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The Extended Linear Complementarity Problem and the Modeling and Analysis of Hybrid Systems

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Abstract. First we give a short description of the Extended Linear Complementarity Problem (ELCP), which is a mathematical programming problem. We briefly discuss how this problem can be used in the analysis of discrete event systems and continuous variable systems. Next we show that the ELCP can also be used to model and to analyze hybrid systems. More specifically, we consider a traffic-light-controlled intersection, which can be considered as a hybrid system. We construct a model that describes the evolution of the queue lengths in the various lanes (as continuous variables) as a function of time and we show that this leads to an ELCP. Furthermore, it can be shown that some problems in the analysis of another class of hybrid systems, the "complementary-slackness systems", also lead to an ELCP.

1 Introduction

The main purpose of this paper is to show that the Extended Linear Complementarity Problem (ELCP) — which is a kind of mathematical programming problem — can be used to model and to analyze certain classes of hybrid systems. The formulation of the ELCP arose from our research on discrete event systems. Furthermore, the ELCP can also be used to analyze some classes of continuous variable systems (i.e., systems that can be modeled using difference or differential equations). Since hybrid systems can be considered as a merge of discrete event systems and continuous variable systems, this leads to the question as to whether the ELCP can also be used in the analysis of hybrid systems. We show that this is indeed the case. More specifically, we consider a traffic-light-controlled intersection — which can be considered as a simple hybrid system. We show that the evolution of the queue lengths at a traffic-light-controlled intersection can be described by an ELCP. Furthermore, the ELCP can also be used to model another class of hybrid systems, the so-called "complementary-slackness systems".

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This paper is organized as follows. In Section 2 we introduce the Extended Linear Complementarity Problem. In Section 3 we briefly discuss how the ELCP can be used to model and to analyze certain classes of discrete event systems, continuous variable systems and hybrid systems. In Section 4 we consider a traffic-light-controlled intersection and we show how the evolution of the queue lengths in this system can be described by an ELCP. Finally, we present some conclusions and directions for future research in Section 5.

2 The Extended Linear Complementarity Problem

The Extended Linear Complementarity Problem (ELCP) is an extension of the Linear Complementarity Problem, which is one of the fundamental problems in mathematical programming [3]. The ELCP is defined as follows:

> Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets ϕ_1, ϕ_2, ..., ϕ_m of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{1}$$
>
> subject to $Ax \geq c$ and $Bx = d$, or show that no such x exists.

Equation (1) represents the *complementarity condition* of the ELCP. One possible interpretation of this condition is the following: since $Ax \geq c$, (1) is equivalent to

$$\forall j \in \{1, 2, \ldots, m\} \, : \, \prod_{i \in \phi_j} (Ax - c)_i = 0 \ .$$

So we could say that each set ϕ_j corresponds to a group of inequalities of $Ax \geq c$ and that in each group at least one inequality should hold with equality, i.e., the corresponding residue should be equal to 0:

$$\forall j \in \{1, 2, \ldots, m\} \, : \, \exists i \in \phi_j \text{ such that } (Ax - c)_i = 0 \ .$$

In general, the solution set of the ELCP defined above consists of the union of faces of the polyhedron defined by the system of linear equations and inequalities ($Ax \geq c$ and $Bx = d$) of the ELCP. In [6, 7] we have developed an algorithm to compute the complete solution set of an ELCP. This algorithm yields a description of the solution set of an ELCP by vertices, extreme rays and a basis of the linear subspace corresponding to the largest affine subspace of the solution set. In that way it provides a geometrical insight in the entire solution set of the ELCP and related problems.

We shall now give a brief description of the ELCP algorithm of [6, 7]. The algorithm consists of two parts:

- First we determine the set of (finite) vertices $\mathcal{X}^{\text{fin}}$, the set of extreme rays $\mathcal{X}^{\text{ext}}$ and a basis $\mathcal{X}^{\text{cen}}$ of the linear subspace corresponding to the largest affine subspace of the solution set of the ELCP. This is done by iteratively solving the system $Ax \geq c$, $Bx = d$, whereby in the kth step ($k = 1, 2, \ldots, p+q$) we compute the intersection of the current solution set with the half-space or hyperplane determined by the kth inequality or equality. We also remove solutions that do not satisfy the complementarity condition.

