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ON MAX-ALGEBRAIC MODELS FOR TRANSPORTATION NETWORKS

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Abstract

We will consider the modeling and analysis of public transportation networks which evolve according to a timetable. Some results are summarized. A way to control these networks is introduced.

1 Introduction

Transportation networks are examples of what are known as Discrete Event Systems (DES). The evolution of these systems is determined by the occurrence of certain events. In a transportion network, e.g. a railway network, examples of discrete events are the departure from or arrival at a station of a train. The evolution of a class of DES, viz. those which involve synchronization constraints, can be described by linear models provided that the max-algebra structure is used. In transportation networks such constraints follow from the demand that trains should connect. The max-algebra consists of the real numbers and minus infinity together with the operations maximization and addition. For an extensive discussion of the max-algebra and its applications in the modeling of DES we refer to [3].

Max-linear models for transportation networks have been studied by several authors. In [1] and [2] the modeling and analysis of transportation networks with a timetable has been treated. As an example the Dutch Intercity network was discussed. In [3] and [4] the effect of different routings on the performance of transportation networks without timetables is studied. In this paper we will summarize some of the previous results and we will extend the analysis of these networks. An extension to the model which was already suggested in [2], is the use of controllable connections. A train will only wait on a connecting train when the latter train has no or only a small delay. Otherwise, the connection is broken. In this way the propagation of a delay through the network can be controlled.

This paper is organized as follows. In section 2 we will give a short introduction to the max-algebraic concepts needed in this paper. In section 3 we show how max-linear models for railway networks can be obtained and discuss some aspects of the timetable according to which the network should evolve. In section 4 we will introduce a way to control such networks. The paper ends with some concluding remarks in section 5.

2 Max-algebra

In this section we will introduce the max-algebra. This structure was introduced in [5]. Also in [3] an extensive discussion of the max-algebra and similar structures can be found.

Let $\varepsilon = -\infty$ and denote by $\mathbb{R}_\varepsilon$ the set $\mathbb{R} \cup \{\varepsilon\}$. For elements $a, b \in \mathbb{R}_\varepsilon$ we define the operations $\oplus$ and $\otimes$ by

$$\begin{aligned} a \oplus b &= \max(a, b), \\ a \otimes b &= a + b. \end{aligned}$$

The structure $\mathbb{R}_\varepsilon$ together with the operations $\oplus$ and $\otimes$ will be called the max-algebra and will be denoted by $\mathbb{R}_{\max}$. We have that ε is the neutral element for the operation $\oplus$ and the absorbing element for $\otimes$. The neutral element for $\otimes$ is 0.

We can extend the max-algebra operations to matrices in the following way. If $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ then

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}, \qquad i = 1, \ldots, m; j = 1, \ldots, n.$$

If $A \in \mathbb{R}_\varepsilon^{m\times p}$ and $B \in \mathbb{R}_\varepsilon^{p\times n}$ then

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj}), \quad (1)$$

for $i = 1, \ldots, m$ and $j = i, \ldots, n$.

3 Modeling of transportation networks

In this section we will discuss the modeling of transportation networks. We are interested in the departure times of the trains from the stations. The modeling issues in this paper will be illustrated by a simple example network which is given in Figure 1. In this small network there is a train service from P via Q to S and vice versa and there is a service from Q to R and back. At station Q trains from P and S have to give connection to the train with destination R and vice versa. All the examples in this paper refer to this network.

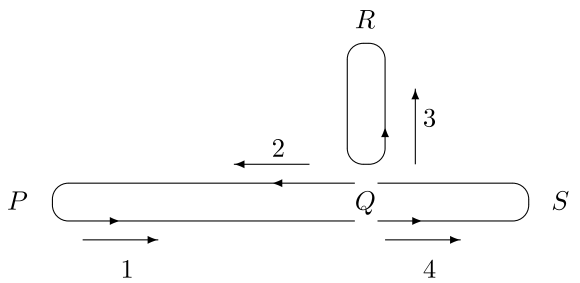

Figure 1: A simple network

In the following let $x_i(1)$ denote the departure time of the first train in direction i, $i = 1, \ldots, n$ where n is the number of different directions in the network. The train which is bound to leave in direction i for the $(k+1)$-st time (with $k = 1, 2, \ldots$) cannot leave before a number of conditions are satisfied. A first condition is that the train must have arrived at the station. Suppose that the train coming from direction j will continue in direction i. Then this gives rise to the following condition

