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Minimal Realizations and State Space Transformations in the Symmetrized Max-Algebra*

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Abstract

Similarity transformations between two different minimal realizations of a given impulse response of a discrete event system are discussed. In the symmetrized max-algebra an explicit expression can be given for the transformation between an arbitrary minimal realization of a given impulse response and a minimal realization of the same impulse response in a standard form. It is conjectured that a more general result holds which gives a transformation matrix between any two minimal realizations of an impulse response. We will illustrate the difficulties encountered when trying to prove this conjecture.

1 Introduction

A class of Discrete Event Systems (DES), e.g. systems which involve synchronization, can be described by linear models provided that the usual addition is replaced by maximization and multiplication by addition. The resulting algebraic structure is called the max-algebra and a max-algebraic system theory has been developed for this class of DES. An extensive exposition of such systems and of the underlying algebraic structure can be found in [1].

One of the problems in the system theory for DES is the minimal realization problem, which can be formulated in the following way. Given an impulse response of a system, find a state space description of minimal dimension of which the behavior is equal to the given impulse response. An overview is given of a number of (partial) solutions for this problem.

The minimal realization problem for max-linear systems was introduced in [11]. The results in this paper were extended in [12] in which the two-dimensional case was studied. In these papers a mapping from the max-algebra to the conventional algebra is used to solve the problem. Cuninghame-Green [5] tries to solve the problem using algebraic techniques valid within the algebra itself. Extensions are given in [15] and in [16]. In [7] and [6] it is shown that the minimal realization problem can be formulated as an Extended Linear Complementarity Problem (ELCP), an extension of the Linear Complementarity Problem which is one of the fundamental problems in mathematical programming. The ELCP approach can then be used to compute a partial realization of a given impulse response even for MIMO systems. A drawback of the general ELCP is that it is an NP-hard problem, so it can probably not be solved in polynomial time. It is not clear yet whether the minimal realization problem is also NP-hard. In [8] the authors present a heuristic procedure which can overcome this drawback of the ELCP method.

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In conventional system theory it is always possible to find a state space transformation between two different minimal realizations of the same impulse response. It is investigated whether a similar statement holds true in the max-algebraic system theory of DES. It can be shown that in some cases state space transformations exist in the max-algebra, see e.g. [6] in which some possible transformations are discussed. A problem in finding transformations for more general cases is that in the max-algebra the inverse of a matrix only exists for a small class of matrices. Therefore, we extend the search for state space transformations to the symmetrized max-algebra, which is the linear closure of the max-algebra. The symmetrized max-algebra structure was first introduced in [10], see also [1] and [9].

This paper is organized as follows. In Section 2 we will discuss the max-algebra and its linear closure, the symmetrized max-algebra. Furthermore we will briefly demonstrate how a class of systems can be described by linear relations in the max-algebra. In Section 3 the minimal realization problem is discussed and some known results are summarized. We show why the similarity transformation problem is of interest. This problem will be discussed in Section 4. We first recall some results from conventional system theory. Then we will introduce possible similarity transformations for max-algebraic systems. Finally, in Section 5 we will make some concluding remarks and state some questions which have remained unanswered.

2 Max-algebra and extensions

In this section we will give a brief overview of the max-algebra and of the symmetrized max-algebra. For a more extensive discussion we refer to [1] or [4].

Let $\varepsilon = -\infty$ and denote by $\mathbb{R}_\varepsilon$ the set $\mathbb{R} \cup \{\varepsilon\}$. For $a, b \in \mathbb{R}_\varepsilon$ the operations $\oplus$ and $\otimes$ are defined by

$$\begin{aligned} a \oplus b &= \max(a, b), \\ a \otimes b &= a + b. \end{aligned}$$

The set $\mathbb{R}_\varepsilon$ together with the operations $\oplus$ and $\otimes$ will be denoted by $\mathbb{R}_{\max}$ and is called the max-algebra or max-plus algebra. In $\mathbb{R}_{\max}$, ε is the neutral element for the operation $\oplus$ and an absorbing element for $\otimes$. The neutral element for $\otimes$ is 0.

We can extend the max-algebra operations to matrices in the following way. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} \qquad \text{for } i = 1, \dots, m \text{ and } j = 1, \dots, n.$$

If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj} \qquad \text{for } i = 1, \dots, m \text{ and } j = 1, \dots, n.$$

We will denote by E_n or just by E the $n \times n$ max-algebraic unit matrix. For this matrix we have

$$\begin{aligned} E_{ij} &= \varepsilon && \text{for } i, j = 1, \dots, n \text{ with } i \neq j \\ E_{ii} &= 0 && \text{for } i = 1, \dots, n. \end{aligned}$$

A problem with $\mathbb{R}_{\max}$ is that it is not a ring since an equation of the form $a \oplus x = b$ does not necessarily have a solution. If $a > b$ no solution exists. The reason for this is that for $a \in \mathbb{R}$ we cannot find an element $b \in \mathbb{R}_\varepsilon$ such that $a \oplus b = \varepsilon$. A solution to overcome this problem, at least in some ways, is presented in [10], see also [1] and [9]. In these references a symmetrization of $\mathbb{R}_{\max}$ is introduced, which results in the closure of $\mathbb{R}_{\max}$ denoted by $\mathbb{S}_{\max}$. This structure will be called the symmetrized max-plus algebra. In this section we will give the basic notions regarding $\mathbb{S}_{\max}$. For a formal derivation and proofs we refer to [10], [1] and [9].

The set $\mathbb{S}$ consists of the three subsets $\mathbb{S}^{\oplus}$, $\mathbb{S}^{\ominus}$ and $\mathbb{S}^{\bullet}$ defined by

$$\begin{aligned} \mathbb{S}^{\oplus} &= \{a \mid a \in \mathbb{R}_{\varepsilon}\} \\ \mathbb{S}^{\ominus} &= \{\ominus a \mid a \in \mathbb{R}_{\varepsilon}\} \\ \mathbb{S}^{\bullet} &= \{a^{\bullet} = a \ominus a \mid a \in \mathbb{R}_{\varepsilon}\}. \end{aligned}$$

The elements in $\mathbb{S}^{\oplus}$ will be called max-positive, the elements in $\mathbb{S}^{\ominus}$ max-negative and the elements in $\mathbb{S}^{\bullet}$ will be called balanced. The elements in the set $\mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$ will be called signed. The set of signed elements is denoted by $\mathbb{S}^{\vee}$. For elements $x, y \in \mathbb{R}_{\varepsilon}$ we have

$$\begin{aligned} x \oplus (\ominus y) &= x && \text{if } x > y, \\ x \oplus (\ominus y) &= \ominus y && \text{if } x < y, \\ x \oplus (\ominus y) &= x^{\bullet} && \text{if } x = y. \end{aligned}$$

Furthermore, for any $x, y \in \mathbb{S}$ we have

$$\begin{aligned} \ominus(x \oplus y) &= (\ominus x) \oplus (\ominus y) \\ x \otimes (\ominus y) &= \ominus(x \otimes y), \\ (\ominus x) \otimes (\ominus y) &= x \otimes y, \\ \ominus(\ominus x) &= x. \end{aligned}$$

These properties allow us to write $a \oplus (\ominus b) = a \ominus b$. Note that the $\ominus$-sign shares many properties with the minus sign in ordinary algebra.

Let $a \in \mathbb{S}$. Define its max-positive part $a^{\oplus}$ and its max-negative part $a^{\ominus}$ as follows. If $a \in \mathbb{S}^{\oplus}$ then $a^{\oplus} = a$ and $a^{\ominus} = \varepsilon$. If $a \in \mathbb{S}^{\ominus}$ then $a^{\oplus} = \varepsilon$ and $a^{\ominus} = a$. Finally, if $a \in \mathbb{S}^{\bullet}$ then there exists $b \in \mathbb{R}$ such that $a^{\oplus} = a^{\ominus} = b$. With these definitions any element $a \in \mathbb{S}$ can then be written as $a = a^{\oplus} \ominus a^{\ominus}$.