– Next we determine the set Λ of maximal *cross-complementary* pairs of subsets of $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$. A pair $(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}})$ is cross-complementary if the sum of any nonnegative combination of the elements of $\mathcal{X}_s^{\text{ext}}$ and any convex combination of the elements of $\mathcal{X}_s^{\text{fin}}$ satisfies the complementarity condition.
The set Λ is determined using a kind of backtracking algorithm: we start with a pair of the form $(\emptyset, \{x_k^{\text{f}}\})$ with $x_k^{\text{f}} \in \mathcal{X}^{\text{fin}}$ and then we keep on adding new elements of $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ to the current pair in a systematic way until we obtain a pair that is not cross-complementary any more[1]. In that case we do a backtracking step. This continues until we have obtained all maximal cross-complementary pairs.

Now any solution x of the ELCP can be written as

$$x = \sum_{x_k^{\text{c}} \in \mathcal{X}^{\text{cen}}} \lambda_k x_k^{\text{c}} + \sum_{x_k^{\text{e}} \in \mathcal{X}_s^{\text{ext}}} \kappa_k x_k^{\text{e}} + \sum_{x_k^{\text{f}} \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k^{\text{f}} \tag{2}$$

for some pair $(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}) \in \Lambda$ with $\lambda_k \in \mathbb{R}$, $\kappa_k \geq 0$, $\mu_k \geq 0$ and $\sum_k \mu_k = 1$. For more information on the ELCP algorithm and for a worked example the interested reader is referred to [6].

In [6, 7] we have also shown that the general ELCP with rational data is an NP-hard problem.

3 The ELCP and discrete event systems and continuous variable systems

In this section we briefly discuss how the ELCP can be used in the analysis of certain classes of discrete event systems (such as max-linear discrete event systems) and of certain classes of continuous variable systems (such as, e.g., piecewise-linear resistive electrical circuits).

3.1 The ELCP and max-linear time-invariant discrete event systems

Typical examples of discrete event systems (DESs) are flexible manufacturing systems, subway traffic networks, parallel processing systems, telecommunication

[1] It can be shown that it is sufficient to test only one combination of the elements of $\mathcal{X}_s^{\text{ext}}$ and $\mathcal{X}_s^{\text{fin}}$ to determine whether the pair $(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}})$ is cross-complementary or not.

networks and logistic systems. The class of the DESs essentially contains man-made systems that consist of a finite number of resources (e.g., machines, communications channels, or processors) that are shared by several users (e.g., product types, information packets, or jobs) all of which contribute to the achievement of some common goal (e.g., the assembly of products, the end-to-end transmission of a set of information packets, or a parallel computation).

One of the most characteristic features of a DES is that its dynamics are event-driven as opposed to time-driven: the behavior of a DES is governed by events rather than by ticks of a clock. An event corresponds to the start or the end of an activity. If we consider a production system then possible events are: the completion of a part on a machine, a machine breakdown, or a buffer becoming empty.

In general, the description of the behavior of a DES leads to a model that is nonlinear in conventional algebra. However, there exists a class of DESs for which the model is "linear" when we express it in the max-plus algebra [1, 2, 4], which has maximization and addition as basic operations. DESs that can be described by such a "linear" model are called *max-linear DESs*. Loosely speaking we could say that the class of max-linear DESs corresponds to the class of deterministic time-invariant DESs in which only synchronization and no concurrency occurs.

The basic operations of the max-plus algebra are maximization (represented by $\oplus$) and addition (represented by $\otimes$). There exists a remarkable analogy between the basic operations of the max-plus algebra on the one hand, and the basic operations of conventional algebra (addition and multiplication) on the other hand. As a consequence many concepts and properties of conventional algebra (such as Cramer's rule, eigenvectors and eigenvalues, the Cayley-Hamilton theorem, ...) also have a max-plus-algebraic analogue (see, e.g., [1]). Furthermore, this analogy also allows us to translate many concepts, properties and techniques from conventional linear system theory to system theory for max-linear time-invariant DESs. However, there are also some major differences that prevent a straightforward translation of properties, concepts and algorithms from conventional linear algebra and linear system theory to max-plus algebra and max-plus-algebraic system theory for DESs.