$$x_i(k+1) \geq a_{ij} \otimes x_j(k), \qquad (2)$$

where $x_i(k+1)$ denotes the $(k+1)$-st departure time in direction i and where a_{ij} is the traveling time on the track from j to i to which the time needed for passengers to leave and board the train is added. The condition given by (2) is a strong one. The train cannot leave before it has arrived.

Another condition is the following. We (and certain passengers) would like the train to wait on possible connecting trains. This yields the conditions

$$x_i(k+1) \geq a_{il} \otimes x_l(k), \qquad (3)$$

where l ranges over the set of predecessors of i. Again, a_{il} denotes the transportation time from l to i, to which the time needed to leave and board the train and to change trains is added. The condition given by (3) can be seen as a weak condition. In contrast to condition (2), the train in direction i could have left without waiting on the other trains. Finally, a third condition follows from the timetable. We do not want the train to leave before its scheduled departure time. This leads to

$$x_i(k+1) \geq d_i(k+1), \qquad (4)$$

in which $d_i(k+1)$ denotes the scheduled departure time for the $(k+1)$-st train in direction i.

Next, we assume that a train leaves immediately as all conditions are satisfied. The departure time of the $(k+1)$-st train in direction i ($i = 1, \ldots, n$) is then given by, in max-algebra notation,

$$x_i(k+1) = a_{ij}(k) \otimes x_j(k) \oplus d_i(k+1) \oplus \bigoplus_l a_{il}(k) \otimes x_l(k). \qquad (5)$$

Let $x(k) = (x_1(k), \ldots, x_i(k), \ldots, x_n(k))^T$. Then the model which gives the departure times of the trains becomes, in max-algebraic matrix-vector notation,

$$x(k+1) = A_1(k) \otimes x(k) \oplus A_2(k) \otimes x(k) \oplus d(k+1). \qquad (6)$$

In (6) the matrix $A_1(k)$ represents the strong conditions given by (2) and $A_2(k)$ represents the weak conditions of (3). For these matrices we have that $a_{ij} = \varepsilon$ if the train from direction j does not give a connection to a train in direction i. With $A(k) = A_1(k) \oplus A_2(k)$, Equation (6) becomes

$$x(k+1) = A(k) \otimes x(k) \oplus d(k+1). \qquad (7)$$

In the following we will be using both the model description given by (5) and the models given by (6) or (7) depending on which is more convenient.

Example 1 Consider the network given in Figure 1. If we denote by $x_i(k)$ ($i = 1, 2, 3, 4$) the departure time of the k-th train ($k \geq 1$) in direction i and by $a_i(k)$ the traveling time on the corresponding track, we obtain the following model (still without timetable information)

$$\begin{aligned}
x_1(k+1) &= a_2(k) \otimes x_2(k) \\
x_2(k+1) &= a_3(k) \otimes x_3(k) \oplus a_4(k) \otimes x_4(k) \\
x_3(k+1) &= a_1(k) \otimes x_1(k) \oplus a_3(k) \otimes x_3(k) \\
&\quad \oplus a_4(k) \otimes x_4(k) \\
x_4(k+1) &= a_1(k) \otimes x_1(k) \oplus a_3(k) \otimes x_3(k).
\end{aligned}$$

We can write the model as $x(k+1) = A_1(k) \otimes x(k) \oplus A_2(k) \otimes x(k)$, or, with $A(k) = A_1(k) \oplus A_2(k)$, as $x(k+1) = A(k) \otimes x(k)$.