In $\mathbb{S}_{\max}$ we still cannot find an element b such that for an element $a \in \mathbb{R}$ we have that $a \oplus b = \varepsilon$. But with the introduction of a new relation, the so-called balance relation, we can get close. For $a, b \in \mathbb{S}$ the balance relation, denoted by ∇, is defined as

$$a \nabla b \;\Leftrightarrow\; a^{\oplus} \oplus b^{\ominus} = a^{\ominus} \oplus b^{\oplus}.$$

From this definition we can derive the following rules:

1. $\forall a, b, c \in \mathbb{S}$: $a \nabla b \oplus c \Leftrightarrow a \ominus b \nabla c$,
2. $\forall a, b \in \mathbb{S}^{\vee}$: $a \nabla b \Leftrightarrow a = b$.

The first rule implies, with $c = \varepsilon$, that: $a \nabla b \Leftrightarrow a \ominus b \nabla \varepsilon$. When $a = b$ we conclude from rules 1 and 2 that $a \ominus a \nabla \varepsilon$ or $a^{\bullet} \nabla \varepsilon$.

The introduction of the max-negative numbers, the balanced numbers and the balance operator allows us to manipulate with max-algebraic numbers almost in the same way as with numbers in the conventional algebra. One exception is that we do not have cancellation of equal terms with opposite signs, since $a \ominus a$ becomes $a^{\bullet}$ which is unequal to ε for $a \neq \varepsilon$. But we do have that $a^{\bullet} \nabla \varepsilon$.

Rule 2 is only valid for signed elements. Let, for instance, $a = 4^\bullet$ and $b = 3$. Then $a \nabla b$ since $4^\bullet \nabla 3 \Leftrightarrow 4 \ominus 4 \nabla 3 \Leftrightarrow 4 \nabla 3 \oplus 4 \Leftrightarrow 4 \nabla 4 \Leftrightarrow 4 = 4$, but $a \neq b$. This implies that equality is a stronger property than balance.

A major difficulty with the balance relation is that it is not transitive, e.g. $1 \nabla 1^\bullet \Leftrightarrow 1^\bullet \nabla \ominus 1$ but $1 \not\nabla \ominus 1$.

The extension of $\mathbb{S}_{\max}$ to matrices is similar to the extension of $\mathbb{R}_{\max}$ to matrices.

In $\mathbb{S}_{\max}$ we can define the determinant of a matrix, see [1]. First, we define the signature of a permutation σ as

$$\operatorname{sgn}(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even} \\ \ominus 0 & \text{otherwise.} \end{cases}$$

Then the determinant of an $n \times n$ matrix A is defined (as usual) as

$$\det(A) = \bigoplus_{\sigma} \operatorname{sgn}(\sigma) \otimes \bigotimes_{i=1}^{n} A_{i\sigma(i)}.$$

Next, we can define the transpose of the matrix of cofactors, denoted by $A^\natural$, by $A^\natural_{ij} = \operatorname{cof}_{ji}(A)$, where $\operatorname{cof}_{ji}(A)$ is equal to the determinant of the matrix obtained from A by deleting its j-th row and i-th column. This matrix satisfies $A \otimes A^\natural \nabla \det(A) \otimes E_n$, according to Theorem 3.76 of [1]. The 'inverse' of a matrix A, denoted by $A^\#$ since it is not the real inverse in max-algebra sense, could then be defined as $A^\# \otimes \det(A) = A^\natural$, provided that $\det(A) \not\nabla \varepsilon$.

The determinant can be used to characterize linear dependency of columns in a matrix. In the symmetrized max-algebra vectors $v_1, v_2, \ldots, v_m$ are said to be linearly dependent if scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{S}^\vee$ which are not all equal to ε, exist such that $\bigoplus_{i=1}^{m} \alpha_i \otimes v_i \nabla \varepsilon$. Let A be an $n \times n$ matrix. Then its columns are linearly dependent if and only if $\det(A) \nabla \varepsilon$. Since $\det(A) = \det(A^T)$ a similar statement is valid for the rows of the matrix A.

Within the max-algebra structure, a class of discrete event systems can be described by linear (in max-algebra sense) equations. Such relations were first described in [3] and [2], see also [1]. Consider for instance a production network which consists of n nodes (machines). Node i can only become active for the $(k+1)$-th time when previous nodes have finished their k-th activity and supplied node i. Let $x_i(k)$ denote the time instant node i becomes active for the k-th time and let a_{ij} denote the production time of node j and the transportation time from node j to node i. Then we have that

$$x_i(k+1) = \max_j (x_j(k) + a_{ij})$$

where j ranges over all nodes preceding node i. In max-algebra notation this relation becomes

$$x_i(k+1) = \bigoplus_j x_j(k) \otimes a_{ij},$$

or, in matrix-vector form

$$x(k+1) = A \otimes x(k).$$

A more general model is the following

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k) \tag{1}$$

$$y(k) = C \otimes x(k). \tag{2}$$

In this model $u(k)$ denotes the time instants outside resources become available and $y(k)$ denotes the time instants at which the k-th production cycle is finished. In the following we shall characterize a model of the form (1)–(2) by the triple (A,B,C) of system matrices.

When we assume that $x(0) = x_0$ then the input/output behavior of the system (1)–(2) is given by

$$y(k) = C \otimes A^k \otimes x_0 \oplus \bigoplus_{i=0}^{k-1} C \otimes A^{k-1-i} \otimes B \otimes u(i). \tag{3}$$

If we apply a unit impulse, defined by $u(k) = \varepsilon$ for $k \neq 0$ and $u(0) = 0$, to the system and if we assume that $x_0 = \varepsilon$, then the output of the system becomes $y(k) = C \otimes A^{k-1} \otimes B$ for $k = 1,2,\ldots$ One could view the application of the unit impulse to the system as the starting of the process, where it is assumed that all the resources are immediately available. Define

$$g_k = C \otimes A^{k-1} \otimes B \qquad k = 1,2,\ldots \tag{4}$$

These values are called the Markov parameters and the sequence $\{g_k\}_{k=1}^{\infty}$ is the impulse response of the system. In this paper we will only consider single input, single output (SISO) systems. For multi input, multi output (MIMO) systems the Markov parameters become matrices.

3 The minimal realization problem

The minimal realization problem can be formulated as follows. Given a sequence of Markov parameters $\{g_k\}_{k=1}^{\infty}$, find matrices A,B,C of appropriate dimensions such that $C \otimes A^{k-1} \otimes B = g_k$ for $k = 1,2,\ldots$ and such that the dimension of A is as small as possible.

A starting point is the construction of the semi-infinite Hankel matrix H corresponding with a sequence of Markov parameters $\{g_k\}_{k=1}^{\infty}$. This matrix is given by

$$H = \begin{pmatrix} g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ g_3 & g_4 & g_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We will denote by $H_{\alpha,\beta}$ the truncated Hankel matrix consisting of the first α rows and the first β columns of H.

The following theorem is an immediate translation from a similar theorem from conventional linear system theory (see e.g. [14]).

Theorem 1 *Given an impulse response $\{g_k\}_{k=1}^{\infty}$ such that for the corresponding Hankel matrix*

$$i\text{-th column} \oplus a_1 \otimes (i-1)\text{-th column} \oplus \ldots \oplus a_n \otimes (i-n)\text{-th column} \,\nabla\, \varepsilon, \tag{5}$$

for $i = n+1, n+2, \ldots$, $a_i \in \mathbb{S}$ and where n is the smallest integer for which this or another dependency of this form is possible. Then the discrete event system characterized by

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \ldots & \varepsilon \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \varepsilon \\ \varepsilon & \varepsilon & \ldots & \varepsilon & 0 \\ \ominus a_n & \ominus a_{n-1} & \ldots & \ldots & \ominus a_1 \end{pmatrix}, \; B = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}, \; C = \begin{pmatrix} 0 & \varepsilon & \ldots & \varepsilon \end{pmatrix} \tag{6}$$

is a minimal realization. In general we will have

$$C \otimes A^{i-1} \otimes B \nabla g_k, \qquad i = 1, 2, \ldots \tag{7}$$

Proof: Direct calculation shows that the impulse response of the given system balances the given impulse response. If there would exist a lower dimensional realization, then there would be a smaller number of successive columns of the Hankel matrix which would be linear independent. This follows from the fact that the resulting A-matrix satisfies its own characteristic equation (see [13]). Hence, we obtain a contradiction with the statement of the theorem. □

Remark: Since we will have in general a relation of the form (7) in which we have a balanced relation instead of (4) in which equality holds, the realization given by (6) will be called a minimal balancing realization. We will refer to the realization given by (6) as the realization in companion form.