If we write down a model for a max-linear DES and if we use the symbols $\oplus$ and $\otimes$ to denote maximization and addition[2] we obtain a description of the following form:

$$x(k+1) = A \otimes x(k) \;\oplus\; B \otimes u(k) \tag{3}$$

$$y(k) = C \otimes x(k) \;\;, \tag{4}$$

where x is the state vector, u the input vector and y the output vector. For a manufacturing system $u(k)$ would typically represent the time instants at which

[2] For matrices A and B these operations are defined by $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ and $(A \otimes B)_{ij} = \bigoplus_k a_{ik} \otimes b_{kj}$. Note that these definitions closely resemble the definitions of matrix sum and matrix product of conventional algebra but with $+$ replaced by $\oplus$ and $\times$ replaced by $\otimes$.

raw material is fed to the system for the $(k+1)$st time; $x(k)$ the time instants at which the machines start processing the kth batch of intermediate products; and $y(k)$ the time instants at which the kth batch of finished products leaves the system. In analogy with the state space model for linear time-invariant discrete-time systems, a model of the form (3) – (4) is called a *max-linear time-invariant state space model.*

Let $x, r \in \mathbb{R}$. The rth max-plus-algebraic power of x is denoted by $x^{\otimes^r}$ and corresponds to rx in conventional algebra.

Now consider the following problem:

> Given $p_1 + p_2$ positive integers $m_1, \ldots, m_{p_1+p_2}$ and real numbers a_{ki}, b_k and c_{kij} for $k = 1, \ldots, p_1 + p_2$, $i = 1, \ldots, m_k$ and $j = 1, \ldots, n$, find $x \in \mathbb{R}^n$ such that
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1, \ldots, p_1 \;, \tag{5}$$
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} \leq b_k \qquad \text{for } k = p_1 + 1, \ldots, p_1 + p_2 \;. \tag{6}$$

We call (5) – (6) a *system of multivariate max-plus-algebraic polynomial equalities and inequalities.* Note that the exponents may be negative or real.

In [6, 10] we have shown that the problem of solving a system of multivariate max-plus-algebraic polynomial equalities and inequalities can be recast as an ELCP. This enables us to solve many important problems that arise in the max-plus algebra and in the system theory for max-linear DESs such as: computing max-plus-algebraic matrix factorizations, performing max-plus-algebraic state space transformations, computing state space realizations of the impulse response of a max-linear time-invariant DES, constructing matrices with a given max-plus-algebraic characteristic polynomial, computing max-plus-algebraic singular value decompositions, computing max-plus-algebraic QR decompositions, and so on [6–10].

Although the analogues of these problems in conventional algebra and linear system theory are easy to solve, the max-plus-algebraic problems are not that easy to solve and for almost all of them the ELCP approach is at present the only way to solve the problem.

For more information on the max-plus-algebra and on max-plus-algebraic system theory for discrete event systems the interested reader is referred to [1, 2, 4, 6, 15] and the references given therein.

3.2 The ELCP and piecewise-linear resistive electrical circuits

In this section we consider electrical circuits that may contain the following elements: linear resistive elements, piecewise-linear (PWL) resistors (the resistors are not required to be either voltage or current controlled), and PWL controlled sources (all four types) with one controlling variable (the characteristics may