When the traveling times $a_i(k)$ ($i = 1, 2, 3, 4$) are deterministic and time-invariant, the behavior of the system $x(k+1) = A \otimes x(k)$ is determined by the max-algebraic eigenvalue of A (see [3] for definitions and the determination of the max-algebraic eigenvalue). When the traveling times are stochastic, the asymptotic behavior can be calculated as in [6] or [4]. Let, e.g. $a_1 = 14,\ a_2 = 17,\ a_3 = 11,\ a_4 = 9$. Then the system becomes $x(k+1) = A \otimes x(k)$ with

$$A = \begin{pmatrix} \varepsilon & 17 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 11 & 9 \\ 14 & \varepsilon & 11 & 9 \\ 14 & \varepsilon & 11 & \varepsilon \end{pmatrix}. \tag{8}$$

The eigenvalue of the matrix A is equal to 14. This means that for an appropriate choice of $x(1)$ every 14 time units a train can depart (see [3] for conditions and details). If we include a timetable in our model, it should be such that the time between two consecutive departures of trains in the same direction should differ by at least 14 time units since otherwise it will not be possible for the system to operate under this timetable. This problem has been treated in [2].

□

Next we will discuss the timetable information. Let $d(1)$ denote the vector which contains the departure times of the first train in each direction. If we want our schedule to be as regular as possible the timetable should satisfy

$$d(k+1) = \tau \otimes d(k), \qquad k = 1, 2, 3, \ldots, \tag{9}$$

where τ represents the period of the timetable. We already explained that the period τ should satisfy $\tau \geq \lambda$ where λ is the eigenvalue of the matrix A.

Example 2 The eigenvalue of the matrix A was equal to 14. So τ must at least be as large as this. Here we will choose $\tau = 15$, which means that according to the timetable every 15 time units a train will leave in each direction. Note that if the trains would run from A to B and back, from B to C and back and from B to D and back the eigenvalue of the resulting system would be $15\frac{1}{2}$. Then we could not have chosen $\tau = 15$. So, the choice of the routing also is of influence on the possible period of the timetable, see also [3] or [4].

□

After the choice of the period, we have to choose the initial value $d(1)$. This value should be such that when $x(1) = d(1)$, then also the following should hold (when there are no delays)

$$x(2) = A \otimes x(1) \oplus d(2) = A \otimes d(1) \oplus d(2) = d(2)$$

or in other words

$$A \otimes d(1) \leq d(2). \tag{10}$$

If condition (10) is satisfied then the timetable is feasible. In [2] it is called realistic. We note that when (10) is satisfied, we also have $A \otimes d(k) \leq d(k+1)$ $(k \geq 1)$ because of (9). When (9) does not hold, we have to require that $A \otimes d(k) \leq d(k+1)$ for all $k \geq 1$.

Example 3 Let $d(1) = (\ 0 \quad 0 \quad 0 \quad 0\ )^T$. Then $A \otimes d(1) = (\ 17 \quad 11 \quad 14 \quad 14\ )^T$, while $d(2) = 15 \otimes d(1) = (\ 15 \quad 15 \quad 15 \quad 15\ )^T$. Hence, condition (10) is not satisfied. With the given initial value, the second train in direction 1 cannot leave on time. When $d(1) = (\ 2 \quad 0 \quad 2 \quad 2\ )^T$ then in each direction the second train is able to leave on time.

□

From (9) and (10) we can derive a set of realistic timetables. Combination of these conditions leads to

$$A \otimes d(1) \leq \tau \otimes d(1). \tag{11}$$

One choice for $d(1)$ which always satisfies (11) is an eigenvector v corresponding to the eigenvalue λ of A. Then we have $A \otimes v = \lambda \otimes v \leq \tau \otimes v$ because of the condition $\tau \geq \lambda$. In general the eigenvector will not be the only choice which satisfies (11).

Example 4 For our example network, with A given by (8) and $\tau = 15$, it follows that the set of realistic timetables $d(1) = (\ d_1 \quad d_2 \quad d_3 \quad d_4\ )^T$ is given by

$$\begin{array}{lll} 2 \otimes d_2 \leq d_1 \leq 5 \otimes d_2 & & \\ 1 \otimes d_2 \leq d_3 \leq 4 \otimes d_2 & \text{and} & 1 \otimes d_3 \geq d_1 \\ 1 \otimes d_2 \leq d_4 \leq 6 \otimes d_2 & \text{and} & 1 \otimes d_4 \geq d_1 \end{array}$$

It is easily checked that the eigenvector $v = (\ 3 \quad 0 \quad 3 \quad 3\ )^T$ of A satisfies these conditions.

