4])

Let \( \{G_k\}_{k=0}^\infty \) be the impulse response of a max-linear time-invariant DES. If

\[
\exists k_0 \in \mathbb{N}, \exists c \in \mathbb{N}_0, \exists \lambda \in \mathbb{R}_+ \text{ such that } \forall k \geq k_0 : G_{k+c} = \lambda^c \otimes G_k ,
\]

(A.1)

then we say that the impulse response \( \{G_k\}_{k=0}^\infty \) is ultimately geometric.

Note that an ultimately geometric sequence \( G = \{G_k\}_{k=0}^\infty \) is also ultimately periodic. Furthermore, the smallest integers \( c \) and \( k_0 \) for which (A.1) holds, correspond to respectively the period of \( G \) and the length of the transient part of \( G \).

Suppose that we have a DES that can be characterized by a triple \( (A, B, C) \). A sufficient but not necessary condition for the impulse response of this DES to be ultimately geometric is that \( A \) is irreducible (cf. Theorem 2.4). This will, e.g., be the case for a DES without separate independent subsystems, and with a cyclic behavior or with feedback from the output to the input (such as, e.g., a flexible production system in which the parts are carried around on a limited number of pallets that circulate in the system [3]).

#### Definition A.2 (Max-plus-algebraic weak column rank [11, 12])

Let \( A \in \mathbb{R}^{m \times n}_+ \). If \( A \neq \mathbb{E}^{m \times n} \), then the max-plus-algebraic weak column rank of \( A \) is defined by

\[
\text{rank}_{\oplus, wc}(A) = \min \left\{ \# I \mid I \subseteq \{1, 2, \ldots, n\} \text{ and } \forall k \in \{1, 2, \ldots, n\}, \right.
\]

\[
\exists \ell \in \mathbb{N}_0, \exists i_1, i_2, \ldots, i_\ell \in I, \exists \alpha_1, \alpha_2, \ldots, \alpha_\ell \in \mathbb{R}_+ \text{ such that } A_{.,k} = \bigoplus_{j=1}^\ell \alpha_j A_{.,i_j} .
\]

By definition we have \( \text{rank}_{\oplus, wc}(\mathbb{E}) = 0 \).

Efficient methods to compute the max-plus-algebraic weak column rank of a matrix are described in [4, 11, A2]. It is easy to verify that for any matrix \( A \in \mathbb{R}^{m \times n}_+ \) we have \( \text{rank}_{\oplus, Schein}(A) \leq \text{rank}_{\oplus, wc}(A) \).

#### Lemma A.3

Let \( G \) be an ultimately geometric sequence with period \( c \). Let \( k_0 \) be the length of the transient part of \( G \). Then we have

\[
\text{rank}_{\oplus, wc} H(G) = \text{rank}_{\oplus, wc} \left( H(G) \right)_{\{1, 2, \ldots, k\}, \{1, 2, \ldots, k\}} \text{ for all } k \geq k_0 + c .
\]

(A.2)
Proof: We shall prove this lemma for a sequence of numbers \( g = \{g_k\}_{k=0}^{\infty} \). The extension of this proof to a sequence of matrices is straightforward.

Define \( H_1 = (H(g))_{1,2,\ldots,k_0+c} \) and \( H_2 = (H(g))_{1,2,\ldots,k_0+c,m} \).

First we show that \( \text{rank}_{\oplus,\text{wc}} H(g) = \text{rank}_{\oplus,\text{wc}} H_1 \).

Let \( k \in \mathbb{N} \). We have

\[
(H(G))_{k_0+k+1} = \begin{bmatrix} g_{k_0+k} \\ g_{k_0+k+1} \\ g_{k_0+k+2} \\ \vdots \end{bmatrix}.
\]

Since \( g \) is ultimately geometric, there exists a number \( \lambda \in \mathbb{R} \) such that \( g_{k_0+c+k} = \lambda^{s+c} \otimes g_{k_0+k} \) for all \( k \in \mathbb{N} \). Hence, \( g_{k_0+c+k} = \lambda^{s+c} \otimes g_{k_0+k} \) for all \( r \in \mathbb{N}_0 \) and \( k \in \mathbb{N} \), and thus also

\[
(H(G))_{k_0+c+k+1} = \lambda^{s+c} \otimes (H(G))_{k_0+k+1} \quad \text{for all } r \in \mathbb{N}_0 \text{ and } k \in \mathbb{N}.
\]

This implies that any column \( (H(G))_{k_0+c+l} \) with \( l \in \mathbb{N}_0 \) can be written as \( \alpha \otimes (H(G))_{k_0+s} \) for some \( s \in \{1,2,\ldots,c\} \) and some \( \alpha \in \mathbb{R} \). As a consequence, we have

\[
\text{rank}_{\oplus,\text{wc}} H(G) = \text{rank}_{\oplus,\text{wc}} (H(G))_{1,2,\ldots,k_0+c} = \text{rank}_{\oplus,\text{wc}} H_1.
\]