Example 1 Consider the following sequence of Markov parameters

$$\{g_k\}_{k=1}^{\infty} = 3, 5, 8, 9, 14, 15, 20, 21, 26, 27, 32, 33, \ldots \tag{8}$$

A relation between those parameters which holds for any four consecutive Markov parameters, is given by

$$g_{i+3} \ominus 2 \otimes g_{i+2} \ominus 6 \otimes g_{i+1} \oplus 8 \otimes g_i \nabla \varepsilon, \qquad i = 1, 2, \ldots \tag{9}$$

There is no relation of the form (5) which holds for any three consecutive parameters. A similar relation holds for the columns of the corresponding Hankel matrix. According to Theorem 1 a minimal balancing realization is given by

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ \ominus 8 & 6 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}, \; C = \begin{pmatrix} 0 & \varepsilon & \varepsilon \end{pmatrix}. \tag{10}$$

The sequence of Markov parameters generated by this triple is equal to

$$\{g'_k\}_{k=1}^{\infty} = 3, 5, 8, 11^{\bullet}, 14, 17^{\bullet}, 20, 23^{\bullet}, \ldots$$

This sequence also satisfies the relation given by (9), but it is not equal to the original sequence $\{g_k\}_{k=1}^{\infty}$. We have that for $k = 1, 2, \ldots$

$$\begin{aligned} g_{2k} &\nabla g'_{2k}, \\ g_{2k-1} &= g'_{2k-1}. \end{aligned}$$

Note that for $k \geq 3$ we have both $g_{k+2} = 6 \otimes g_k$ and $g'_{k+2} = 6 \otimes g'_k$.

The system description in state space form, characterized by the matrices given in (10), can also be written in the following way

$$\begin{aligned} x_1(k+1) &= x_2(k) \oplus 3 \otimes u(k) \\ x_2(k+1) &= x_3(k) \oplus 5 \otimes u(k) \\ x_3(k+1) \oplus 8 \otimes x_1(k) &= 6 \otimes x_2(k) \oplus 2 \otimes x_3(k) \oplus 8 \otimes u(k) \\ y(k) &= x_1(k). \end{aligned}$$

In matrix-vector form these relations read

$$x(k+1) \oplus A^{\ominus} \otimes x(k) = A^{\oplus} \otimes x(k) \oplus Bu(k) \tag{11}$$

where the matrix A from (10) is written as

$$A = A^{\oplus} \ominus A^{\ominus}.$$

It is not clear yet how to interpret a system of the form (11). □

The use of Theorem 1 seems to be rather limited since the entries of the matrix A in (6) are not necessarily in $\mathbb{R}_\varepsilon$. A special case of Theorem 1 is given in [11] and in [12]. It is repeated here as Theorem 2.

Theorem 2 *Suppose that the columns in the Hankel matrix satisfy the following relation*

$$i\text{-}th\ column = c_1 \otimes (i-1)\text{-}th\ column \oplus \ldots \oplus c_n \otimes (i-n)\text{-}th\ column,$$

for $i = n+1, n+2, \ldots$ with $c_j \in \mathbb{R}_\varepsilon$ for $j = 1, \ldots, n$, and suppose n is the smallest integer for which this or another dependency of the same form is possible. Then the discrete-event system characterized by

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \ldots & & \varepsilon \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \varepsilon \\ \varepsilon & \varepsilon & \ldots & \ldots & \varepsilon & 0 \\ c_n & c_{n-1} & \ldots & \ldots & \ldots & c_1 \end{pmatrix}, \ B = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}, \ C = \begin{pmatrix} 0 & \varepsilon & \ldots & \varepsilon \end{pmatrix}$$

is a minimal realization.

So, in this case we have that $c_j = \ominus a_j$ for $j = 1, \ldots, n$. Note that all the entries of the matrices are in $\mathbb{R}_\varepsilon$.

In [12] also the following theorem is proved.

Theorem 3 *Given a series $\{g_k\}_{k=1}^{\infty}$ such that for the corresponding Hankel matrix any three successive columns are linearly dependent. Then a realization, represented by the triple (A, B, C), of at most state dimension 2 exists for which the given series is the impulse response and such that all entries of the matrices A, B and C are in $\mathbb{R}_\varepsilon$.*

In the work by Cuninghame-Green *et al.* (see e.g. [5], [15] and [16]) a sufficient condition for the existence of a minimal realization is derived for a certain behavior of the impulse response. These results are summarized in [6] as follows.

Theorem 4 *A minimal realization exists if the impulse response $G = \{g_k\}_{k=1}^{\infty}$ of a SISO DES exhibits a 'uniformly up-terrace' behavior, i.e. if the sequence G consists of M subsequences with lengths $n_1, n_2, \ldots, n_M$ and increments $c_1, c_2, \ldots, c_M$ respectively such that*

$$g_{k+1} = g_k + c_i \qquad \text{for } i = 1, 2, \ldots, M \text{ and } k = t_i, \ldots, t_i + n_i - 1,$$

with $n_M = +\infty$, $t_1 = 0$, $t_{i+1} = t_i + n_i$ and $c_{i+1} > c_i$ for $i = 1, 2, \ldots, M-1$.

The results which we mentioned above all deal with specific cases. No general theory exists yet. In [7] and [6] it is shown that the realization problem can be formulated as an Extended Linear Complementarity Problem (ELCP). In this reference a lower and upper bound for the system order is used. The lower bound is equal to the smallest value r such that r consecutive columns of the Hankel matrix are linearly dependent. Note that the order of the realization given in Theorem 1 is equal to this lower bound. With the ELCP approach it is also possible to find realizations for MIMO systems, while the other results which we mentioned in this section all deal with SISO systems. A drawback of this approach is that up until now no efficient algorithms have been developed which solve the problem in all cases.

The problem now becomes whether we can derive from (6) a realization of the given impulse response such that the entries of the resulting matrices are all in $\mathbb{R}_\varepsilon$. In conventional system theory the approach would be to look for a state space or similarity transformation which would transform the realization in companion form to a desired form. Therefore, we will study similarity transformations in the max-algebraic system theory in the next section and we will try to find a relation between a triple (A,B,C) given by (6) and a yet unknown triple (A',B',C') such that the entries of the latter triple are in $\mathbb{R}_\varepsilon$.

Before we will discuss state space transformations we conclude this section with the following proposition which we will use in the following section. The proposition provides another similarity between conventional system theory and the max-algebraic system theory for DES.

Proposition 5 *Let the triple (A,B,C) be a minimal balancing realization of order n of a given sequence of Markov parameters $\{g_k\}_{k=1}^{\infty}$. Define the matrices O and R as follows*

$$O = \begin{pmatrix} C \\ C \otimes A \\ \vdots \\ C \otimes A^{n-1} \end{pmatrix} \text{ and } R = \begin{pmatrix} B & A \otimes B & \dots & A^{n-1} \otimes B \end{pmatrix}. \tag{12}$$

Then $\det(O) \not\nabla \varepsilon$ *and* $\det(R) \not\nabla \varepsilon$.