□

In order to analyze the model we will introduce the variable $z(k)$ defined as follows. Let $z_i(k)$ $(k = 1, 2, \ldots;\ i = 1, 2, \ldots, n)$ be the difference between the actual departure time of the k-th train in direction i and the scheduled departure time of this train, or $z_i(k) = x_i(k) - d_i(k)$. So, $z_i(k)$ gives the delay of the k-th train in direction i. Since a train will not leave before its scheduled departure time, see (4), it follows that we always have $z_i(k) \geq 0$.

The choice of the initial timetable $d(1)$ can be of influence on the propagation of delays.

Example 5 Let $d(1) = (\ 2 \quad 0 \quad 1 \quad 1\ )^T$ and assume that the first train in direction 2 has a delay of one time unit, hence $z_2(1) = 1$. This delay will propagate through the network. It follows that $z(2) = (\ 1 \quad 0 \quad 0 \quad 0\ )^T$ and $z(3) = (\ 0 \quad 0 \quad 1 \quad 1\ )^T$. Next, let $d(1) = (\ 3 \quad 0 \quad 4 \quad 5\ )^T$ and assume again that $z_2(1) = 1$. This delay will not propagate through the network. We have that $z_i(k) = 0$ for $k \geq 2$ and $i = 1, 2, 3, 4$. This means that the latter timetable is less sensitive for a delay of the trains in direction 2. Hence, if for some reason the probability of delays in direction 2 is larger than in direction 3, the latter initial values for the timetable should be preferred. It can be shown, however, that the former timetable is less sensitive to delays of trains in direction 3.

□

In general the sensitivity to delays is determined by the 'slack' time present in the network. This slack time is defined as the difference between the time instant the trains are ready to leave which follows from conditions (2) and (3), and the time instant the trains are allowed to leave given by the time table. Depending on the timetable, a delay can be 'absorbed' by this slack time.

4 Control of the network

In this section we will discuss some extensions of the model given by (5) and we will introduce a way to control it.

We do not want a train to wait too long on a connecting train. This can be modeled as follows. The train in direction i leaves when at time $d_i(k+1)+M_{il}$, with $M_{il} > 0$, a connecting train from direction l has not arrived yet. The quantity M_{il} can be interpreted as the maximal time the train in direction i has to wait for the train from direction l after its scheduled departure time. The model becomes

$$\begin{aligned} x_i(k+1) &= a_{ij}(k) \otimes x_j(k) \oplus d_i(k+1) \oplus \\ &\quad \bigoplus_l (a_{il}(k) \otimes x_l(k) \oplus' M_{il} \otimes d_i(k+1)), \end{aligned}$$

in which $a \oplus' b = \min(a, b)$, (see also [5]). A drawback of this approach is that the $(k+1)$-st train in direction i will always wait on the k-th train from direction l until $d_i(k+1) + M_{il}$, also when the delay of this train is larger than this quantity. So, it is possible that the train in direction i will wait unnecessarily on the train from direction l.

A solution to avoid unnecessary waiting times and also a way to minimize spreading of the delay through the network is to make the connections controllable. We will introduce a decision variable $u_{il}(k)$ which indicates whether the k-th train in direction i will wait for a connecting train from direction l. This decision can be based on several measures which will be discussed later. The introduction of $u_{il}(k)$ leads to the following model,

$$\begin{aligned} x_i(k+1) &= a_{ij}(k) \otimes x_j(k) \oplus d_i(k+1) \oplus \\ &\quad \bigoplus_l a_{il}(k) \otimes x_l(k) \otimes u_{il}(k), \end{aligned} \tag{12}$$

where $u_{il}(k)$ is given by

$$u_{il} = \begin{cases} 0 & \text{if } i \text{ will wait for } l \\ \varepsilon & \text{otherwise} \end{cases} \tag{13}$$

So, when $u_{il}(k) = \varepsilon$, the train in direction i will not wait for the train from direction l and the corresponding term vanishes from (12). We can only control the weak conditions given by (3). The connections given by (2) cannot be controlled. The train in direction i always has to wait for the train from direction j since it is the same train.