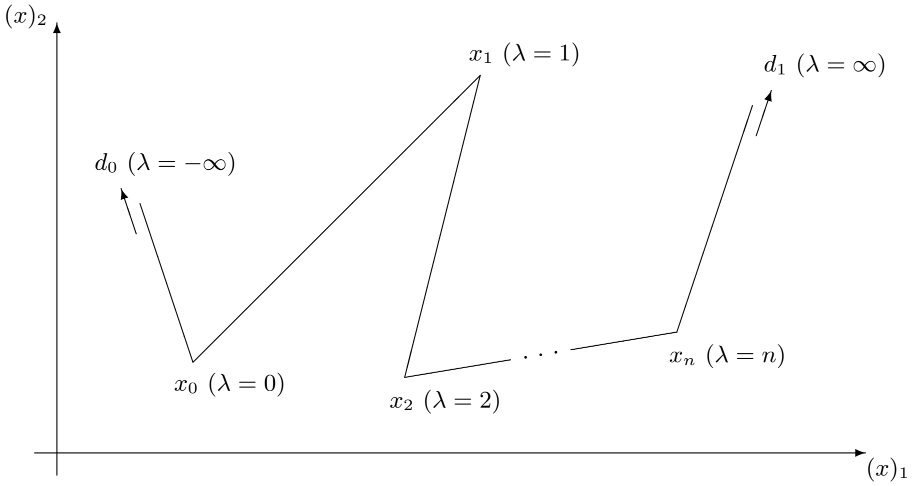

Fig. 1. A one-dimensional PWL curve in $\mathbb{R}^2$ characterized by $n + 1$ breakpoints x_0, $\ldots$, x_n and two directions d_0 and d_1. The points on this curve can be parameterized by (7) where λ is a real continuous parameter.

be multi-valued). These electrical circuits can be considered as examples of continuous variable systems (i.e., systems that can be modeled using difference or differential equations). In this section we shall show that by using an intelligent parameterization of the PWL characteristics the equations that describe the relations between the voltages and currents in these electrical circuits can be reformulated as (a special case of) an ELCP. For sake of simplicity we consider only two-terminal resistors since they can be described by a one-dimensional PWL manifold[3].

If x is a vector, then we define $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$, where the operations are performed componentwise. An equivalent definition is:

$$x = x^+ - x^- , \qquad x^+, x^- \geq 0 , \qquad (x^+)^T x^- = 0 \ .$$

It is easy to verify that a one-dimensional PWL curve in $\mathbb{R}^2$ characterized by $n+1$ breakpoints x_0, $\ldots$, x_n and two directions d_0 and d_1 (see Figure 1) can be parameterized as follows [5, 23]:

$$\begin{aligned} x \ = \ & x_0 + d_0\lambda^- + (x_1 - x_0)\lambda^+ + \sum_{k=2}^{n} (x_k - 2x_{k-1} + x_{k-2})(\lambda - k + 1)^+ + \\ & (d_1 - x_n + x_{n-1})(\lambda - n)^+ \ , \end{aligned} \qquad (7)$$

[3] If we allow multi-terminal nonlinear resistors, which can be modeled by higher-dimensional PWL manifolds, we shall also obtain an ELCP (See [5]).

with $\lambda \in \mathbb{R}$. Introducing auxiliary variables $\lambda_i = \lambda - i$ yields a description of the following form:

$$\begin{aligned} x &= x_0 + Ay^- + By^+ \\ C(y^+ - y^-) &= d \\ y^+, y^- &\geq 0 \\ (y^+)^T y^- &= 0 \end{aligned}$$

where

$$\begin{aligned} y^- &= \begin{bmatrix} \lambda^- & \lambda_1^- & \dots & \lambda_n^- \end{bmatrix}^T \\ y^+ &= \begin{bmatrix} \lambda^+ & \lambda_1^+ & \dots & \lambda_n^+ \end{bmatrix}^T \\ A^- &= [\, d_0 \;\; 0 \;\; \dots \;\; 0 \,] \\ A^+ &= [\, 0 \;\; x_1 - x_0 \;\; x_2 - 2x_1 + x_0 \;\; \dots \;\; d_1 - x_n + x_{n-1} \,] \\ B &= \begin{bmatrix} 1 & -1 & 0 \dots & 0 \\ 1 & 0 & -1 \dots & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 1 & 0 & 0 \dots & -1 \end{bmatrix} \\ c &= [\, 1 \;\; 2 \;\; \dots \;\; n \,]^T \quad . \end{aligned}$$

If we extract all nonlinear resistors out of the electrical circuit, the resulting N-port contains only linear resistive elements and independent sources. As a consequence, the relation between the branch currents and voltages of this N-port is described by a system of linear equations. If we combine these equations with the PWL descriptions (7) of the nonlinear resistors, we finally get a system of the form:

$$Mw^+ + Nw^- = q\,, \qquad w^+, w^- \geq 0\,, \qquad (w^+)^T (w^-) = 0 \;\;, \tag{8}$$

where the vector w contains the parameters λ and λ_i of the PWL descriptions of all the nonlinear resistors. It is easy to verify that (8) can be considered as (a special case of) an ELCP. If we solve (8), we get the complete set of operating points of the electrical circuit.