Proof: Suppose $\det(R) \nabla \varepsilon$ (When $\det(O) \nabla \varepsilon$ the proof is analogous.) This implies that the columns of R are linearly dependent. From (12) it follows that the i-th column of R is equal to $A^{i-1} \otimes B$ ($i = 1, \dots, n$). So, there exist scalars α_i ($i = 1, \dots, n$) not all equal to ε such that

$$\bigoplus_{i=1}^{n} \alpha_i \otimes A^{i-1} \otimes B \nabla \varepsilon.$$

If we multiply this relation with $C \otimes A^k$, $k = 0, 1, \dots$, we obtain

$$\bigoplus_{i=1}^{n} \alpha_i \otimes C \otimes A^{k+i-1} \otimes B \nabla \varepsilon, \qquad \text{for } k = 0, 1, \dots$$

or, from the definition of the Markov parameters (see (4)),

$$\bigoplus_{i=1}^{n} \alpha_i \otimes g_{k+i} \nabla \varepsilon, \qquad \text{for } k = 0, 1, \dots$$

This implies that there is a relation between any n consecutive Markov parameters and according to Theorem 1 a realization of dimension lower than n would exist. This contradicts the assumption that the triple (A,B,C) is a minimal balancing realization of order n. $\square$

Remarks:

- Recall that in conventional system theory a minimal realization is both reachable and observable (see e.g. [14]). For SISO systems this means that the determinant of neither the reachability matrix nor the observability matrix is equal to zero. A similar statement is made in Proposition 5 for max-algebraic systems. However, there is no interpretation in terms of reachability or observability of the system yet.
- Proposition 5 is valid for general matrices in $\mathbb{S}_{\max}$. For matrices with entries in $\mathbb{R}_\varepsilon$ this proposition only holds if the minimal balancing realization and a minimal realization for which all entries of the system matrices are in $\mathbb{R}_\varepsilon$, are of the same order.

The opposite of Proposition 5 is not necessarily true. Let the triple (A,B,C) be an n-dimensional realization of a given impulse response which is not minimal and let H be the corresponding Hankel matrix. Then according to Proposition 6.3.3 from [6] we have that $\det(H_{n,n})\,\nabla\,\varepsilon$. According to Proposition 2.1.7 from [9] there holds

$$\det(H_{n,n}) = \det(O \otimes R) = \det(O) \otimes \det(R) \oplus a^{\bullet} \qquad \text{with } a \in \mathbb{R}_\varepsilon \tag{13}$$

in which O and R are as defined in (12). From (13) it follows that $\det(H_{n,n})\,\nabla\,\varepsilon$ does not necessarily imply that either $\det(O)\,\nabla\,\varepsilon$ or $\det(R)\,\nabla\,\varepsilon$. To illustrate this consider the system given by the following matrices

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}, \; B = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \; C = \begin{pmatrix} 3 & 2 \end{pmatrix}. \tag{14}$$

Then

$$O = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix} \text{ and } R = \begin{pmatrix} -3 & 3 \\ 2 & 5 \end{pmatrix}.$$

We have $\det(O) = 8\,\not\nabla\,\varepsilon$ and $\det(R) = \ominus 5\,\not\nabla\,\varepsilon$. But the system given by (14) is not minimal. The sequence of Markov parameters $\{g_k\}_{k=1}^{\infty}$ associated with (14) is given by

$$\{g_k\}_{k=1}^{\infty} = 4, 7, 10, 13, 16, 19, 22, \ldots \tag{15}$$

From this sequence we conclude that $g_{k+1} = 3 \otimes g_k$ for $k = 1, 2, \ldots$ and $\det(H_{2,2}) = 14^{\bullet}\,\nabla\,\varepsilon$. Hence, a one-dimensional realization of (15) exists. An example of a system which has (15) as its Markov parameters is given by the triple (A', B', C') with $A' = 3$, $B' = 2$, $C' = 2$.

We will encounter the matrices O and R defined in (12) again in the next section in which we will discuss similarity transformations.

4 Similarity transformations

In this section we will first recall some results on similarity transformations from conventional system theory. Then we will show that when a certain transformation exists between two realizations, these realizations exhibit the same behavior. Unfortunately, such a transformation does not always exists between two minimal realizations of the same impulse response. Therefore, we will look for similarity transformations under less restrictive constraints. We will show that a transformation between the realization in companion form and a realization with elements in $\mathbb{R}_\varepsilon$ exists under certain conditions. It is conjectured that a similar result is valid for more general realizations.

4.1 Results from conventional system theory

An important question is whether we are able to find state space transformations between two different realizations of a given impulse response. In conventional system theory, see e.g. [14], the following results are known:

1. When a similarity transformation is applied to the system, represented by the triple (A_1, B_1, C_1), then the resulting system (A_2, B_2, C_2) will have the same behavior as the original system.

2. Between any two minimal realizations (A_1, B_1, C_1) and (A_2, B_2, C_2) of a given impulse response a similarity transformation exists.

If we represent the similarity transformation by an invertible matrix T then the relation between the two systems is in both cases given by

$$A_1 = TA_2T^{-1},\ B_1 = TB_2 \text{ and } C_1 = C_2T^{-1}. \tag{16}$$

In the max-algebra the inverse of a matrix only exists for a small class of matrices, viz. matrices which can be written as the product of a diagonal matrix and a permutation matrix. Only for such matrices state space transformations can be defined in a similar way as in the conventional system theory. Therefore, we will look for a more general formulation of (16) in which we do not need inverse matrices.

4.2 A similarity transformation in the max-algebra

In [6] two transformations are proposed, the so-called L- and M-transformations, which make it possible to derive system equivalence for a broader class of triples of system matrices. But in the same reference it is also shown that such transformations may not exist between two different realizations of the same impulse response. Here we introduce a different transformation, the T-transformation, which is more general than the L- and M-transformations.

Proposition 6 *Let the triples (A_1, B_1, C_1) and (A_2, B_2, C_2) be such that:*

$$\begin{aligned} T \otimes A_2 &= A_1 \otimes T, \\ T \otimes B_2 &= B_1, \text{ and} \\ C_2 &= C_1 \otimes T, \end{aligned}$$

for some matrix T. Then both triples are equivalent (i.e. they exhibit the same input/output behavior).

Proof: Let $k \in \mathbb{N}$. Then

$$\begin{aligned} C_2 \otimes A_2^k \otimes B_2 &= C_1 \otimes T \otimes A_2^k \otimes B_2 \\ &= C_1 \otimes A_1 \otimes T \otimes A_2^{k-1} \otimes B_2 \\ &= \cdots = C_1 \otimes A_1^k \otimes T \otimes B_2 \\ &= C_1 \otimes A_1^k \otimes B_1. \end{aligned}$$

□

In the same way it is shown that two triples (A_1, B_1, C_1) and (A_2, B_2, C_2) are equivalent when a matrix S exists such that $S \otimes A_1 = A_2 \otimes S$, $S \otimes B_1 = B_2$ and $C_1 = C_2 \otimes S$. The matrices S and T do not have to be square. It is easily shown that these transformations, which will be called the T-transformation respectively the S-transformation, include the M-transformation respectively the L-transformation. But unfortunately, also these transformations may not exist between two different realizations of the same impulse response. This will be demonstrated in the following example.

Example 2 Consider the triple (A_1, B_1, C_1) given by

$$A_1 = \begin{pmatrix} 6 & 9 \\ 0 & 5 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \; C_1 = \begin{pmatrix} 9 & 15 \end{pmatrix}, \tag{17}$$

and the triple (A_2, B_2, C_2) given by

$$A_2 = \begin{pmatrix} 6 & 10 \\ -1 & 5 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \; C_2 = \begin{pmatrix} 9 & 15 \end{pmatrix}. \tag{18}$$

Both triples are minimal realizations of the following sequence of Markov parameters

$$\{g_k\}_{k=1}^{\infty} = 11, 16, 21, 27, 33, 39, 45, 51, 57, \ldots, \tag{19}$$

see Example 6.4.1 in [6]. Between the triples (A_1, B_1, C_1) and (A_2, B_2, C_2) a state space transformation T exists such that $T \otimes A_2 = A_1 \otimes T$, $T \otimes B_2 = B_1$ and $C_2 = C_1 \otimes T$. The matrix T given by

$$T = \begin{pmatrix} 0 & 4 \\ -6 & 0 \end{pmatrix} \tag{20}$$

satisfies these relations. It can be shown that no matrix $S \in \mathbb{R}_{\varepsilon}^{2\times 2}$ exists such that $S \otimes A_1 = A_2 \otimes S$, $S \otimes B_1 = B_2$ and $C_1 = C_2 \otimes S$.