Using a similar reasoning as the one that has been used above, it can be shown that any row \( (H_1)_{k_0+c+l} \) with \( l \in \mathbb{N}_0 \) can be written as \( \alpha \otimes (H_1)_{k_0+s} \) for some \( s \in \{1,2,\ldots,c\} \) and some \( \alpha \in \mathbb{R} \). So if we have

\[
(H_1)_{k} = \bigoplus_{j=1}^{l} \alpha_j (H_2)_{i,j}
\]

for some \( l, k, i_1, i_2, \ldots, i_l \in \{1,2,\ldots,k_0+c\} \) and \( \alpha_1, \alpha_2, \ldots, \alpha_l \in \mathbb{R} \), then we also have

\[
(H_1)_{k} = \bigoplus_{j=1}^{l} \alpha_j (H_1)_{i,j}.
\]

This implies that \( \text{rank}_{\oplus} H_1 = \text{rank}_{\oplus,\text{wc}} (H_1)_{1,2\ldots,k_0+c} = \text{rank}_{\oplus,\text{wc}} H_2 \).

Hence, \( \text{rank}_{\oplus,\text{wc}} H(G) = \text{rank}_{\oplus,\text{wc}} H_2 \). As a consequence, (A.2) holds. \( \square \)

Remark A.4 Note that Lemma A.3 implies that if \( G \) is an ultimately geometric sequence then \( \text{rank}_{\oplus,\text{wc}} H(G) \) is finite and can be determined using a finite number of elementary operations.

The max-plus-algebraic sum of sequences is defined as follows. If \( G = \{G_k\}_{k=0}^{\infty} \) and \( H = \{H_k\}_{k=0}^{\infty} \) with \( G_k, H_k \in \mathbb{R}^{l_x \times m} \) for all \( k \in \mathbb{N} \), then \( G \oplus H \) is a sequence with \( (G \oplus H)_k = G_k \oplus H_k \) for all \( k \in \mathbb{N} \).

From Theorem 3.1 it follows that the impulse response of a max-linear time-invariant DES can always be considered as the max-plus-algebraic sum of a finite number of ultimately geometric impulse responses (see also [1, 11, 12]).

Theorem A.5 Let \( g \) be the impulse response of a max-linear time-invariant SISO DES with \( g \neq \{\varepsilon\}_{k=0}^{\infty} \). Let \( g_1, g_2, \ldots, g_s \) be ultimately geometric sequences such that \( g = g_1 \oplus g_2 \oplus \cdots \oplus g_s \).

Then there exists a state space realization of \( g \) of order \( \sum_{i=1}^{s} \text{rank}_{\oplus,\text{wc}} (H(g_i)) \).
Proof: See [11, 12]. □

Proposition A.6 For any ultimately periodic sequence $G$ we can compute a finite upper bound for the minimal system order of the max-linear time-invariant DES the impulse response of which coincides with $G$ using a finite number of elementary operations.

Proof: This is a direct consequence of Lemma A.3 and Theorem A.5. □

B The ultimate behavior of the sequence of consecutive max-plus-algebraic matrix powers

If we permute the rows or the columns of the max-plus-algebraic identity matrix, we obtain a max-plus-algebraic permutation matrix. If $P \in \mathbb{R}^{n \times n}_\epsilon$ is a max-plus-algebraic permutation matrix, then we have $P \otimes P^T = P^T \otimes P = E_n$. A matrix $R \in \mathbb{R}^{m \times n}_\epsilon$ is a max-plus-algebraic upper triangular matrix if $r_{ij} = \epsilon$ for all $i, j$ with $i > j$.

Lemma B.1 If $A \in \mathbb{R}^{n \times n}_\epsilon$ then there exists a max-plus-algebraic permutation matrix $P \in \mathbb{R}^{n \times n}_\epsilon$ such that the matrix $\hat{A} = P \otimes A \otimes P^T$ is a max-plus-algebraic block upper triangular matrix of the form

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \ldots & \hat{A}_{1l} \\ \epsilon & \hat{A}_{22} & \ldots & \hat{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon & \epsilon & \ldots & \hat{A}_{ll} \end{bmatrix} \tag{A.3}$$

with $l \geq 1$ and where the matrices $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ are square and irreducible. The matrices $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ are uniquely determined to within simultaneous permutation of their rows and columns, but their ordering in (A.3) is not necessarily unique.

Proof: See, e.g., [1]. This lemma is also the max-plus-algebraic equivalent of a result of [A5]. A proof of the uniqueness assertion can be found in [A1] (Theorem 3.2.4).

The form in (A.3) is called the max-plus-algebraic Frobenius normal form of the matrix $A$. Note that if $A$ is irreducible then there is only one block in (A.3) and then $A$ is a max-plus-algebraic Frobenius normal form of itself.

