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G.J. Olsder and B. De Schutter

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K.U.Leuven
Leuven, Belgium
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Chapter 32

The minimal realization problem in the max-plus algebra

Geert Jan Olsder* and Bart De Schutter**

*Department of Mathematics and Informatics
Delft University of Technology
2600 GA Delft
THE NETHERLANDS
g.j.olsder@math.tudelft.nl

**ESAT-SISTA
K.U.Leuven
3001 Leuven
BELGIUM
Bart.DeSchutter@esat.kuleuven.ac.be

32.1 Description of the problem

Given an arbitrary real sequence $\{g_i\}_{i=1}^{\infty}$ elegant necessary and sufficiency conditions are known for the existence of an $n \times n$ matrix A, an $n \times 1$ vector b and a $1 \times n$ vector c, for some appropriate n, such that

$$g_i = cA^{i-1}b \qquad \text{for } i = 1, 2, \ldots \tag{32.1}$$

The elements of A, b and c are supposed to be real numbers. An additional requirement might be that n, which determines the sizes of A, b and c, must be as small as possible. In that case n is called the *minimal system order* and the triple A, b and c is a *minimal realization*. Efficient algorithms to calculate a minimal realization are known (see, e.g., [11]).

The problem considered in this chapter arises when the underlying algebra is the so-called max-plus algebra [2, 3] rather than the conventional algebra tacitly used above. One obtains the max-plus algebra from the conventional algebra by replacing addition by maximization and multiplication by addition. These operations are indicated by $\oplus$ (maximization) and $\otimes$ (addition). In the max-plus algebra one for instance has

$$\begin{pmatrix} 1 & 4 \\ -3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} (1 \otimes 5) \oplus (4 \otimes 1) \\ (-3 \otimes 5) \oplus (0 \otimes 1) \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} .$$

32.2 Motivation

In conventional system theory the sequence $\{g_i\}_{i=1}^{\infty}$ arises as the impulse response of the linear, finite-dimensional, discrete-time, time-invariant SISO[1] state space description

$$x(k+1) = Ax(k) + bu(k) \ , \quad y(k) = cx(k) \ .$$

The problem considered here is to compute a minimal realization and to characterize the minimal system order for max-plus linear systems, i.e., systems of the form

$$x(k+1) = A \otimes x(k) \oplus b \otimes u(k) \ , \quad y(k) = c \otimes x(k) \ . \tag{32.2}$$

In spite of its misleading simple formulation, this problem has met with formidable difficulties.

32.3 History and partial results

32.3.1 Characterization of max-plus-algebraic impulse responses

A necessary and sufficient condition for a sequence $\{g_i\}_{i=1}^{\infty}$ to be the impulse response of a system that can be described by a model of the form (32.2) is that the sequence is *ultimately periodic* [8, 9], i.e.,

$$\begin{aligned} &\exists m, \lambda_0, \ldots, \lambda_{m-1}, k_0 \text{ such that } \forall k \geq k_0 \ : \\ &\qquad g_{km+m+s} \ = \ \lambda_s^{\otimes^c} \otimes g_{km+s} \quad \text{for } s = 0, 1, \ldots, m-1 \ . \end{aligned}$$

where $\lambda^{\otimes^m} = \lambda \times m$.

32.3.2 The minimal system order

We define so-called Hankel matrix $H(\alpha, \beta)$ of size $\alpha \times \beta$ as

$$H(\alpha, \beta) = \begin{pmatrix} g_1 & g_2 & \cdots & g_\beta \\ g_2 & g_3 & \cdots & g_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ g_\alpha & g_{\alpha+1} & \cdots & g_{\alpha+\beta-1} \end{pmatrix} . \tag{32.3}$$

[1] SISO: single-input single output. Generalizations to multiple-input multiple-output systems exist, but will not be emphasized here.

In conventional system theory the minimal system order is given by the rank of the Hankel matrix $H(\infty, \infty)$. However, in contrast to linear algebra the different notions of rank (like column rank, row rank, minor rank, ...) are in general not equivalent in the max-plus algebra[2].

Let $H = H(\infty, \infty)$. It can be shown [8] that the minimal system order is equal to the smallest integer r for which there exist an $\infty \times r$ matrix U. an $r \times \infty$ matrix V and an $r \times r$ matrix A such that $H = U \otimes V$ and $U \otimes A = \overline{U}$, where $\overline{U}$ is the matrix obtained by removing the first row of U.

The different notions of matrix rank in the max-plus-algebra can be used to obtain lower and upper bounds for the minimal system order. The so-called max-plus-algebraic minor rank and Schein rank of H provide lower bounds [8, 9]. At present, there are no efficient (i.e., polynomial time) algorithms to compute the max-plus-algebraic minor rank or the Schein rank of a matrix. The max-plus-algebraic weak column rank of H provides an upper bound [8, 9]. Efficient methods to compute this rank are described in [3, 8].

32.3.3 Minimal state space realization: partial results

Transformation to conventional algebra

There exists a transformation from the max-plus algebra to the linear algebra that is based on the following equivalences:

$$x \oplus y = z \quad \Leftrightarrow \quad e^{xs} + e^{ys} \sim ce^{zs}\ ,\ s \to \infty \tag{32.4}$$

$$x \otimes y = z \quad \Leftrightarrow \quad e^{xs} \cdot e^{ys} = e^{zs} \quad \text{for all } s > 0 \tag{32.5}$$

with $c = 2$ if $x = y$ and $c = 1$ otherwise.

Using this transformation the minimal realization problem in the max-plus algebra can be mapped to a minimal realization problem for matrices with exponentials as entries and with conventional addition and multiplication as basic operations [12, 13]. This implies that we can use the techniques from conventional realization theory to obtain a minimal realization afterwards (try to) transform the results back to the max-plus algebra. However, only realizations with positive coefficients for the leading exponentials can be mapped back to the max-plus algebra, and it is not always obvious how and whether such a realization can be constructed.

Partial state space realization

The *partial* minimal realization problem is defined as follows: given a finite sequence $g_1,\ g_2,\ \ldots,\ g_N$, find A, b and c such that $g_i = c \otimes A^{\otimes^{i-1}} \otimes b$ for $i = 1, 2, \ldots, N$. It can be shown that this leads to a system of so-called max-plus-algebraic polynomial equations and that such a system can be recast as an Extended Linear Complementarity Problem (ELCP) [5, 6]. This enables us to solve the partial minimal realization problem and by applying some limit arguments this results in a realization of the entire impulse response. However, it can be shown that the general ELCP is NP-hard.

[2] An overview of the relations between the different ranks in the max-plus algebra can found on p. 122 of [8].

Special sequences of Markov parameters

For some special cases, e.g., if the sequence $\{g_i\}_{i=1}^{\infty}$ exhibits uniformly up-terrace behavior [16, 17], or if the sequence exhibits a convex transient behavior and a so-called ultimately geometric behavior with period 1 [4, 10], there exist methods to efficiently compute minimal state space realizations.

32.4 Related fields

Based on the relations (32.4) and (32.5) it is easy to verify that there exists a connection between the minimal realization problem in the max-plus algebra and the minimal realization problem for nonnegative systems. Indeed, some of the results obtained in system theory for nonnegative systems also hold in the max-plus algebra (see, e.g., [7]). For more information on the minimal realization problem for nonnegative systems the reader is referred to [1, 15].

Remark: For a more extended overview of known results, open problems and additional references in connection with the minimal realization problem in the max-plus algebra the interested reader is referred to [14].

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