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THE EXTENDED LINEAR COMPLEMENTARITY PROBLEM AND LINEAR COMPLEMENTARY-SLACKNESS SYSTEMS

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Abstract

First we introduce the Extended Linear Complementarity Problem, which is a kind of mathematical programming problem. Next we show how this problem arises when we want to compute stationary points of linear complementary-slackness systems or when we want to determine the uniqueness of smooth continuations and the associated mode selection problem for linear complementary-slackness systems. We also briefly discuss the role of the Extended Linear Complementarity Problem in the analysis of other classes of hybrid systems such as discrete event systems and traffic signal controlled intersections.

1 The Extended Linear Complementarity Problem

1.1 The Linear Complementarity Problem

One of the possible formulations of the Linear Complementarity Problem (LCP) is the following [1]:

> Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, find $w,\, z \in \mathbb{R}^n$ such that
>
> $$\begin{aligned} w &= q + Mz & (1)\\ w, z &\geq 0 & (2)\\ z^T w &= 0\ . & (3) \end{aligned}$$

Note that if w and z are solutions of the LCP then it follows from (2) and (3) that $z_i w_i = 0$ for $i = 1, 2, \ldots, n$. So for each index $i \in \{1, 2, \ldots, n\}$ at least one of the following conditions should hold:

$$z_i = 0 \quad \text{and} \quad w_i \geq 0 \qquad (4)$$

or

$$z_i \geq 0 \quad \text{and} \quad w_i = 0\ . \qquad (5)$$

Hence, we have

$$w_i > 0 \ \Rightarrow\ z_i = 0 \qquad \text{and} \qquad z_i > 0 \ \Rightarrow\ w_i = 0$$

for $i = 1, 2, \ldots, n$, i.e., the zero patterns of w and z are complementary. Therefore, condition (3) is called the *complementarity condition* of the LCP.
The LCP has numerous applications such as quadratic programming problems, determination of the Nash equilibrium of a bimatrix game problem, the market equilibrium problem, the optimal invariant capital stock problem, the optimal stopping problem, etc. [1]. This makes the LCP one of the fundamental problems of mathematical programming. For more information on the LCP and its applications the interested reader is referred to [1, 8].

1.2 The Extended Linear Complementarity Problem

The Extended Linear Complementarity Problem (ELCP) is an extension of the LCP and is defined as follows [3, 4]:

> Given $P \in \mathbb{R}^{p \times n}$, $Q \in \mathbb{R}^{q \times n}$, $r \in \mathbb{R}^p$, $s \in \mathbb{R}^q$ and m subsets $\phi_1,\ \phi_2,\ \ldots,\ \phi_m$ of $\{1, 2, \ldots, p\}$, find $z \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pz - r)_i = 0 \qquad (6)$$
>
> subject to $Pz \geq r$ and $Qz = s$.

Condition (6) is called the complementarity condition of the ELCP. Since this condition is equivalent to

$$\forall j \in \{1, 2, \ldots, m\}\,, \exists i \in \phi_j \text{ such that } (Pz - r)_i = 0\ ,$$

the ELCP can be considered as a system of linear equations and inequalities ($Pz \geq r$, $Qz = s$), where there are m groups of linear inequalities (one group for each index set ϕ_j) such that in each group at least one inequality should hold with equality.

In [3, 4] we have developed an algorithm to compute the complete solution set of an ELCP. In [3, 4] we have also shown that the general ELCP with rational data is NP-hard.

2 The ELCP and complementary-slackness systems

2.1 Complementary-slackness systems

In general the behavior of a linear complementary-slackness system (CSS) can be described by a model of the following form [10]:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

subject to

$$y(t) \geq 0,\ u(t) \geq 0,\ y^T(t)\, u(t) = 0\ . \tag{7}$$

Typical examples of CSS are linear mechanical systems subject to geometric inequality constraints, or electrical networks consisting of linear resistors, capacitors, inductors, transformers, gyrators and ideal diodes.