In [6] also the following triple is computed as a minimal realization of the sequence given by (19),

$$A_5 = \begin{pmatrix} 6 & 10 \\ 0 & 5 \end{pmatrix}, \; B_5 = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \; C_5 = \begin{pmatrix} 8 & 15 \end{pmatrix}. \tag{21}$$

When we try to solve the equations $T \otimes A_5 = A_1 \otimes T$, $T \otimes B_5 = B_1$ and $C_5 = C_1 \otimes T$ with the entries of the matrix T as the unknowns, it turns out that no solution can be found. Hence, there does not exist a T-transformation between the triples (A_1, B_1, C_1) and (A_5, B_5, C_5). In a similar way it can be shown that no S-transformation exists between those triples. In [6] it was already shown that no M- or L-transformation exists between these two realizations. $\square$

From the previous examples we conclude that it is not always possible to find a state space transformation in the max-algebra. Therefore, we will extend our search for state space transformations between different realizations of a given impulse to the symmetrized max-algebra in the following sections.

4.3 Transforming the companion form

From conventional system theory it is known (see [14, Theorem 20]) that between two minimal realizations (A_1, B_1, C_1) and (A_2, B_2, C_2) of an impulse response a unique state space transformation exists, represented by a matrix T, where T is given by (for SISO systems) $T = (O_1)^{-1} O_2 = R_1 (R_2)^{-1}$ in which O_i and R_i $(i = 1, 2)$ are the observability respectively the controllability matrices of the given systems. In the following we will show that similar results are valid, under certain conditions, for systems in the max-algebra. A drawback however is that we obtain balances instead of the equalities as in Proposition 6.

Proposition 7 *Let the triple (A',B',C') be an n-dimensional realization of a sequence of Markov parameters $\{g_k\}_{k=1}^{\infty}$ such that all entries of the matrices are in $\mathbb{R}_\varepsilon$. Let (A,B,C) be the n-dimensional minimal balancing realization of the same sequence according to Theorem 1 and such that the matrix A' satisfies the characteristic equation of A. Then a state space transformation matrix T such that $T \otimes A' \nabla A \otimes T$, $T \otimes B' \nabla B$ and $C' \nabla C \otimes T$ is given by*

$$T = \begin{pmatrix} C' \\ C' \otimes A' \\ \vdots \\ C' \otimes (A')^{n-1} \end{pmatrix}. \tag{22}$$

Proof : Some easy computations show that

$$T \otimes A' = \begin{pmatrix} C' \otimes A' \\ C' \otimes (A')^2 \\ \vdots \\ C' \otimes (A')^n \end{pmatrix},$$

while

$$A \otimes T = \begin{pmatrix} C' \otimes A' \\ C' \otimes (A')^2 \\ \vdots \\ \ominus a_1 \otimes C' \otimes (A')^{n-1} \ominus \cdots \ominus a_{n-1} \otimes C' \otimes A' \ominus a_n \otimes C' \end{pmatrix}.$$

The characteristic equation of A is equal to $\lambda^n \oplus a_1 \otimes \lambda^{n-1} \oplus \cdots \oplus a_n \nabla \varepsilon$. Since we have assumed that A' satisfies the characteristic equation of A, it follows immediately that $T \otimes A' \nabla A \otimes T$.

Furthermore, we have

$$T \otimes B' = \begin{pmatrix} C' \otimes B' \\ C' \otimes A' \otimes B' \\ \vdots \\ C' \otimes (A')^{n-1} \otimes B' \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} = B,$$

since both triples are realizations of the same sequence of Markov parameters. Finally

$$C \otimes T = \begin{pmatrix} 0 & \varepsilon & \dots & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} C' \\ C' \otimes A' \\ \vdots \\ C' \otimes (A')^{n-1} \end{pmatrix} = C',$$

which concludes the proof. □

Remarks:

- Note that we even proved that $T \otimes B' = B$ and $C \otimes T = C'$, i.e. here we have an equality instead of a balance relation. The equalities follow immediately from the fact that all the entries of the matrices involved are in $\mathbb{R}_\varepsilon$. For the relation between A and A' it will in general not be possible to obtain equality instead of a balance relation, since the matrix $A \otimes T$ may contain signed entries.

- One could try to use the result of Proposition 7 as the starting point of a construction of a minimal realization. But it seems that the resulting problem which has to be solved is just the minimal realization problem.

- The matrix A' does not necessarily need to have the same characteristic equation as A. In the proof we only used that A' satisfied the characteristic equation of A, see also Example 6 in Section 4.4.

A question is whether we can transform the minimal balancing realization obtained in Theorem 1, with entries which are not necessarily in $\mathbb{R}_\varepsilon$, to a triple for which the entries are in $\mathbb{R}_\varepsilon$ and such that the dimension of both systems are equal. In [12] it was shown that when the minimal balancing realization is two-dimensional, always a two-dimensional realization can be found of which all the entries of the system matrices are in $\mathbb{R}_\varepsilon$, see also Theorem 3. For higher dimensional systems this will not always be the case as is shown in the following example.

Example 3 Consider the following sequence of Markov parameters

$$\{g_k\}_{k=1}^{\infty} = 14, 20, 30, 33, 44, 47, 58, 61, 72, 75, 86, 89, \ldots$$

A relation between the Markov parameters is the following

$$g_{i+3} \oplus 4 \otimes g_{i+2} \ominus 14 \otimes g_{i+1} \ominus 18 \otimes g_i \,\nabla\, \varepsilon,$$

for $i = 1, 2, 3, \ldots$ According to Theorem 1 a minimal balancing realization is given by

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ 18 & 14 & \ominus 4 \end{pmatrix},\ B = \begin{pmatrix} 14 \\ 20 \\ 30 \end{pmatrix},\ C = \begin{pmatrix} 0 & \varepsilon & \varepsilon \end{pmatrix}.$$

For this realization we have

$$\{C \otimes A^{k-1} \otimes B\}_{k=1}^{\infty} = 14, 20, 30, 34^{\bullet}, 44, 48^{\bullet}, 58, 62^{\bullet}, 72, 76^{\bullet}, \ldots \nabla \{g_k\}_{k=1}^{\infty}.$$

It turns out, for instance with the ELCP approach from [6], that no realization of dimension 3 of the sequence $\{g_k\}_{k=1}^{\infty}$ exists with all its entries in $\mathbb{R}_\varepsilon$. A minimal realization in $\mathbb{R}_\varepsilon$ is given by the following 4-dimensional system

$$A' = \begin{pmatrix} -3 & 2 & -6 & -1 \\ -2 & -3 & 4 & 9 \\ 10 & 1 & -6 & 1 \\ -1 & 5 & -4 & -1 \end{pmatrix},\ B' = \begin{pmatrix} 0 \\ 8 \\ -4 \\ -4 \end{pmatrix},\ C' = \begin{pmatrix} -2 & -2 & 10 & -5 \end{pmatrix}.$$

The reason why it is not possible to transform the three-dimensional triple (A, B, C) to a three-dimensional triple with entries in $\mathbb{R}_\varepsilon$ could be the following. The characteristic polynomial of the matrix A does not satisfy the necessary and sufficient conditions given by [6, Proposition 5.3.3], for being the characteristic polynomial of a 3×3 matrix with entries in $\mathbb{R}_\varepsilon$. To conclude this example we note that $\det(R') = 65^{\bullet} \,\nabla\, \varepsilon$ and so according to Proposition 5 the system (A', B', C') is not minimal in $\mathbb{S}_{\max}$. □

4.4 A balancing similarity transformation

Next, we will give a proposition in which we describe a possible similarity transformation between two minimal realizations of a given impulse response. For a part of this proposition we do not have a proof yet and the result is only conjectured. We will illustrate this conjecture with some small examples.