A possible criterion for deciding whether a train will wait on a delayed connecting train is the following. The train in direction i will only wait if the delay of the train from direction l is below a certain threshold value, or if

$$a_{il} \otimes x_l(k) \leq d_i(k+1) \otimes M_{il}. \tag{14}$$

The threshold value M_{il} can be different for different connections depending for instance on the number of passengers which will usually change over from the train from direction l to the train in direction i.

Condition (14) is one possibility for a choice of the controls. In general we would like to choose the values of u_{il} such that the delays of all trains are minimal while at the same time maintaining as many connections as possible. Minimizing the total delay of all trains in the network now means choosing the controls in such a way that

$$\sum_k \sum_j z_j(k) \tag{15}$$

is minimal. On the other hand we want to maintain as many connections as possible. Define $u'_{il}(k)$ as follows: $u'_{il}(k) = 1$ if $u_{il}(k) = 0$ and $u'_{il}(k) = 0$ if $u_{il}(k) = \varepsilon$. Then maximizing the number of connections means that

$$\sum_k \sum_{i,l} u'_{il}(k) \tag{16}$$

should be maximal. If some connections are more important to maintain than others, for instance because a lot of passengers use this connection, this can be modeled by introducing weight factors w_{il}. Then (16) becomes

$$\sum_k \sum_{i,l} w_{il} u'_{il}(k). \tag{17}$$

The larger the value of w_{il} is, the more important it is to maintain the corresponding connection.

If we combine the criteria given by (15) and (17) we obtain the following. We should choose the controls $u_{il}(k)$ such that J defined by

$$J = \frac{(\sum_k \sum_j z_j(k))^\alpha}{1 + \sum_k \sum_{i,l} w_{il} u'_{il}(k)}, \tag{18}$$

is minimal. In (18) $\alpha > 0$ is a weight factor indicating which of our objectives (minimizing the total delay or maximizing the number of connections) is the more important. We have also added 1 to (17) in order to avoid that the denominator becomes zero.

An alternative criterion is to choose the controls such that J' defined by

$$J' = \alpha \sum_k \sum_j z_j(k) - \sum_k \sum_{i,l} w_{il} u'_{il}(k) \qquad (19)$$

is minimal. Again $\alpha > 0$ indicates the relative importance of our criteria.

We have not indicated the ranges for the indices in the given criteria since usually we do not have to take all values into account. We will make more comments on this in the remainder of the paper.

Remark: Theorem 2.2.24 in [2] shows that when the timetable is realistic, i.e. it satisfies (10), then the delays will not increase. In the same reference it is shown that under certain conditions the delays eventually decrease and become zero after finite time. The delays decrease because of the slack time which may be present in the network. As a consequence (15) will remain finite.

We will now give a procedure to minimize J. We assume that the entries of the matrix $A(k)$ are deterministic and constant. Furthermore, let the only delay in the system be in the initial value $x(1)$. This is not a restriction. Assume the trains will leave on time until $k = k_0 - 1$ for some value k_0 and there is a delay in $x(k_0)$. Then we only have to consider the delays and controls in (15), (17) and (18) for $k > k_0$.

Starting point of the procedure is the situation when no control is applied. First we determine the set U of control variables which can be used to optimize J. These are the controls $u_{ij}(k)$ which correspond to delayed trains, so for which $a_{ij}(k) \otimes x_j(k) \otimes u_{ij}(k) > d_i(k+1)$ when $u_{ij}(k) = 0$. Only with these controls we can influence the total delay in the system. Next we compose different control strategies by setting one or more of the controls from U equal to ε. This means that the corresponding connections are broken. We apply these control strategies to the system and analyse the resulting behavior. From these results we determine the value of our objective function J. In this way we can consider all possible combinations and hence are able to find the control strategy which minimizes J. A problem however is that the number of possible control strategies grows combinatorially with the number of elements of U. When U consists of n elements there will be 2^n different control strategies.