Let m be the number of inputs and outputs of the CSS. Note that condition (7) implies that at each time instant t there exists an index set $I \subseteq \{1, 2, \ldots, m\}$ such that

$$y_i(t) = 0 \quad \text{for} \quad i \in I \tag{8}$$

$$u_i(t) = 0 \quad \text{for} \quad i \notin I\ . \tag{9}$$

Each index set $I \subseteq \{1, 2, \ldots, m\}$ corresponds to a *mode* of the system. So in principle there are 2^m different possible modes, but note that some of them may not be feasible (due to the other constraints on u and y).

Now assume that we have a CSS that is driven by a constant but yet unknown input u. In order to compute the stationary points of this CSS, we have to add the condition $\dot{x}(t) = 0$. If we eliminate the output y, this leads to:

$$\begin{aligned} Ax + Bu &= 0 \\ Cx + Du &\geq 0 \\ u &\geq 0 \\ (Cx + Du)^T u &= 0\ . \end{aligned}$$

It is easy to verify that this problem is an ELCP with

$$z = \begin{bmatrix} x \\ u \end{bmatrix},\quad P = \begin{bmatrix} C & D \\ O & I \end{bmatrix},\quad Q = \begin{bmatrix} A & B \end{bmatrix},$$

$r = 0$, $s = 0$ and $\phi_j = \{j, j+m\}$ for $j = 1, 2, \ldots, m$ where m is the number of inputs and outputs of the system.

It is easy to verify that if we replace (7) by more general conditions of the form $w_i(t) \geq 0$, $\tilde{w}_i(t) \geq 0$, $w_i(t)\, \tilde{w}_i(t) = 0$ for $i \in I$, where w_i and $\tilde{w}_i$ are components of u, y or x and where I is a set of indices, then we also get an ELCP if we want to compute the stationary points of the system when it is driven by a constant input.

2.2 The ELCP and the Linear Dynamic Complementarity Problem

Let us first introduce some definitions.

We say that a vector $a \in \mathbb{R}^n$ is lexicographically nonnegative, denoted by $a \succeq 0$, if either $a_i = 0$ for all i or the first nonzero component of a is positive. So we have $[1\ \ {-1}]^T \succeq 0$ and $[0\ \ 0\ \ 2]^T \succeq 0$ but $[0\ \ {-1}\ \ 2]^T \not\succeq 0$.

The sign decomposition a^+, a^- of a vector $a \in \mathbb{R}^n$ is defined as follows: $a^+, a^- \in \mathbb{R}^n$, $a = a^+ - a^-$ with

$$a^+ \geq 0,\ a^- \geq 0 \ \text{ and } \ (a^+)^T a^- = 0\ . \tag{10}$$

So if $a = [2\ \ {-3}\ \ 0\ \ {-4}]^T$ then we have $a^+ = [2\ \ 0\ \ 0\ \ 0]^T$ and $a^- = [0\ \ 3\ \ 0\ \ 4]^T$. Note the resemblance between the conditions (2) – (3), (7) and (10).

The Linear Dynamic Complementarity Problem (LDCP) is defined as follows [10, 12]:

> Given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$, find for a given $x_0 \in \mathbb{R}^n$ sequences $\{y_l\}_{l=0}^{n-1}$, $\{u_l\}_{l=0}^{n-1}$ with $y_l, u_l \in \mathbb{R}^k$ for all l such that
>
> $$y_0 = Cx_0 + Du_0 \tag{11}$$
>
> $$y_1 = CAx_0 + CBu_0 + Du_1 \tag{12}$$
>
> $$\vdots$$
>
> $$y_{n-1} = CA^{n-1}x_0 + CA^{n-2}Bu_0 + \ldots + CBu_{n-2} + Du_{n-1} \tag{13}$$
>
> and such that for each index $i \in \{1, 2, \ldots, k\}$ at least one of the following statements holds:
>
> $$\begin{bmatrix} (y_0)_i \\ (y_1)_i \\ \vdots \\ (y_{n-1})_i \end{bmatrix} = 0 \ \text{ and } \ \begin{bmatrix} (u_0)_i \\ (u_1)_i \\ \vdots \\ (u_{n-1})_i \end{bmatrix} \succeq 0 \tag{14}$$
>
> or
>
> $$\begin{bmatrix} (y_0)_i \\ (y_1)_i \\ \vdots \\ (y_{n-1})_i \end{bmatrix} \succeq 0 \ \text{ and } \ \begin{bmatrix} (u_0)_i \\ (u_1)_i \\ \vdots \\ (u_{n-1})_i \end{bmatrix} = 0\ . \tag{15}$$

Conditions (14) – (15) are called the complementarity conditions of the LDCP.