Conjecture 8 *Let the triples (A_1,B_1,C_1) and (A_2,B_2,C_2) be two minimal realizations of the same sequence of Markov parameters and let (A,B,C) be a minimal balancing realization of the same sequence. If the matrices A_1 and A_2 are of the same order as A and satisfy the characteristic equation of A, then a state space transformation matrix T exists such that $T \otimes B_2 \nabla B_1$ and $C_2 \nabla C_1 \otimes T$. Under certain conditions (to be derived in the proof) T also satisfies $T \otimes A_2 \nabla A_1 \otimes T$. One matrix T is given by $T = T_o = O_1^{\#} \otimes O_2$ with*

$$O_i = \begin{pmatrix} C_i \\ C_i \otimes A_i \\ \vdots \\ C_i \otimes A_i^{n-1} \end{pmatrix}, \qquad i = 1,2, \tag{23}$$

and where the matrix $O_1^{\#}$ is as defined in Section 2. Another transformation matrix is given by $T = T_r = R_1 \otimes (R_2)^{\#}$ with

$$R_i = \begin{pmatrix} B_i & A_i \otimes B_i & \ldots & A_i^n \otimes B_i \end{pmatrix}, \qquad i = 1,2.$$

Proof: We first note that, since the triples are minimal realizations, according to Proposition 5 $\det(O_1) \not\nabla \varepsilon$ and $\det(R_2) \not\nabla \varepsilon$ and hence the matrices $(O_1)^{\#}$ and $(R_2)^{\#}$ exist.

Since both triples are realizations of the same sequence of Markov parameters it follows immediately that $O_1 \otimes B_1 = O_2 \otimes B_2$ and hence $(O_1)^{\#} \otimes O_1 \otimes B_1 = (O_1)^{\#} \otimes O_2 \otimes B_2$. Since $(O_1)^{\#} \otimes O_1 \nabla E$, we conclude that $B_1 \nabla T_o \otimes B_2$. In a similar way it can be shown that $C_2 \nabla C_1 \otimes T_r$.

Let (A,B,C) be the realization of the same Markov parameters according to Theorem 1. In the proof of Proposition 7, we concluded that $C_1 = C \otimes O_1$ and $C_2 = C \otimes O_2$. If we multiply both sides of the former equality with $(O_1)^{\#} \otimes O_2$ we obtain

$$\begin{aligned} C_1 \otimes (O_1)^{\#} \otimes O_2 &= C \otimes O_1 \otimes (O_1)^{\#} \otimes O_2 \\ &\nabla C \otimes O_2 \\ &= C_2. \end{aligned}$$

In an analogous way it can be shown that $B_1 \nabla T_r \otimes B_2$.

The relations $T \otimes A_2 \nabla A_1 \otimes T$ with $T = T_o$ respectively $T = T_r$ remain to be shown. A major problem in this case is the fact that the balance relation is not necessarily transitive. In the following we derive which conditions should be satisfied.

If we assume that both A_1 and A_2 satisfy the characteristic equation of A, then we know from Proposition 7 (with $T = O_1$ respectively $T = O_2$) that the following relations hold

$$O_1 \otimes A_1 \nabla A \otimes O_1 \tag{24}$$

$$O_2 \otimes A_2 \nabla A \otimes O_2. \tag{25}$$

If we multiply both sides of (24) with $(O_1)^{\#}$ we obtain

$$O_1 \otimes A_1 \otimes (O_1)^{\#} \nabla A \otimes O_1 \otimes (O_1)^{\#}. \tag{26}$$

Since $O_1 \otimes (O_1)^{\#} \nabla E$ we have that the right-hand side of (26) balances A. The first question now is whether also the left-hand side of (26) balances A, so whether

$$O_1 \otimes A_1 \otimes (O_1)^{\#} \nabla A. \tag{27}$$

If (27) is true, then also, after right-multiplication with O_2,

$$O_1 \otimes A_1 \otimes (O_1)^{\#} \otimes O_2 \nabla A \otimes O_2. \tag{28}$$

The second question now becomes, from (28) and (25), whether

$$O_1 \otimes A_1 \otimes (O_1)^{\#} \otimes O_2 \nabla O_2 \otimes A_2. \tag{29}$$

If (29) holds, then we also have, after multiplication from the left with $(O_1)^{\#}$, that

$$(O_1)^{\#} \otimes O_1 \otimes A_1 \otimes (O_1)^{\#} \otimes O_2 \nabla (O_1)^{\#} \otimes O_2 \otimes A_2. \tag{30}$$

The left-hand side of (30) balances $A_1 \otimes (O_1)^{\#} \otimes O_2$ and hence the third and final question becomes whether we can conclude that the following holds

$$A_1 \otimes (O_1)^{\#} \otimes O_2 \nabla (O_1)^{\#} \otimes O_2 \otimes A_2. \tag{31}$$

If (31) holds then the matrix T_o defined by $T_o = (O_1)^{\#} \otimes O_2$ satisfies $A_1 \otimes T_o \nabla T_o \otimes A_2$. The problem is that it is not clear under which conditions we can answer the questions posed, represented by the relations (27), (29) and (31), positively. Similar conditions as (27), (29) and (31) can be derived for the relation $T_r \otimes A_2 \nabla A_1 \otimes T_r$ to hold. □

Remarks:

- Note that we have for the triple (A,B,C) given by (6) in Theorem 1 that

  $$O = \begin{pmatrix} C \\ C \otimes A \\ \vdots \\ C \otimes A^{n-1} \end{pmatrix} = E_n.$$

  So the transformation in Proposition 7 can be seen as a special case of the transformation suggested in Conjecture 8.

- A different question is the following. Consider the triples (A_1,B_1,C_1) and (A_2,B_2,C_2) and suppose that a matrix T can be found such that $T \otimes A_2 \nabla A_1 \otimes T$, $T \otimes B_2 \nabla B_1$ and $C_2 \nabla C_1 \otimes T$. Are both systems equivalent? This question seems hard to prove because the balance relation is not transitive. Therefore, we cannot proceed as in the proof of Proposition 6.

In the following examples we will illustrate the given results.

Example 4 Consider the realizations given in Example 2 by (17) and (21). It is shown in [6] that no M- or L-transformation exists between the two realizations. Also no S- or T-transformation exists, see Example 2. But there does exist a transformation matrix T_o, viz.

$$T_o = \begin{pmatrix} 0 & 5 \ominus 5 \\ \ominus(-6) & 0 \end{pmatrix}, \tag{32}$$

such that

$$T_o \otimes A_5 \nabla A_1 \otimes T_o,\ B_1 \nabla T_o \otimes B_5 \text{ and } C_5 \nabla C_1 \otimes T_o. \tag{33}$$

When we use Theorem 1 we find the following realization

$$A = \begin{pmatrix} \varepsilon & 0 \\ \ominus 11 & 6 \end{pmatrix},\ B = \begin{pmatrix} 11 \\ 16 \end{pmatrix},\ C = \begin{pmatrix} 0 & \varepsilon \end{pmatrix}. \tag{34}$$

Between this realization and the realization given by (17) respectively (21) there exist transformation matrices T^i $(i = 1,5)$ such that

$$T^i \otimes A_i \nabla A \otimes T^i,\ T^i \otimes B_i \nabla B \text{ and } C_i \nabla C \otimes T^i, \qquad i = 1,5.$$

The following matrices satisfy these relations.

$$T^1 = \begin{pmatrix} 9 & 15 \\ 15 & 20 \end{pmatrix} \text{ respectively } T^5 = \begin{pmatrix} 8 & 15 \\ 15 & 20 \end{pmatrix},$$

between the triples (A,B,C) and (A_1,B_1,C_1) respectively between the triples (A,B,C) and (A_5,B_5,C_5). Note that indeed for the matrices T^1 and T^5 we have

$$T^i = \begin{pmatrix} C_i \\ C_i \otimes A_i \end{pmatrix} \qquad i = 1,5.$$

If we compute the matrix $(T^1)^\#$, see Section 2 for the definition, we obtain

$$(T^1)^\# = \begin{pmatrix} \ominus(-10) & -15 \\ -15 & \ominus(-21) \end{pmatrix}$$

and hence

$$(T^1)^\# \otimes T^5 = \begin{pmatrix} 0 & 5^\bullet \\ \ominus(-6) & 0 \end{pmatrix} = T_o$$

which supports Conjecture 8.