An alternative procedure is the following. We start again from the situaton where all connections are maintained. First we will break the connection which gives the largest improvement of J. Then we break one of the other connections and see whether J still decreases etc. We stop when we cannot further improve J. In this way we only have to consider $O(n^2)$ possible control strategies. A drawback is that in general the result we find will be a suboptimal solution since we do not consider all possibilities.

Note that we can find a lower bound for the total delay in the system caused by an initial delay if we do not take the conditions given by (3) into account. So we only consider the system $x(k+1) = A_1(k) \otimes x(k) \oplus d(k+1)$. This is the system which remains when we set in (12) $u_{il}(k) = \varepsilon$ for all controls.

Example 6 With controls $u_{il}(k)$ the model for our example network becomes

$$\begin{aligned}
x_1(k+1) &= a_2(k) \otimes x_2(k) \oplus d_1(k+1) \\
x_2(k+1) &= a_3(k) \otimes x_3(k) \otimes u_{23}(k) \oplus a_4(k) \otimes x_4(k) \\
&\quad \oplus d_2(k+1) \\
x_3(k+1) &= a_1(k) \otimes x_1(k) \otimes u_{31}(k) \oplus a_3(k) \otimes x_3(k) \\
&\quad \oplus a_4(k) \otimes x_4(k) \otimes u_{34}(k) \oplus d_3(k+1) \\
x_4(k+1) &= a_1(k) \otimes x_1(k) \oplus a_3(k) \otimes x_3(k) \otimes u_{43}(k) \\
&\quad \oplus d_4(k+1).
\end{aligned}$$

Let $d(1) = \begin{pmatrix} 2 & 0 & 3 & 4 \end{pmatrix}^T$ and assume that the first train in direction 3 has a delay of six time units, hence $x_3(1) = 9$ and $z(1) = \begin{pmatrix} 0 & 0 & 6 & 0 \end{pmatrix}^T$. If we do not control the connections then $x(2) = \begin{pmatrix} 17 & 20 & 20 & 20 \end{pmatrix}^T$, and hence $z(2) = \begin{pmatrix} 0 & 5 & 2 & 1 \end{pmatrix}^T$. The delay will propagate through the network in the following way

$$z(3) = \begin{pmatrix} 5 & 1 & 0 & 0 \end{pmatrix}^T, \qquad z(4) = \begin{pmatrix} 1 & 0 & 3 & 2 \end{pmatrix}^T,$$
$$z(5) = \begin{pmatrix} 0 & 2 & 0 & 0 \end{pmatrix}^T, \qquad z(6) = \begin{pmatrix} 2 & 0 & 0 & 0 \end{pmatrix}^T.$$

The first value for k for which $z_i(k) = 0$ for all i, so where all trains will leave on time, is $k = 7$. The total value for (15) is equal to 24, where we did not include $z(1)$ because we cannot influence this value.

Next, we determine the set U. It consists of the following controls: $u_{23}(1)$, $u_{43}(1)$, $u_{23}(2)$, $u_{31}(3)$ and $u_{43}(4)$. This yields a total of 32 different control strategies we could apply. But it turns out that when $u_{23}(1) = \varepsilon$ we do not have to consider $u_{31}(3) = \varepsilon$ and/or $u_{23}(4) = \varepsilon$. Breaking these connections does not change the total delay since in this case the corresponding trains are on time. When $u_{31}(3) = \varepsilon$ we do not have to take $u_{23}(4) = \varepsilon$ into account for a similar reason. This leaves us with the following control strategies.