In [10, 12] the LDCP has been used to determine the existence and uniqueness of smooth continuations for linear CSS for a given initial point x_0, and to solve the associated mode selection problem (i.e., determining an index set I such that the conditions (8) and (9) hold). The mode selection problem will have a unique solution if and only if the LDCP (14) – (15) has a unique solution.

In [10] it has been shown that under fairly mild assumptions[1]

[1] I.e., if for some $j \in \mathbb{N}$ we have $D = CB = \cdots = CA^{j-1}B = O$ and the principal minors of CA^jB are positive.

the LDCP can be reduced to a series of LCPs. Let us now show that the LDCP is a special case of the ELCP.

Lemma 2.1 *Composing complementarity conditions of an ELCP by nested combinations of logical operators such as logical "and" ($\wedge$), logical "or" ($\vee$), negation, implication ($\Rightarrow$) or equivalence ($\Leftrightarrow$) results again in a complementarity condition of the form (6).*

Proof: See [6]. $\square$

Theorem 2.2 *The LDCP is a special case of the ELCP.*

Proof: Consider the LDCP defined by (11) – (15). If we define

$$y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}, \; u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}, \; q = \begin{bmatrix} Cx_0 \\ CAx_0 \\ \vdots \\ CA^{n-1}x_0 \end{bmatrix}$$

and

$$P = \begin{bmatrix} D & O & O & \dots & O \\ CB & D & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{n-2}B & CA^{n-3}B & CA^{n-4}B & \dots & D \end{bmatrix},$$

then we have $y = Pu + q$. Consider the sign decomposition of y and u:

$$\begin{gathered} y = y^+ - y^- \\ u = u^+ - u^- \\ y^+, y^-, u^+, u^- \geq 0 \\ (y^+)^T y^- = 0 \text{ and } (u^+)^T u^- = 0 \; . \end{gathered}$$

The last condition is equivalent to $(y^+)^T y^- + (u^+)^T u^- = 0$ since we have $(y^+)^T y^- \geq 0$ and $(u^+)^T u^- \geq 0$. So if we define

$$x = \begin{bmatrix} y^+ \\ u^+ \\ y^- \\ u^- \end{bmatrix} \text{ and } Z = \begin{bmatrix} I & -P & -I & P \end{bmatrix},$$

then the system (11) – (13) can be rewritten as

$$Zx = q \tag{16}$$

$$x \geqslant 0 \tag{17}$$

$$\sum_{i=1}^{2nk} x_i x_{i+2nk} = 0 \quad , \tag{18}$$

which is (a special case of) an ELCP.
Now we show that combining condition (18) with the complementarity conditions (14) – (15) of the LDCP results in a complementarity condition of the form (6). We shall do this for the case with $n = 2$ and $k = 1$. The general case can be dealt with in a similar way.
Consider condition (14) for $i = 1$. Now we show that the condition

$$\begin{bmatrix} (y_0)_1 \\ (y_1)_1 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} (u_0)_1 \\ (u_1)_1 \end{bmatrix} \succeq 0 \tag{19}$$

leads to a logical combination of a number of complementarity conditions. Clearly, the first part of this condition is equivalent to $(y_0^+)_1 = (y_0^-)_1 = (y_1^+)_1 = (y_1^-)_1 = 0$. Since $(y_0^+)_1, (y_0^-)_1, (y_1^+)_1, (y_1^-)_1 \geq 0$, this condition is equivalent to

$$(y_0^+)_1 + (y_0^-)_1 + (y_1^+)_1 + (y_1^-)_1 = 0 \; . \tag{20}$$

The second part of condition (19) is equivalent to

$$((u_0)_1 \geq 0) \wedge \Big(((u_0)_1 = 0) \Rightarrow ((u_1)_1 \geq 0) \Big) \; ,$$

which can be rewritten as

$$((u_0^-)_1 = 0) \wedge \\ \Big(\big(((u_0^+)_1 = 0) \wedge ((u_0^-)_1 = 0) \big) \Rightarrow ((u_1^-)_1 = 0) \Big) \; .$$