Another transformation matrix between the triples (A_1,B_1,C_1) and (A_5,B_5,C_5) is given by the following matrix T_r

$$T_r = \begin{pmatrix} 0 & 4^\bullet \\ -5^\bullet & 0 \end{pmatrix}.$$

Note that $T_r \nabla T_o$. The matrix T_r is equal to $R_1 \otimes R_5^\#$ where R_1 and R_5 are given by

$$R_i = \begin{pmatrix} B_i & A_i \otimes B_i \end{pmatrix},$$

for $i = 1,5$.

In a similar way we can derive a transformation matrices T_o and T_r between the triple (A_1,B_1,C_1) and the triple (A_2,B_2,C_2) given by (18). These transformation matrices turn out to be

$$T_o = \begin{pmatrix} 0 & 5^\bullet \\ -6^\bullet & 0 \end{pmatrix} \text{ and } T_r = \begin{pmatrix} 0 & 4^\bullet \\ -5^\bullet & 0 \end{pmatrix}.$$

Note that $T_o \nabla T_r$. With these matrices we have the following relations $T_o \otimes A_2 \nabla A_1 \otimes T_o$, $T_o \otimes B_2 \nabla B_1$ and $C_2 \nabla C_1 \otimes T_o$ for T_o and similar relations with T_r. Note that in Example 2 we found a matrix T such that we had equalities in these relations. The matrix T from Example 2, which is given by (20), satisfies both $T \nabla T_o$ and $T \nabla T_r$. □

Based on the last remarks in Example 4 we have the following conjecture.

Conjecture 9 *Let T be a transformation matrix between the triples (A_1, B_1, C_1) and (A_2, B_2, C_2) such that $T \otimes A_2 \nabla A_1 \otimes T$, $T \otimes B_2 \nabla B_1$ and $C_2 \nabla C_1 \otimes T$. If T' is another matrix such that these relations hold, then $T' \nabla T$.*

Remark: The matrix T_o given in (32) can be written as

$$T_o = T_p \ominus T_n$$

with

$$T_p = \begin{pmatrix} 0 & 5 \\ \varepsilon & 0 \end{pmatrix} \text{ and } T_n = \begin{pmatrix} \varepsilon & 5 \\ -6 & \varepsilon \end{pmatrix}.$$

With these matrices the balanced relations given by (33) can be transformed to the following equalities

$$\begin{aligned} T_p \otimes A_5 \oplus A_1 \otimes T_n &= A_1 \otimes T_p \oplus T_n \otimes A_5, \\ B_1 \oplus T_n \otimes B_5 &= T_p \otimes B_5, \\ C_5 \oplus C_1 \otimes T_n &= C_1 \otimes T_p, \end{aligned} \tag{35}$$

since all entries of the matrices involved are in $\mathbb{R}_\varepsilon$. So, we could say that two triples (A_1, B_1, C_1) and (A_5, B_5, C_5) with entries in $\mathbb{R}_\varepsilon$ are equivalent if matrices T_p and T_n with entries in $\mathbb{R}_\varepsilon$ can be found such that (35) holds. It is, however, not clear yet how we can prove, using the relations given by (35), that the triples (A_1, B_1, C_1) and (A_5, B_5, C_5) yield the same Markov parameters.

The following example is a continuation of Example 1.

Example 5 In Example 1 it was shown that a balancing realization of the following sequence of Markov parameters

$$\{g_k\}_{k=1}^{\infty} = 3, 5, 8, 9, 14, 15, 20, 21, 26, 27, 32, 33, \ldots$$

was given by

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ \ominus 8 & 6 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}, \; C = \begin{pmatrix} 0 & \varepsilon & \varepsilon \end{pmatrix}. \tag{36}$$

A problem with this realization is that not all entries of A are in $\mathbb{R}_\varepsilon$.

A realization of the sequence $\{g_k\}_{k=1}^{\infty}$ which does have all its entries in $\mathbb{R}_\varepsilon$ is given in [6] by the triple

$$A_1 = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & 3 \\ 0 & 3 & \varepsilon \end{pmatrix}, \; B_1 = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix}, \; C_1 = \begin{pmatrix} 2 & \varepsilon & 2 \end{pmatrix}. \tag{37}$$

According to Proposition 7 a similarity transformation exists between the triples (A, B, C) and (A_1, B_1, C_1). It is given by

$$T = \begin{pmatrix} C_1 \\ C_1 \otimes A_1 \\ C_1 \otimes A_1^2 \end{pmatrix} = \begin{pmatrix} 2 & \varepsilon & 2 \\ 4 & 5 & \varepsilon \\ 6 & 6 & 8 \end{pmatrix}.$$

With this matrix T we have

$$T \otimes A_1 = \begin{pmatrix} 4 & 5 & \varepsilon \\ 6 & 6 & 8 \\ 8 & 11 & 9 \end{pmatrix} \text{ while } A \otimes T = \begin{pmatrix} 4 & 5 & \varepsilon \\ 6 & 6 & 8 \\ 10^\bullet & 11 & 10^\bullet \end{pmatrix}$$

and hence $T \otimes A_1 \nabla A \otimes T$. Since the matrix $A \otimes T$ contains signed entries we do not have equality in this relation. Furthermore, we have

$$T \otimes B_1 = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} = B \text{ and } C \otimes T = \begin{pmatrix} 2 & \varepsilon & 2 \end{pmatrix} = C_1.$$

Another triple which realizes the Markov parameters of (8) and of which all entries are in $\mathbb{R}_\varepsilon$ is given by

$$A_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix},\ C_2 = \begin{pmatrix} 3 & 1 & 3 \end{pmatrix}. \tag{38}$$

There exists a transformation matrix T_o between the triple (A_1, B_1, C_1) given by (37) and the triple (A_2, B_2, C_2) given by (38) such that $T_o \otimes A_2 \nabla A_1 \otimes T_o$, $T_o \otimes B_2 \nabla B_1$ and $C_2 \nabla C_1 \otimes T_o$. This matrix is given by

$$T_o = \begin{pmatrix} 1 & -1^\bullet & 1^\bullet \\ 0^\bullet & 1 & 0^\bullet \\ -1^\bullet & -1^\bullet & 1 \end{pmatrix},$$

As in Example 4 the matrix T_o is equal to $O_1^\# \otimes O_2$ with $O_i = \begin{pmatrix} C_i \\ Ci \otimes A_i \\ C_i \otimes A_i^2 \end{pmatrix}$, for $i = 1, 2$.

The matrix T_o represents the following relation between the state variables associated with the systems given by (37) and (38). Let $x^1(k)$ and $x^2(k)$ denote the state variable corresponding to (37) respectively (38). Then we have

$$x^1(k) \nabla T_o \otimes x^2(k). \tag{39}$$

It turns out that when we compute the impulse response of the systems given by (37) and (38) the corresponding state variables $x^1(k)$ and $x^2(k)$ indeed satisfy (39) for all $k = 1, 2, \ldots$ Furthermore it is easily shown that $x^1(k) = T \otimes x(k)$, where $x(k)$ denotes the state variable corresponding to (36).

Another transformation matrix between the systems given by (37) and (38) is given by

$$T_r = \begin{pmatrix} 1 & 1^\bullet & 0 \\ -2^\bullet & 1 & -1^\bullet \\ 0^\bullet & 0^\bullet & 1 \end{pmatrix},$$

which is equal to $R_1 \otimes R_2^\#$ with $R_i = \begin{pmatrix} B_i & A_i \otimes B_i & A_i^2 \otimes B_i \end{pmatrix}$, for $i = 1, 2$. Also in this example we have that $T_o \nabla T_r$. □

It is not completely clear under which conditions Conjecture 8 is valid. We will now present an example for which it does not hold true.