In a similar way we can determine the driving-point characteristic (i.e., the relation between the input current and the input voltage) and the transfer characteristics of the electrical circuit [23].

In general, the behavior of an electrical network consisting of linear resistors, capacitors, inductors, transformers, gyrators and ideal diodes can be described by a model of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

subject to the conditions

$$y(t) \geq 0, \quad u(t) \geq 0, \quad (y(t))^T u(t) = 0 \tag{9}$$

(see, e.g., [19]). In order to compute the *stationary points* of such an electrical circuit, we add the condition $\dot{x}(t) = 0$, which leads to a Linear Complementarity Problem [19]. If we replace (9) by more general conditions of the form $w_i \geq 0$, $z_i \geq 0$, $w_i z_i = 0$, where w_i and z_i are components of u, y or x, then we get (a special case of) an ELCP.

3.3 The ELCP and hybrid systems

In Sections 3.1 and 3.2 we have shown that the ELCP arises in the analysis of certain classes of discrete event systems and continuous variable systems. Since hybrid systems arise from the interaction between discrete event systems and continuous variable systems, and since they exhibit characteristics of both discrete event systems and continuous variable systems, this leads to the question as to whether the ELCP can also play a role in the modeling and analysis of certain classes of hybrid systems. In the next section we shall show that this is indeed the case: we study a traffic-light-controlled intersection, which can be considered as a simple hybrid system. The evolution of the queue lengths in this hybrid system can be described by an ELCP.

Furthermore, in [19, 21, 22] Schumacher and van der Schaft consider another class of hybrid systems — the "complementary-slackness systems" — typical examples of which are electrical networks with diodes, or mechanical systems subject to geometric inequality constraints. They develop a method to determine the uniqueness of smooth continuations and to solve the associated mode selection problem for complementary-slackness systems. When the underlying system is a linear system, then this leads to a Linear Dynamic Complementarity Problem which can also be considered as a special case of the ELCP [11].

Hence, the ELCP can indeed be used in the analysis of certain classes of hybrid systems.

4 Traffic-light-controlled intersections

4.1 The set-up and the model of the system

Consider a single intersection of two two-way streets with controllable traffic lights on each corner (see Figure 2). For sake of brevity and simplicity we make the following assumptions:

- the traffic lights can either be red or green,
- the average arrival and departure rates of the cars are constant or slowly time-varying,
- the queue lengths are continuous variables.

These assumptions deserve a few remarks:

- Adding an all-red or amber phase leads to a similar, but more complex model (see [12]).
- If we keep in mind that one of the main purposes of the model that we shall derive, is the design of optimal traffic light switching schemes, then assuming that the average arrival and departure rates are constant is not a serious restriction, provided that we use a moving horizon strategy: we compute the optimal traffic light switching scheme for, say, the next 10 cycles, based on a prediction of the average arrival and departure rates (using data measured during the previous cycles) and we apply this scheme during the first of the 10 cycles, meanwhile we update our estimates of the arrival and departure rates and compute a new optimal scheme for the next 10 cycles, and so on.
- Designing optimal traffic light switching schemes is only useful if the arrival and departure rates of vehicles at the intersection are high. In that case, approximating the queue lengths by continuous variables only introduces small errors. Furthermore, in practice there is also some uncertainty and variation in time of the arrival and departure rates, which makes that in general computing the exact optimal traffic light switching scheme is utopian. Moreover, in practice we are more interested in quickly obtaining a good approximation of the optimal traffic light switching scheme than in spending a large amount of time to obtain the exact optimal switching scheme.