Since this condition is a logical combination of elementary complementarity conditions, it can be rewritten as one complementarity condition of the form (6) by Proposition 2.1. In a similar way we can show that condition (15) also leads to one complementarity condition of the form (6). If we now take the logical "or" of the complementarity conditions that correspond to (14) and (15) then by Proposition 2.1 we get again one complementarity condition of the form (6).
This implies that the complementarity conditions of the LDCP together with condition (18) lead to one large complementarity condition of the form (6). If we combine this condition with (16) and (17), we finally get an ELCP. Any solution of this ELCP will — after extraction of the vectors $y_0, y_1, \dots, y_{n-1}, u_0, u_1, \dots, u_{n-1}$ — yield a solution of the LDCP. So we can say that the LDCP is a special case of the ELCP. $\square$

3 A worked example

Let

$$A = \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -2 & -3 \\ 4 & -1 \end{bmatrix}, \; C = \begin{bmatrix} -2 & -2 \\ -1 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} -2 & -1 \\ 5 & -3 \end{bmatrix}, \; x_0 = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

and consider the following LDCP:

> Find $y_0, y_1, u_0, u_1 \in \mathbb{R}^2$ such that
>
> $$\begin{aligned} y_0 &= Cx_0 + Du_0 \\ y_1 &= CAx_0 + CBu_0 + Du_1 \end{aligned}$$
>
> and such that at least one of the following statements is true for each index $i \in \{1, 2\}$:
>
> $$\begin{bmatrix} (y_0)_i \\ (y_1)_i \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} (u_0)_i \\ (u_1)_i \end{bmatrix} \succeq 0$$

or

$$\begin{bmatrix} (y_0)_i \\ (y_1)_i \end{bmatrix} \succeq 0 \quad \text{and} \quad \begin{bmatrix} (u_0)_i \\ (u_1)_i \end{bmatrix} = 0 \ .$$

Using the reasoning given in the proof of Theorem 2.2 and in the proof of Lemma 2.1 (see [6]) this LDCP can be transformed into the ELCP of Table 1 with

$$x = \begin{bmatrix} (y_0^+)_1 \\ (y_0^+)_2 \\ (y_1^+)_1 \\ (y_1^+)_2 \\ (u_0^+)_1 \\ (u_0^+)_2 \\ (u_1^+)_1 \\ (u_1^+)_2 \\ (y_0^-)_1 \\ (y_0^-)_2 \\ (y_1^-)_1 \\ (y_1^-)_2 \\ (u_0^-)_1 \\ (u_0^-)_2 \\ (u_1^-)_1 \\ (u_1^-)_2 \end{bmatrix}$$

and where $\tilde{x}$ is a vector of "mirror variables": $\tilde{x}_i$ is equal to 0 if and only if x_i is different from 0. If we solve the ELCP of Table 1 using the algorithm of [3, 4] we obtain a solution set that consists of two solutions:

$$y_0 = \begin{bmatrix} 4 \\ 11 \end{bmatrix}, \ y_1 = \begin{bmatrix} 20 \\ 49 \end{bmatrix}, \ u_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ ,$$

and

$$\tilde{y}_0 = \begin{bmatrix} 0 \\ 21 \end{bmatrix}, \ \tilde{y}_1 = \begin{bmatrix} 0 \\ 99 \end{bmatrix}, \ \tilde{u}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \ \tilde{u}_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \ .$$

Since the LDCP considered here has more than one solution, the associated mode selection problem for the given initial point x_0 will not have a unique solution either. For more information on the connection between the mode selection problem for CSS and the LDCP and for some worked examples the interested reader is referred to [9, 10, 11, 13].

4 The ELCP and other hybrid systems

Hybrid systems arise from the interaction between discrete event systems (i.e., asynchronous systems in which the state transitions are initiated by events) and continuous variable systems (i.e., systems that can be modeled using difference or differential equations). In general we could say that a hybrid system can be in one of several modes whereby in each mode the behavior of the system can be described by a system of difference or differential equations, and that the system switches from one mode to another due to the occurrence of events. For a CSS a mode corresponds to a particular choice of the index set I such that the conditions (8) and (9) hold.