Let $A \in \mathbb{B}^{n \times n}$ (or $A \in \mathbb{R}^{n \times n}_\epsilon$). If $\hat{A} = P \otimes A \otimes P^T$ is the max-plus-algebraic Frobenius normal form of $A$, then we have $A = P^T \otimes \hat{A} \otimes P$. Hence,

$$A^\otimes k = (P^T \otimes \hat{A} \otimes P)^\otimes k = P^T \otimes \hat{A}^\otimes k \otimes P$$

for all $k \in \mathbb{N}$. Therefore, we may consider without loss of generality the sequence $\{\hat{A}^\otimes k\}_{k=0}^\infty$ instead of the sequence $\{A^\otimes k\}_{k=0}^\infty$. Furthermore, since the transformation from $A$ to $\hat{A}$ corresponds to a simultaneous reordering of the rows and columns of $A$ (or to a reordering of the vertices of $G(A)$), we have $c(A) = c(\hat{A})$.

The following lemma is an extension of Theorem 2.4 and a corrected version of a lemma that can be found in [A6]:

\[1\] Although this theorem is stated for $(0, 1)$-matrices, there is a one-to-one correspondence between a max-plus-algebraic boolean matrix and a $(0, 1)$-matrix if we let $0$ and $\epsilon$ correspond with $1$ and $0$ respectively.
Lemma B.2 Let $\hat{A} \in \mathbb{R}_n^{n \times n}$ be a matrix of the form (A.3) where the matrices $\hat{A}_{11}, \ldots, \hat{A}_{ll}$ are square and irreducible. Let $\lambda_i$ and $c_i$ be respectively the max-plus-algebraic eigenvalue and the cyclicity of $\hat{A}_{ii}$ for $i = 1, \ldots, l$. Define sets $\alpha_1, \ldots, \alpha_l$ such that $\hat{A}_{\alpha_i \alpha_j} = \hat{A}_{ij}$ for all $i, j$ with $i \leq j$.

Define

$S_{ij} = \{ \{i_0, \ldots, i_s\} \subseteq \{1, \ldots, l\} \mid i = i_0 < i_1 < \ldots < i_s = j \}$

$\Gamma_{ij} = \bigcup_{\gamma \in S_{ij}} \gamma$

$\Lambda_{ij} = \begin{cases} \{\lambda_t \mid t \in \Gamma_{ij}\} & \text{if } \Gamma_{ij} \neq \emptyset, \\ \{\varepsilon\} & \text{if } \Gamma_{ij} = \emptyset, \end{cases}$

$c_{ij} = \begin{cases} \text{lcm} \{c_t \mid t \in \Gamma_{ij}\} & \text{if } \Gamma_{ij} \neq \emptyset \text{ and } c_t \neq 0 \text{ for some } t \in \Gamma_{ij}, \\ 1 & \text{otherwise}, \end{cases}$

for all $i, j$ with $i < j$. We have

\[ \forall i, j \in \{1, \ldots, l\} \text{ with } i > j : \left(\hat{A}^k\right)_{\alpha_i \alpha_j} = \varepsilon_{n_i \times n_j} \text{ for all } k \in \mathbb{N}. \quad (A.4) \]

Moreover, there exists an integer $K \in \mathbb{N}$ such that

\[ \forall i \in \{1, \ldots, l\} : \left(\hat{A}^{k+c_i}\right)_{\alpha_i \alpha_i} = \lambda_i \otimes \left(\hat{A}^k\right)_{\alpha_i \alpha_i} \text{ for all } k \geq K \]  

and

\[ \forall i, j \in \{1, \ldots, l\} \text{ with } i < j, \forall p \in \alpha_i, \forall q \in \alpha_j, \exists \gamma_0, \ldots, \gamma_{c_{ij}-1} \in \Lambda_{ij} \text{ such that} \]

\[ \left(\hat{A}^{kc_{ij}+c_{ij}+s}\right)_{pq} = \gamma_s \otimes \left(\hat{A}^{kc_{ij}+s}\right)_{pq} \text{ for all } k \geq K \text{ and for } s = 0, \ldots, c_{ij} - 1. \]  

Furthermore, for each combination $i, j, p, q$ with $i < j, p \in \alpha_i$ and $q \in \alpha_j$, there exists at least one index $s \in \{0, \ldots, c_{ij} - 1\}$ such that the smallest $\gamma_s$ for which (A.6) holds is equal to $\max \Lambda_{ij}$.

Proof: See [A3].

If $G = \{G_k\}_{k=0}^\infty$ is the impulse response of a max-linear time-invariant DES and if the triple $(A, B, C)$ is a state space realization of the DES, then it follows from Lemmas B.1 and B.2 that the period of $G$ is a divisor of the cyclicity $c(A)$ of the system matrix $A$.

Additional references