Example 6 Consider the systems

$$A_1 = \begin{pmatrix} 5 & -1 & 0 \\ 3 & -3 & 5 \\ -3 & -3 & -4 \end{pmatrix}, \; B_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \; C_1 = \begin{pmatrix} -3 & 0 & 2 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} \varepsilon & \varepsilon & 5 \\ -3 & \varepsilon & 0 \\ 0 & \varepsilon & 5 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix}, \; C_2 = \begin{pmatrix} 2 & 7 & 2 \end{pmatrix}.$$

Both systems are minimal realizations of the sequence

$$\{g_k\}_{k=1}^{\infty} = 2,5,7,12,17,22,27,32,37,42,\ldots$$

If we compute a transformation matrix T_o as in Conjecture 8 we obtain

$$T_o = \begin{pmatrix} -1 & \ominus 1 & 4 \\ 2^\bullet & 7 & 4^\bullet \\ 0 & \ominus 1 & 2^\bullet \end{pmatrix},$$

and we have

$$T_o \otimes A_2 = \begin{pmatrix} 4 & \varepsilon & 9 \\ 5 & \varepsilon & 9^\bullet \\ 2^\bullet & \varepsilon & 7^\bullet \end{pmatrix} \text{ while } A_1 \otimes T_o = \begin{pmatrix} 4 & 6^\bullet & 9 \\ 5 & \ominus 6 & 7^\bullet \\ 1^\bullet & 4 & 1^\bullet \end{pmatrix}.$$

So, in this case $T_o \otimes A_2 \not\nabla A_1 \otimes T_o$. A reason could be that between any four consecutive Markov parameters several relations are possible. In fact, the given sequence of Markov parameters satisfies any relation of the form

$$g_{i+3} \ominus 5 \otimes g_{i+2} \oplus a \otimes g_{i+1} \oplus b \otimes g_i \nabla \varepsilon, \qquad i = 1,2,\ldots,$$

where $a,b \in \mathbb{S}$ with $a \oplus 7 = 7$ and $b \oplus 10 = 10$. Therefore, it is possible that triples (A_1,B_1,C_1) and (A_2,B_2,C_2) of which A_1 and A_2 have different characteristic polynomials, as is the case in this example, both satisfy $C_i \otimes A_i^{k-1} \otimes B_i = g_k$ for $i = 1,2$ and $k = 1,2,\ldots$. Note that we do have $T \otimes B_2 \nabla B_1$ and $C_2 \nabla C_1 \otimes T$.

One of the relations between the Markov parameters is the following

$$g_{i+3} \ominus 5 \otimes g_{i+2} \ominus 2 \otimes g_{i+1} \oplus 7^\bullet \nabla \varepsilon \qquad i = 1,2,\ldots$$

A minimal realization according to Theorem 1 is given by

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ 7^\bullet & 2 & 5 \end{pmatrix}, \; B = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}, \; C = \begin{pmatrix} 0 & \varepsilon & \varepsilon \end{pmatrix}.$$

Note that, since we have that $C \otimes A^{i-1} \otimes B = g_i$ for all i, this realization is not only a minimal balancing realization. Both A_1 and A_2 satisfy the characteristic equation of A, although their respective characteristic polynomials are different. Let T_i ($i = 1,2$) be given by (23). Then we have that $T_i \otimes A_i \nabla A \otimes T_i$, $B = T_i \otimes B_i$ and $C_i = C \otimes T_i$ ($i = 1,2$). From these relations we easily derive (see also the proof of Conjecture 8) that $B_1 \nabla T_o \otimes B_2$ and $C_2 \nabla C_1 \otimes T$ where $T_o = (T_1)^\# \otimes T_2$. We already showed that $T_o \otimes A_2 \not\nabla A_1 \otimes T_o$. For the matrices in this example the relations (27) and (29) are true, but relation (31) does not hold. $\square$

5 Concluding remarks

In this paper we have summarized some results on the minimal realization problem in the max-algebraic system theory for discrete event systems. We also tried to find similarity transformations between different realizations of a given impulse response. In certain cases the existence of a similarity transformation could be proved. The transformations that we found resemble the transformations which exist between two equivalent minimal realizations in the conventional system theory. We have not proved a general result yet. The intransitivity of the balance relation is the major obstacle. It is not obvious how to solve this problem, since the intransitivity of the balance relation seems to follow immediately from its definition.

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References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat, *Synchronization and Linearity*. New York: John Wiley & Sons, 1992.

[2] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot, "A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing," *IEEE Transactions on Automatic Control*, vol. 30, no. 3, pp. 210–220, Mar. 1985.

[3] G. Cohen, P. Moller, J.P. Quadrat, and M. Viot, "Linear system theory for discrete event systems," in *Proceedings of the 23rd IEEE Conference on Decision and Control*, Las Vegas, Nevada, pp. 539–544, Dec. 1984.

[4] R.A. Cuninghame-Green, *Minimax Algebra*, vol. 166 of *Lecture Notes in Economics and Mathematical Systems*. Berlin, Germany: Springer-Verlag, 1979.

[5] R.A. Cuninghame-Green, "Algebraic realization of discrete dynamic systems," in *Proceedings of the 1991 IFAC Workshop on Discrete Event System Theory and Applications in Manufacturing and Social Phenomena*, Shenyang, China, pp. 11–15, June 1991.

[6] B. De Schutter, *Max-Algebraic System Theory for Discrete Event Systems*. PhD thesis, Faculty of Applied Sciences, K.U.Leuven, Leuven, Belgium, Feb. 1996.

[7] B. De Schutter and B. De Moor, "Minimal realization in the max algebra is an extended linear complementarity problem," *Systems & Control Letters*, vol. 25, no. 2, pp. 103–111, May 1995.

[8] B. De Schutter and B. De Moor, “Matrix factorization and minimal state space realization in the max-plus algebra,” in *Proceedings of the 1997 American Control Conference*, Albuquerque, New Mexico, pp. 3136–3140, June 1997.

[9] S. Gaubert, *Théorie des Systèmes Linéaires dans les Dioïdes*. PhD thesis, Ecole Nationale Supérieure des Mines de Paris, France, July 1992.

[10] Max Plus, “Linear systems in (max,+) algebra,” in *Proceedings of the 29th IEEE Conference on Decision and Control*, Honolulu, Hawaii, pp. 151–156, Dec. 1990.

[11] G.J. Olsder, “On the characteristic equation and minimal realizations for discrete-event dynamic systems,” in *Proceedings of the 7th International Conference on Analysis and Optimization of Systems* (Antibes, France) (A. Bensoussan and J.L. Lions, eds.), vol. 83 of *Lecture Notes in Control and Information Sciences*, pp. 189–201, Berlin, Germany: Springer-Verlag, 1986.

[12] G.J. Olsder and R.E. de Vries, “On an analogy of minimal realizations in conventional and discrete-event dynamic systems,” in *Discrete Event Systems: Models and Applications* (Proceedings of the IIASA Conference, Sopron, Hungary, Aug. 1987) (P. Varaiya and A.B. Kurzhanski, eds.), vol. 103 of *Lecture Notes in Control and Information Sciences*, Berlin, pp. 149–161, Springer, 1988.

[13] G.J. Olsder and C. Roos, “Cramer and Cayley-Hamilton in the max algebra,” *Linear Algebra and Its Applications*, vol. 101, pp. 87–108, 1988.

[14] E.D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, vol. 6 of *Textbooks in Applied Mathematics*. New York: Springer-Verlag, 1990.

[15] L. Wang, X. Xu, P. Butkovič, and R.A. Cuninghame-Green, “Combinatorial aspect of minimal realization of discrete event dynamic system,” in *Proceedings of the Joint Workshop on Discrete Event Systems (WODES ’92)*, Prague, Czechoslovakia, pp. 59–62, Aug. 1992.

[16] L. Wang, X. Xu, and R.A. Cuninghame-Green, “Realization of a class of discrete event sequence over max-algebra,” in *Proceedings of the 1995 American Control Conference*, Seattle, Washington, pp. 3146–3150, June 1995.