In [3, 5] we have shown that the ELCP can be used to solve many problems that arise in the system theory for max-linear time-invariant discrete event systems, i.e., discrete event systems that can be described by a time-invariant model that is linear in the max-plus algebra [2], which has maximization and addition as its basic operations. In [14] it has been shown that the Generalized LCP (which is also a special case of the ELCP) plays a role in the modeling and analysis of piecewise-linear resistive electrical circuits. In [7] we have used the ELCP in a model that describes the evolution of the queue lengths at a traffic signal controlled intersection.

Since all these systems can be considered as special cases of hybrid systems, this seems to indicate that the ELCP will play an important role in many analysis problems for hybrid systems.

5 Conclusions and topics for further research

In this paper we have discussed how the Extended Linear Complementarity Problem (ELCP) can be used to compute stationary points of linear complementary-slackness systems. We have also shown that the Linear Dynamic Complementarity Problem, which can be used to determine uniqueness of smooth continuations and the associated mode selection problem for linear complementary-slackness systems, is a special case of the ELCP. We have also given some other classes of hybrid systems that can be analyzed using an ELCP. As a consequence, ELCP can be considered as a general framework for the analysis of many classes of hybrid systems.

It is obvious that each class of hybrid systems that can be analyzed using the ELCP will lead to a special case of the ELCP that is especially suited for the class of systems under consideration. The computational complexity of these ELCPs and the development of efficient algorithms to solve them or to determine whether their solution set is non-empty or whether their solution is a unique solution are still open problems. One possible approach to tackle the complexity problem is to use approximations and/or to develop procedures to efficiently obtain suboptimal solutions (see, e.g., [7]).

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Find $x, \tilde{x} \in \mathbb{R}^{16}$ such that

$$
\begin{aligned}
x, \tilde{x} &\geq 0 \\
x_i + \tilde{x}_i &\geq 1 \quad \text{for } i = 1, 2, \ldots, 16
\end{aligned}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & -5 & 3 & 0 & 0 & 0 & -1 & 0 & 0 & 5 & -3 & 0 & 0 \\
0 & 0 & 1 & 0 & 4 & -8 & 2 & 1 & 0 & 0 & -1 & 0 & -4 & 8 & -2 & -1 \\
0 & 0 & 0 & 1 & -10 & -1 & -5 & 3 & 0 & 0 & 0 & -1 & 10 & 1 & 5 & -3
\end{bmatrix} x = \begin{bmatrix} 4 \\ 11 \\ 20 \\ 49 \end{bmatrix}
$$

$$
\begin{aligned}
& x_1 x_5 + x_1 x_7 + x_1 x_{15} + x_1 \tilde{x}_1 + x_2 x_6 + x_2 x_8 + x_2 x_{16} + x_2 \tilde{x}_2 + x_3 x_5 + x_3 x_7 + x_3 x_{11} + x_3 x_{15} + x_3 \tilde{x}_1 \tilde{x}_9 + \\
& x_3 \tilde{x}_3 + x_4 x_6 + x_4 x_8 + x_4 x_{12} + x_4 x_{16} + x_4 \tilde{x}_2 \tilde{x}_{10} + x_4 \tilde{x}_4 + x_5 x_{11} + x_5 \tilde{x}_5 + x_6 x_{12} + x_6 \tilde{x}_6 + x_7 x_{11} + \\
& x_7 x_{15} + x_7 \tilde{x}_7 + x_8 x_{12} + x_8 x_{16} + x_8 \tilde{x}_8 + x_9 + x_{10} + x_{11} x_{15} + x_{11} \tilde{x}_1 \tilde{x}_9 + x_{11} \tilde{x}_{11} + x_{12} x_{16} + \\
& x_{12} \tilde{x}_2 \tilde{x}_{10} + x_{12} \tilde{x}_{12} + x_{13} + x_{14} + x_{15} \tilde{x}_5 \tilde{x}_{13} + x_{15} \tilde{x}_{15} + x_{16} \tilde{x}_6 \tilde{x}_{14} + x_{16} \tilde{x}_{16} = 0 \ ,
\end{aligned}
$$

Table 1: The ELCP of the example of Section 3.

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