j	$u_{23}(1)$	$u_{43}(1)$	$u_{23}(2)$	$u_{31}(3)$	$u_{23}(4)$
1	ε	ε	ε	0	0
2	ε	ε	0	0	0
3	ε	0	ε	0	0
4	0	ε	ε	ε	0
5	0	ε	ε	0	ε
6	0	ε	ε	0	0
7	0	0	ε	ε	0
8	0	0	ε	0	ε
9	0	0	ε	0	0
10	0	ε	0	ε	0
11	0	ε	0	0	ε
12	0	ε	0	0	0
13	ε	0	0	0	0
14	0	0	0	ε	0
15	0	0	0	0	ε
16	0	0	0	0	0

Next, we study what happens if one of these strategies is applied. The results are summarized in the following table. The second column indicates how many of the five connections are maintained, see (16). In (16) we do not have to count the other connections, because they do not influence the total delay. The third column gives the total delay with the given control strategy. The last three columns give $J_i(j)$ $(i = 1, 2, 3)$, the value of J_i for control strategy j, where $J_1 = \frac{\sum z}{1+\sum u'}$, $J_2 = \frac{(\sum z)^{1/2}}{1+\sum u'}$ and $J_3 = \frac{\sum z}{1+\sum w_{ij} u'_{ij}}$ with $w_{43} = 2$ while $w_{ij} = 1$ for all other weights.

j	$\sum u'$	$\sum z$	$J_1(j)$	$J_2(j)$	$J_3(j)$
1	2	2	0.67	0.47	0.67
2	3	4	1.00	0.50	1.00
3	3	3	0.75	0.43	0.60
4	2	14	4.67	1.25	4.67
5	2	17	5.67	1.37	5.67
6	3	21	5.25	1.15	5.25
7	3	15	3.75	0.97	3.00
8	3	18	4.50	1.06	3.60
9	4	22	4.40	0.94	3.67
10	3	16	4.00	1.00	4.00
11	3	19	4.75	1.09	4.75
12	4	23	4.60	0.96	4.60
13	4	5	1.00	0.45	0.83
14	4	17	3.60	0.82	2.83
15	4	20	4.00	0.89	3.33
16	5	24	4.00	0.82	3.43

From this table we can conclude the following. With J_1 control strategy 1 is the optimal one. We note that in this case the total delay is equal to the minimal total delay obtained by setting all control variables in the system equal to ε. So strategy 1 is also the strategy which maximizes the number of connections while the total delay is minimal.

When we consider J_2 in which the maximization of the number of connections is more important than with J_1, we see that strategy 3 is the optimal one. The total delay with strategy 3 is larger but more connections are intact than with strategy 1. Note that also strategy 13, with only one broken connection, gives a better result than strategy 1.

With J_3, in which the connection of trains from direction 3 to direction 4 is more important than the other connections, again strategy 3 gives the best result. For strategy 3 $u_{43}(1) = 0$ whereas for strategy 1, the optimal strategy when all connections have the same weight, this connection is broken.

In this example we considered all possible control strategies and hence we were able to find the global minimum for J_i $(i = 1, 2, 3)$. If we apply the alternative procedure we described, then this yields the following. Suppose we want to minimize J_2. First we break one connection. So, we will consider strategies 9, 12, 13, 14 and 15. Setting $u_{23}(1) = \varepsilon$ (strategy 13) gives the largest improvement of J_2 compared with the situation where all connections are maintained (strategy 16). Next we break one of the remaining controls, i.e. we set $u_{43}(1) = \varepsilon$ or $u_{23}(2) = \varepsilon$. This coincides with strategy 2 respectively strategy 3. As already explained we do not have to consider $u_{31}(3) = \varepsilon$ or $u_{23}(4) = \varepsilon$ when $u_{23}(1) = \varepsilon$. Strategy 3 gives the largest reduction of J_2. Finally, we set $u_{23}(2) = \varepsilon$ (strategy 1). Now J_2 increases. So strategy 3 is a suboptimal solution of our problem. In this case it is also the optimal solution.

$\square$

5 Concluding remarks

We described the modeling of transportation networks with max-linear models and discussed conditions for the timetable. Furthermore, we have introduced a way to control the connections in the network. With the controls the propagation of a delay can be minimized. Further research includes the application to larger models and improvements of the optimization procedure with respect to its efficiency.

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