Let us now continue with the description of the set-up of the system. There are four lanes L_1, L_2, L_3 and L_4, and on each corner of the intersection there are traffic lights (T_1, T_2, T_3 and T_4). The average arrival rate of cars in lane L_i is λ_i. When the traffic light is green, the average departure rate in lane L_i is μ_i. Let t_0, t_1, t_2, t_3, $\ldots$ be the time instants at which the traffic lights switch from green to red or vice versa. The traffic light switching scheme is shown in Table 1. Define $\delta_k = t_{k+1} - t_k$. Let $l_i(t)$ be the queue length (i.e., the number of cars waiting) in lane L_i at time instant t.

Let us now write down the equations that describe the relation between the switching time instants and the queue lengths as continuous variables.

Consider lane L_1. When the traffic light T_1 is red, there are arrivals at lane L_1 and no departures. As a consequence, we have

$$\frac{dl_1(t)}{dt} = \lambda_1 \tag{10}$$

Period	T_1	T_2	T_3	T_4
$t_0 - t_1$	red	green	red	green
$t_1 - t_2$	green	red	green	red
$t_2 - t_3$	red	green	red	green
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1. The traffic light switching scheme.

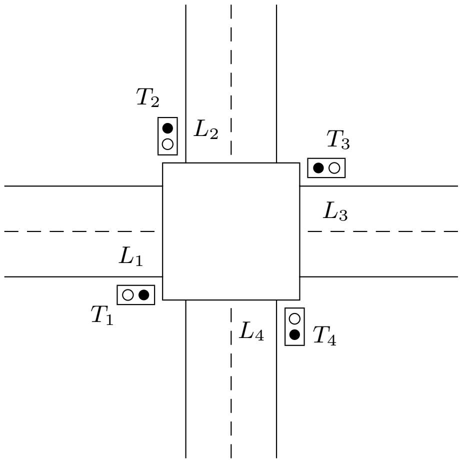

Fig. 2. A traffic-light-controlled intersection of two two-way streets.

for $t \in (t_{2k}, t_{2k+1})$ with $k \in \mathbb{N}$, and

$$l_1(t_{2k+1}) = l_1(t_{2k}) + \lambda_1 \delta_{2k}$$

for $k = 0, 1, 2, \ldots$
When the traffic light T_1 is green, there are arrivals and departures at lane L_1. Since then the net queue growth rate is $\lambda_1 - \mu_1$ and since the queue length $l_1(t)$ cannot be negative, we have

$$\frac{dl_1(t)}{dt} = \begin{cases} \lambda_1 - \mu_1 & \text{if } l_1(t) > 0 \\ 0 & \text{if } l_1(t) = 0 \end{cases} \tag{11}$$

for $t \in (t_{2k+1}, t_{2k+2})$ with $k \in \mathbb{N}$. So

$$l_1(t_{2k+2}) = \max\bigl(l_1(t_{2k+1}) + (\lambda_1 - \mu_1)\delta_{2k+1}, 0\bigr)$$

for $k = 0, 1, 2, \ldots$
Note that we also have

$$l_1(t_{2k+1}) = \max\bigl(l_1(t_{2k}) + \lambda_1 \delta_{2k}, 0\bigr)$$

for $k = 0, 1, 2, \ldots$ since $l_1(t) \geq 0$ for all t.
We can write down similar equations for $l_2(t_k)$, $l_3(t_k)$ and $l_4(t_k)$.
So if we define

$$x_k = \begin{bmatrix} l_1(t_k) \\ l_2(t_k) \\ l_3(t_k) \\ l_4(t_k) \end{bmatrix}, \quad b_1 =, \begin{bmatrix} \lambda_1 \\ \lambda_2 - \mu_2 \\ \lambda_3 \\ \lambda_4 - \mu_4 \end{bmatrix}, \quad b_2 = \begin{bmatrix} \lambda_1 - \mu_1 \\ \lambda_2 \\ \lambda_3 - \mu_3 \\ \lambda_4 \end{bmatrix},$$

then we have

$$x_{2k+1} = \max(x_{2k} + b_1\delta_{2k}, 0) \tag{12}$$

$$x_{2k+2} = \max(x_{2k+1} + b_2\delta_{2k+1}, 0) \tag{13}$$

for $k = 0, 1, 2, \ldots$

Remarks:

- The traffic-light-controlled intersection can be considered as a hybrid system that has the time and the queue lengths as state variables, and that can operate in two regimes characterized by differential equations of the form (10) or (11) depending on the value of a discrete control variable that can have the value "red" or "green".
- The model we have derived is different from the models used by most other researchers due to the fact that we consider red-green cycle lengths that may vary from cycle to cycle. Furthermore, we consider non-saturated intersections, i.e., we allow queue lengths to become 0 during the green cycle. Some authors (see, e.g., [16, 20]) only consider models for oversaturated intersections, i.e., they do not allow queue lengths to become equal to 0 during the green cycle. In that case the maximum operator that appears in (12) – (13) is not necessary any more, which leads to a simpler description of the behavior of the system. However, in [13] we have shown that, when we want to design optimal traffic light switching schemes, applying a model for oversaturated intersections to a non-saturated intersection in general does not lead to an optimal traffic light switching scheme.

4.2 Link with the ELCP

Let us now show that the system (12) – (13) can be reformulated as an ELCP. First consider (12) for an arbitrary index k. This equation can be rewritten as follows:

$$\begin{aligned} x_{2k+1} &\geq x_{2k} + b_1\delta_{2k} \\ x_{2k+1} &\geq 0 \\ (x_{2k+1})_i = (x_{2k} + b_1\delta_{2k})_i \quad &\text{or} \quad (x_{2k+1})_i = 0 \qquad \text{for } i = 1, 2, 3, 4\,, \end{aligned}$$

or equivalently

$$\begin{aligned} x_{2k+1} - x_{2k} - b_1\delta_{2k} &\geq 0 \\ x_{2k+1} &\geq 0 \\ (x_{2k+1} - x_{2k} - b_1\delta_{2k})_i\,(x_{2k+1})_i &= 0 \qquad \text{for all } i\,. \end{aligned}$$

Since a sum of nonnegative numbers is equal to 0 if and only if all the numbers are equal to 0, this system of equations is equivalent to:

$$x_{2k+1} - x_{2k} - b_1\delta_{2k} \geq 0$$

$$x_{2k+1} \geq 0$$

$$\sum_{i=1}^{4} (x_{2k+1} - x_{2k} - b_1 \delta_{2k})_i \, (x_{2k+1})_i = 0 \ .$$

We can repeat this reasoning for (13) and for each index k.
If we consider N switching time instants and if we define

$$x^* = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \quad \text{and} \quad \delta^* = \begin{bmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_{N-1} \end{bmatrix} \ ,$$

we finally get a description of the form

$$Ax^* + B\delta^* + c \geq 0 \tag{14}$$

$$x^* \geq 0 \tag{15}$$

$$(Ax^* + B\delta^* + c)^T x^* = 0 \ . \tag{16}$$

It is easy to verify that the system (14) – (16) is a special case of an ELCP.

Now we can compute traffic light switching schemes that minimize objective functions such as

– (weighted) average queue length over all queues:

$$J_1 = \sum_{i=1}^{4} w_i \, \frac{\displaystyle\int_{t_0}^{t_N} l_i(t) \, dt}{t_N - t_0} \ ,$$

– (weighted) worst case queue length:

$$J_2 = \max_{i,\,t} \, (w_i \, l_i(t)) \ ,$$

– (weighted) average waiting time over all queues:

$$J_3 = \sum_{i=1}^{4} w_i \, \frac{\displaystyle\int_{t_0}^{t_N} l_i(t) \, dt}{\lambda_i (t_N - t_0)} \ ,$$

and so on, where $w_i > 0$ for all i. Furthermore, we can impose extra conditions such as minimum and maximum durations for the green and the red time[4], maximum queue lengths[5], and so on. This leads to the following problem:

$$\text{minimize } J \tag{17}$$

[4] A green time that is too short is wasteful. If the red time is too long, drivers tend to believe that the signals have broken down.

[5] This could correspond to an upper bound on the available storage space due to the distance to the preceding junction or to the layout of the intersection.

subject to

$$\delta_{\min,\mathrm{r}} \leq \delta_{2k} \leq \delta_{\max,\mathrm{r}} \qquad \text{for } k \in \alpha(N) \tag{18}$$

$$\delta_{\min,\mathrm{g}} \leq \delta_{2k+1} \leq \delta_{\max,\mathrm{g}} \qquad \text{for } k \in \beta(N) \tag{19}$$

$$x_k \leq x_{\max} \qquad \text{for } k = 1, 2, \ldots, N \tag{20}$$

$$x_{2k+1} = \max(x_{2k} + b_1\delta_{2k}, 0) \qquad \text{for } k \in \alpha(N) \tag{21}$$

$$x_{2k+2} = \max(x_{2k+1} + b_2\delta_{2k+1}, 0) \qquad \text{for } k \in \beta(N)\ , \tag{22}$$

with

$$\alpha(N) = \left\{0, 1, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor \right\} \quad \text{and} \quad \beta(N) = \left\{0, 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor - 1 \right\}\ ,$$

where $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.
It can be shown [13] that the objective function J_2 (i.e., the (weighted) worst case queue length) is convex as a function of the δ_k's, which implies that problem (17) – (22) with $J = J_2$ can be solved efficiently (if there is no upper bound on the queue lengths, or if we deal with constraint (20) by introducing a convex penalty term if some components of $x_{\max}$ are finite). However, the objective functions J_1 and J_3 are neither convex nor concave. Our computational experiments have shown that in order to solve problem (17) – (22) with $J = J_1$ or $J = J_3$ using constrained optimization (with, e.g., sequential quadratic programming) several initial starting points are necessary to obtain the global minimum.

Using the procedure given above the system (18) – (22) can be rewritten as a system of the form

$$Ax^* + B\delta^* + c \geq 0 \tag{23}$$

$$x^* \geq 0 \tag{24}$$

$$Ex^* + D\delta^* + f \geq 0 \tag{25}$$

$$(Ax^* + B\delta^* + c)^T x^* = 0\ , \tag{26}$$

which is again a special case of an ELCP. In order to determine the optimal traffic light switching scheme we could first determine the solution set of the ELCP and then minimize the objective function J over this solution set. Our computational experiments have shown that the determination of the minimum value of the objective functions J_1 and J_3 is a well-behaved problem in the sense that using a local minimization routine (that uses, e.g., sequential quadratic programming) starting from different initial points always yields the same numerical result (within a certain tolerance). Furthermore, it can be shown [13] that J_2 is a convex function of the parameters λ_k, κ_k and μ_k that characterize the solution set of the ELCP (cf. (2)).

The algorithm of [6, 7] to compute the solution set of a general ELCP requires exponential execution times. This implies that the approach sketched above is not feasible if the number of switching cycles N is large. However, in [12] we have developed efficient methods to determine suboptimal traffic light switching schemes for the model (23) – (26): for the objective functions J_1 (i.e., (weighted) average queue length) and J_3 (i.e., (weighted) average waiting time) we can make some approximations that transform the problem into an optimization problem over a convex feasible set, or even into a linear programming problem. This approach is computationally very efficient and yields suboptimal solutions that approximate the global optimal solution very well.

For more information on other models that describe the evolution of the queue lengths at a traffic-light-controlled intersection and on optimal traffic light control the interested reader is referred to [14, 17, 18, 20] and the references given therein.

5 Conclusions and further research

We have introduced the Extended Linear Complementarity Problem (ELCP) and indicated how it can be used in the modeling and analysis of certain classes of discrete event systems, continuous variable systems and hybrid systems. More specifically, we have shown that for a traffic-light-controlled intersection the evolution of the queue lengths at the switching time instants can be described by an ELCP.

Topics for further research include: development of efficient algorithms for the special cases of the ELCP that appear in the analysis of specific classes of hybrid systems, investigation of the use of the ELCP to model and to analyze other classes of hybrid systems, and extension of our model for a traffic-light-controlled intersection to networks of intersections.

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